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Riemannian Manifolds Satisfying Certain Conditions on Pseudo-Projective Curvature Tensor

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Abstract. In this paper we determine some properties of pseudo-projective curvature tensor denoted by $\bar{P}$ on some Riemannian manifolds, especially on generalized quasi Einstein manifolds in the sense of Chaki. Firstly, we consider a pseudo-projectively Ricci semisymmetric generalized quasi Einstein manifold. After that, we study pseudo-projective flatness of this manifold. Moreover, we construct a non-trivial example for a generalized quasi Einstein manifold to prove the existence.

1. Introduction

In 2002, Prasad [14] defined and studied a tensor field $\bar{P}$ on a Riemannian manifold of dimension $n(> 2)$ which includes the projective curvature tensor P. This tensor field $\bar{P}$ is known as pseudo-projective curvature tensor and it is given by

$$\bar{P}(X,Y)Z = \alpha R(X,Y)Z + \beta[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{\alpha}{n-1} + \beta\right)[g(Y,Z)X - g(X,Z)Y], \tag{1}$$

where α, β are non-zero constants, R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

Note that the pseudo-projective curvature tensor satisfies the following symmetry properties:

(i) $\bar{P}(X,Y,Z,W) = -\bar{P}(Y,X,Z,W)$,
(ii) $\bar{P}(X,Y,Z,W) \neq \mp\bar{P}(X,Y,W,Z)$

for all $X, Y, Z, W \in TM$.

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leqslant i \leqslant n$. Now, from (1), we have

$$\sum_{i=1}^{n} \bar{P}(X,Y,e_i,e_i) = \sum_{i=1}^{n} \bar{P}(e_i,e_i,Z,W) = 0, \tag{2}$$

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$$\sum_{i=1}^{n} \bar{P}(e_i, Y, Z, e_i) = [\alpha + \beta(n-1)] \underbrace{[S(Y,Z) - \frac{r}{n} g(Y,Z)]}_{:=P(Y,Z)} \tag{3}$$

and

$$\sum_{i=1}^{n} \bar{P}(X, e_i, e_i, W) = (\alpha - \beta) \underbrace{[S(X,W) - \frac{r}{n} g(X,W)]}_{:=P(X,W)} . \tag{4}$$

Also,

$$\sum_{i=1}^{n} P(e_i, e_i) = 0. \tag{5}$$

If $\alpha = 1$ and $\beta = -\frac{1}{n-1}$ in (1), then the pseudo-projective curvature tensor takes the form

$$\bar{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] = P(X,Y)Z, \tag{6}$$

where P denotes the projective curvature tensor, [11]. Thus, the projective curvature tensor is the particular case of the pseudo-projective curvature tensor.

The projective curvature tensor is an important tensor in differential geometry. In a Riemannian manifold M, if there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor P vanishes. By virtue of (6), M is projectively flat if and only if it is of constant curvature. Hence the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. For recent developments on projective curvature tensor, we refer to [16] and [21].

In 2000, M.C. Chaki and R.K. Maity introduced the notion of quasi Einstein manifolds as generalization of Einstein manifolds. According to them, a Riemannian manifold (M^n, g) $(n > 2)$ is said to be quasi Einstein manifold [3] if its Ricci tensor of type $(0,2)$ is not identically zero and it satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{7}$$

for all $X, Y \in \mathfrak{X}(M^n)$ where a and b are real valued, non-zero scalar functions of which $b \neq 0$ on (M^n, g) and A is a non-zero 1-form, equivalent to the unit vector field U, that is,

$$g(X,U) = A(X)\ ,\ g(U,U) = 1. \tag{8}$$

A is called an associated 1-form and U is called a generator of (M^n, g). If $b = 0$, then the manifold reduces to an Einstein manifold. Such an $n-$dimensional manifold is denoted by $(QE)_n$.

The notion of generalized quasi Einstein manifold has been first introduced by M.C. Chaki in 2001 [4]. A Riemannian manifold (M^n, g) $(n > 2)$ is said to be generalized quasi Einstein manifold if its Ricci tensor of type $(0,2)$ is not identically zero and it satisfies the following condition [4]

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \tag{9}$$

for all $X, Y \in \mathfrak{X}(M^n)$ where $a,\ b,\ c$ are real valued, non-zero scalar functions of which $b \neq 0$, $c \neq 0$ on (M^n, g), A and B are two non-zero 1-forms such that

$$g(X,U) = A(X)\ \ ,\ g(X,V) = B(X)\ \ ,\ g(U,V) = 0\ \ g(U,U) = g(V,V) = 1. \tag{10}$$

That is, U and V are orthonormal vector fields corresponding to the 1-forms A and B, respectively. Similarly, a, b and c are called associated scalars, A and B are called associated 1-forms and U and V are generators of manifold. Such an n-dimensional manifold has been denoted by $G(QE)_n$. If $c = 0$, then the manifold reduces to a quasi Einstein manifolds and if $b = c = 0$, then the manifold reduces to an Einstein manifold. Also, an operator Q defined by $g(QX, Y) = S(X, Y)$ is called the Ricci operator.

Contracting (9) over X and Y, the scalar curvature function of this manifold is given by

$$r = an + b. \tag{11}$$

In view of the equations (9) and (10), in a generalized quasi Einstein manifold, we have

$$S(Y, U) = (a + b)A(Y) + cB(Y) \;\; and \;\; S(Y, V) = aB(Y) + cA(Y). \tag{12}$$

Let R denote the Riemannian curvature tensor of M. The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined as

$$N(k) : p \to N_p(k) = \Big\{Z \in T_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y], \;\; \forall X, Y \in TM\Big\}, \tag{13}$$

where k is some smooth function, [17]. In a quasi Einstein manifold M, if the generator U belongs to some k-nullity distribution $N(k)$, then M said to be $N(k)$-quasi Einstein manifold [19]. According to C. Özgür and M. M. Triphati [13], in an n-dimensional $N(k)$-quasi Einstein manifold, the function k equals $\frac{a+b}{n-1}$.

Many authors have been investigated the pseudo-projective curvature tensor on Kenmotsu manifold, LP-Sasakian manifolds and otained some conditions for these manifolds to be of Einstein, η-Einstein and pseudo-projectively flat, [1, 2, 20]. Moreover, J.P. Jaiswal and R.H. Ojha, [10] have been studied weakly pseudo projective symmetric and Ricci-symmetric manifolds and investigated the nature of scalar curvature in these manifolds. On the other hand, in [9] authors have been determined some properties of the generalized quasi Einstein manifolds satisfying some Ricci conditions on some curvature tensors such as conformal, concircular and projective curvature tensors. Motivated by these studies, in this paper we study pseudo-projective curvature tensor with certain conditions on some Riemannian manifolds, especially on generalized quasi Einstein manifolds.

This paper is organized as follows: Firstly, we prove that any pseudo-projectively Ricci semisymmetric generalized quasi Einstein manifold (i.e. it satisfies the condition $\bar{P} \cdot S = 0$) is an $N(k)$-quasi Einstein manifold introduced in [19]. After that, we study the pseudo-projective flatness of this manifold and we prove some results about it. Moreover, we construct a non-trivial example for the generalized quasi Einstein manifold to prove the existence.

2. Pseudo-Projectively Ricci Semisymmetric $G(QE)_n$

For a $(0, k)$-tensor T, where $k \geq 1$, a $(0, k + 2)$-tensor $R \cdot T$ is defined by

$$\begin{aligned}(R \cdot T)(X_1, X_2, \ldots, X_k; X, Y) &= (R(X, Y) \cdot T)(X_1, X_2, \ldots, X_k)\\ &= -T(R(X, Y)X_1, \ldots, X_k) - \ldots - T(X_1, \ldots, R(X, Y)X_k),\end{aligned} \tag{14}$$

where R denotes the Riemannian curvature tensor.

Let A be a symmetric $(0, 2)$-tensor and T a $(0, k)$-tensor. Then the tensor $Q(A, T)$ is called a Tachibana tensor [6] of A and T and it is given by

$$\begin{aligned}Q(A, T)(X_1, X_2, \ldots, X_k; X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, X_2, \ldots, X_k)\\ &= -T((X \wedge_A Y)X_1, \ldots, X_k) - \ldots - T(X_1, \ldots, (X \wedge_A Y)X_k).\end{aligned} \tag{15}$$

For symmetric $(0, 2)$-tensors E and F, their Kulkarni-Nomizu product $E \wedge F$ is defined by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) =& E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4)\\ &- E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3)\end{aligned} \tag{16}$$

for all $X_1, X_2, X_3, X_4 \in TM$.

Definition 2.1. ([7]) An n-dimensional Riemannian manifold (M^n, g) is called *Ricci-pseudosymmetric* if and only if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. That means, the equation

$$(R(X,Y) \cdot S)(Z,W) = L_S Q(g,S)(Z,W;X,Y) \tag{17}$$

holds on U_S where $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$ and L_S is a certain function on U_S.

Analogously, we can give the following definition:

Definition 2.2. An n-dimensional Riemannian manifold (M^n, g) is called *pseudo-projectively Ricci semisymmetric* if the pseudo-projective curvature tensor satisfies the condition $\bar{P} \cdot S = 0$.

That is, in a pseudo-projectively Ricci semisymmetric manifold, the condition

$$(\bar{P}(X,Y) \cdot S)(Z,W) = -S(\bar{P}(X,Y)Z,W) - S(Z,\bar{P}(X,Y)W) = 0; \quad \forall X,Y,Z \in \mathfrak{X}(M^n) \tag{18}$$

holds.

In this section, we consider pseudo-projectively Ricci semisymmetric $G(QE)_n$. The main purposes of this section is to prove the following theorem.

Theorem 2.3. *Every pseudo-projectively Ricci semisymmetric $G(QE)_n$ is an $N(k)$-quasi Einstein manifold.*

Proof. Combining (9) and (18), in a pseudo-projectively Ricci semisymmetric $G(QE)_n$ we get

$$\begin{aligned} &a[\bar{P}(X,Y,Z,W) + \bar{P}(X,Y,W,Z)] + b[A(\bar{P}(X,Y)Z)A(W) + A(Z)A(\bar{P}(X,Y)W)] \\ &+ c[A(\bar{P}(X,Y)Z)B(W) + A(W)B(\bar{P}(X,Y)Z) + A(Z)B(\bar{P}(X,Y)W) + A(\bar{P}(X,Y)W)B(Z)] = 0, \end{aligned} \tag{19}$$

where $g(\bar{P}(X,Y)Z,W) = \bar{P}(X,Y,Z,W)$. In view of (1), (19) yields

$$\begin{aligned} &a\beta[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &+ b[\bar{P}(X,Y,Z,U)A(W) + A(Z)\bar{P}(X,Y,W,U)] + c[\bar{P}(X,Y,Z,U)B(W) \\ &+ A(W)\bar{P}(X,Y,Z,V) + A(Z)\bar{P}(X,Y,W,V) + \bar{P}(X,Y,W,U)B(Z)] = 0. \end{aligned} \tag{20}$$

Putting $Z = U$ and $W = V$ in (20) and using (10), we get

$$\begin{aligned} &a\beta[S(Y,U)B(X) - S(X,U)B(Y) + S(Y,V)A(X) - S(X,V)A(Y)] + b\bar{P}(X,Y,V,U) \\ &+ c[\bar{P}(X,Y,U,U) + \bar{P}(X,Y,V,V)] = 0. \end{aligned} \tag{21}$$

By virtue of (1), we have the following three relations:

- $$\bar{P}(X,Y,V,U) = \alpha R(X,Y,V,U) + \beta[S(Y,V)A(X) - S(X,V)A(Y)] - \frac{r}{n}(\frac{\alpha}{n-1} + \beta)[A(X)B(Y) - A(Y)B(X)], \tag{22}$$

- $$\bar{P}(X,Y,U,U) = \beta[S(Y,U)A(X) - S(X,U)A(Y)] \tag{23}$$

and

- $$\bar{P}(X,Y,V,V) = \beta[S(Y,V)B(X) - S(X,V)B(Y)]. \tag{24}$$

By using (22), (23) and (24) in (21) (as b is different than zero), in a $G(QE)_n$ satisfying the condition $\bar{P} \cdot S = 0$, we obtain the following relation:

$$R(X,Y,U,V) = \frac{r}{\alpha n}(\frac{\alpha}{n-1} + \beta)[A(Y)B(X) - A(X)B(Y)]. \tag{25}$$

Contracting (20) over X and W and using the equation (4), we get

$$\begin{aligned} &a\beta[nS(Y,Z) - rg(Y,Z)] + b[\bar{P}(U,Y,Z,U) - A(Z)(\alpha-\beta)\{S(Y,U) - \frac{r}{n}A(Y)\}] \\ &+c[\bar{P}(V,Y,Z,U) + \bar{P}(U,Y,Z,V) - A(Z)(\alpha-\beta)\{S(Y,V) - \frac{r}{n}B(Y)\} \\ &-B(Z)(\alpha-\beta)\{S(Y,U) - \frac{r}{n}A(Y)\}] = 0. \end{aligned} \tag{26}$$

Putting $Z = U$ in (26), we get

$$\begin{aligned} &a\beta[nS(Y,U) - rA(Y)] + b[\bar{P}(U,Y,U,U) - (\alpha-\beta)\{S(Y,U) - \frac{r}{n}A(Y)\}] \\ &+c[\bar{P}(V,Y,U,U) + \bar{P}(U,Y,U,V) - (\alpha-\beta)\{S(Y,V) - \frac{r}{n}B(Y)\}] = 0. \end{aligned} \tag{27}$$

By using (10), (12), (22), (23) and (25) in (27), we obtain

$$\Big[-\alpha b(a+b) + \frac{br}{n}(\frac{\alpha}{n-1} + \beta)(n-1) - \alpha c^2\Big]A(Y) + \Big[-\alpha c(a+b) + \frac{cr}{n}(\frac{\alpha}{n-1} + \beta)(n-1)\Big]B(Y) = 0. \tag{28}$$

Putting $Y = U$ in (28), we get

$$\alpha[b(a+b) + c^2] = \frac{br}{n}(\frac{\alpha}{n-1} + \beta)(n-1). \tag{29}$$

Putting $Y = V$ in (28), we get

$$c[-\alpha(a+b) + \frac{r}{n}(\frac{\alpha}{n-1} + \beta)(n-1)] = 0. \tag{30}$$

Then by (30),we have either $c = 0$ or $\alpha(a+b) = \frac{r}{n}(\frac{\alpha}{n-1} + \beta)(n-1)$. If $c = 0$, then by (29), we get $\alpha b(a+b) = \frac{br}{n}(\frac{\alpha}{n-1} + \beta)(n-1)$ and so $b[\alpha(a+b) - \frac{r}{n}(\frac{\alpha}{n-1} + \beta)(n-1)] = 0$. Thus $b = 0$ or $\alpha(a+b) = \frac{r}{n}(\frac{\alpha}{n-1} + \beta)(n-1)$. If $b = 0$, then as both of b and c are zero, the manifold reduces to an Einstein manifold, but this is a contradiction. Thus $b \neq 0$ and so $\alpha(a+b) = \frac{r}{n}(\frac{\alpha}{n-1} + \beta)(n-1)$.

On the other hand, if $c \neq 0$, then again we obtain $\alpha(a+b) = \frac{r}{n}(\frac{\alpha}{n-1} + \beta)(n-1)$. Using this in (29), we get $c^2\alpha = 0$. Since $\alpha \neq 0$, we get $c = 0$.

That is, we conclude that

$$c = 0 \ \ and \ \ \frac{a+b}{n-1} = \frac{r}{\alpha n}(\frac{\alpha}{n-1} + \beta). \tag{31}$$

Since $c = 0$, the manifold reduces to a quasi Einstein manifold and from (31) and (25), we have

$$R(X,Y)U = \frac{a+b}{n-1}[A(Y)X - A(X)Y], \tag{32}$$

which means that the generator U belongs to the some k-nullity distribution, $k = \frac{a+b}{n-1}$. Hence the manifold under this consideration is an $N(k)$-quasi Einstein manifold. Hence, the proof is completed. □

Additionally, contracting (32), we get $S(Y,U) = (a+b)A(Y)$; i.e., $QY = (a+b)Y$, which means that $a+b$ is an eigenvalue of the Ricci operator Q. Hence we can state:

Theorem 2.4. *In a pseudo-projectively Ricci semisymmetric $G(QE)_n$, $(a+b)$ is an eigenvalue of the Ricci operator Q.*

Remark 2.5. By using the equations, (14), (15) and (16), the following relation holds:

$$(\bar{P} \cdot S) = \alpha(R \cdot S) - \frac{r}{n}[\frac{\alpha}{n-1} + \beta]Q(g, S). \tag{33}$$

Thus, if in a Riemannian manifold, the condition $\bar{P} \cdot S = 0$, then by (33), as $\alpha \neq 0$ we get

$$(R \cdot S) = \frac{r}{n\alpha}[\frac{\alpha}{n-1} + \beta]Q(g, S). \tag{34}$$

That is, the manifold reduces to a Ricci-pseudosymmetric manifold.

Thus, from above remark and Theorem (2.3), the following corollary obtained in [9] can be stated by a simple way:

Corollary 2.6. ([9]) *Every non-Einstein Ricci-pseudosymmetric $G(QE)_n$ is an $N(k)$-quasi Einstein manifold.*

3. Pseudo Projectively Flat and Pseudo-Projectively Conservative Riemannian Manifolds

In [16], projective curvature tensors of a non-symmetric affine connection space are expressed as functions of the affine connection coefficients and Weyl projective tensor of the corresponding associated affine connection space. Moreover, projective flatness of non-symmetric affine connection spaces were analysed.

Accordingly, in this section, we determine some properties of pseudo-projective curvature tensor and then we investigate n-dimensional ($n > 2$) pseudo-projectively flat manifold (M^n, g). In this case, the tensor $\bar{P}$ vanishes. Then by (1), we have

$$\alpha R(X, Y, Z, W) + \beta[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] = \frac{r}{n}(\frac{\alpha}{n-1} + \beta)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \tag{35}$$

Contracting (35) over X and W, we obtain

$$[\alpha + \beta(n-1)].[S(Y, Z) - \frac{r}{n}g(Y, Z)] = 0. \tag{36}$$

Then we get

$$\alpha + \beta(n-1) = 0 \ \ or \ \ S(Y, Z) = \frac{r}{n}g(Y, Z). \tag{37}$$

If $S(Y, Z) = \frac{r}{n}g(Y, Z)$, then the manifold under consideration becomes an Einstein manifold. Next, if $\alpha + \beta(n-1) = 0$ holds, then using this relation in (1) we obtain $\bar{P} = \alpha P$, where P denotes the projective curvature tensor given in (6). Thus, in this case in a pseudo-projectively flat manifold, as $\alpha \neq 0$, the projective curvature tensor vanishes and so this manifold is of constant curvature, i.e. this manifold is again Einstein. Hence, every pseudo projectively flat manifold is an Einstein manifold.

Now, let (M^n, g), $(n > 2)$ be a pseudo projectively conservative manifold so $div\bar{P} = 0$. Taking covariant derivative of (1), we obtain

$$\begin{aligned}(\nabla_W \bar{P})(X, Y)Z = {} & \alpha(\nabla_W R)(X, Y)Z + \beta[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y] \\ & -\frac{dr(W)}{n}(\frac{\alpha}{n-1} + \beta)[g(Y, Z)X - g(X, Z)Y].\end{aligned} \tag{38}$$

Contracting (38) over W, we get

$$\begin{aligned}(div\bar{P})(X, Y)Z = {} & \alpha(divR)(X, Y)Z + \beta[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] \\ & -\frac{1}{n}(\frac{\alpha}{n-1} + \beta)[g(Y, Z)dr(X) - g(X, Z)dr(Y)].\end{aligned} \tag{39}$$

It is known that

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{40}$$

Combining (39) and (40), the divergence of the pseudo-projective curvature tensor can be expressed as

$$(div\bar{P})(X,Y)Z = (\alpha+\beta)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \frac{1}{n}(\frac{\alpha}{n-1}+\beta)[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{41}$$

Thus, in a pseudo projectively conservative manifold (M^n, g), $(n > 2)$, the following relation holds:

$$(\alpha+\beta)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] = \frac{1}{n}(\frac{\alpha}{n-1}+\beta)[g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{42}$$

Contracting (42) over Y and Z and using contracted second Bianchi identity (as $n > 2$), we get for all $X \in TM$, $(\alpha-\beta)dr(X) = 0$ and so either $\alpha = \beta$ or the manifold has constant scalar curvature. Moreover, if $\alpha = \beta$, then by (42), we get

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)], \tag{43}$$

which means that $divC = 0$, where C denotes the conformal curvature tensor. Thus, in this case this manifold is conformally conservative. On the other hand, if $\alpha \neq \beta$, then the scalar curvature is constant and so from (42), we obtain

$$(\alpha+\beta)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] = 0. \tag{44}$$

Thus, if $\alpha + \beta \neq 0$, then by (44) the Ricci tensor of this manifold satisfies the condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{45}$$

which means that the manifold has the Codazzi type Ricci tensor, [8]. Hence, firstly we can state the following theorem.

Theorem 3.1. *Every pseudo projectively flat manifold is Einstein. Let (M^n, g), $(n > 2)$ be a pseudo-projectively conservative manifold. Then, either it is conformally conservative or it has constant scalar curvature. Moreover, if (M^n, g) is of constant scalar curvature and the scalars α and β appeared in $\bar{P}$ satisfies the relation $\alpha + \beta \neq 0$, then the Ricci tensor of this manifold is of Codazzi type.*

Now, we consider (M^n, g), $(n > 2)$ be a pseudo projectively conservative generalized quasi Einstein manifold whose associated scalars are constants. In this case, the scalar curvature $r = an + b$ is constant. Now, we also assume that $\alpha + \beta \neq 0$. Then, by Theorem (3.1), the Ricci tensor of this manifold is of Codazzi type. Thus, taking covariant derivative of the Ricci tensor of $G(QE)_n$, we get

$$\begin{aligned}(\nabla_Z S)(X,Y) = & b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] \\ & + c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)].\end{aligned} \tag{46}$$

Combining, the equations (46) and (45), we obtain

$$\begin{aligned}& b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y) - (\nabla_X A)(Z)A(Y) - A(Z)(\nabla_X A)(Y)] \\ & + c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X) \\ & - (\nabla_X A)(Z)B(Y) - A(Z)(\nabla_X B)(Y) - (\nabla_X A)(Y)B(Z) - A(Y)(\nabla_X B)(Z)] = 0.\end{aligned} \tag{47}$$

Putting $Y = U$ in (47), we get

$$b[(\nabla_Z A)(X) - (\nabla_X A)(Z)] + c[A(X)(\nabla_Z B)(U) + (\nabla_Z B)(X) - A(Z)(\nabla_X B)(U) - (\nabla_X B)(Z)] = 0 \tag{48}$$

and putting $X = U$ in (48), we get

$$-b(\nabla_U A)(Z) + c[2(\nabla_Z B)(U) - A(Z)(\nabla_U B)(U) - (\nabla_U B)(Z)] = 0. \tag{49}$$

Next, putting $Z = V$ in (49), we get

$$-b(\nabla_U A)(V) + 2c(\nabla_V B)(U) = 0. \tag{50}$$

On the other hand, putting $Y = Z = V$ and $X = U$ in (47), we obtain

$$b(\nabla_V A)(V) + 2c(\nabla_U B)(U) = 0. \tag{51}$$

Since $g(U, V) = 0$, from the properties of connection ∇, we have

$$g(\nabla_U U, V) + g(U, \nabla_U V) = 0 \ \ and \ \ g(\nabla_V U, V) + g(U, \nabla_V V) = 0. \tag{52}$$

Thus, by using (52) in (50) and (51) we obtain the following system of equation:

$$\begin{cases} -bg(\nabla_U U, V) + 2cg(\nabla_V V, U) = 0 \\ -bg(\nabla_V V, U) - 2cg(\nabla_U U, V) = 0 \end{cases} \tag{53}$$

Hence, from the above system, we obtain $c[(g(\nabla_U U, V))^2 + (g(\nabla_V V, U))^2] = 0$ and so we have either $c = 0$ or $(g(\nabla_U U, V))^2 + (g(\nabla_V V, U))^2 = 0$. If $c = 0$, then from the system (53), we obtain $g(\nabla_U U, V) = g(\nabla_V V, U) = 0$. (Otherwise, $b = 0$ and this means that the manifold reduces to an Einstein manifold.) On the other hand, if $(g(\nabla_U U, V))^2 + (g(\nabla_V V, U))^2 = 0$, then again we obtain $g(\nabla_U U, V) = g(\nabla_V V, U) = 0$. Thus, in each case, the generators of the manifold satisfy the following relations:

$$g(\nabla_U U, V) = g(\nabla_V V, U) = 0. \tag{54}$$

Now, putting $X = V$ in (48) and using (54), we get

$$b[(\nabla_Z A)(V) - (\nabla_V A)(Z)] - c(\nabla_V B)(Z) = 0. \tag{55}$$

Similarly, putting $Y = V$ and $Z = U$ in (47) and again using (54), we get

$$-b(\nabla_X A)(V) + c(\nabla_U A)(X) = 0. \tag{56}$$

Also, contracting (46) over X and Z, we obtain

$$b[div(A)A(Y) + (\nabla_U A)(Y)] + c[div(A)B(Y) + div(B)A(Y) + (\nabla_U B)(Y) + (\nabla_V A)(Y)] = 0. \tag{57}$$

Putting $Y = U$ in (57) and using (54) we obtain

$$bdiv(A) + cdiv(B) = 0. \tag{58}$$

Furthermore, putting $Y = V$ in (57) and by virtue of (54), we obtain $cdiv(A) = 0$ so we have either $c = 0$ or $div(A) = 0$. If $c = 0$, then by (58), (in this case we may take $b \neq 0$) again we get $div(A) = 0$. Thus, in each case the 1-form A is divergence-free. Moreover, in this case from (58), we have either $c = 0$ or $div(B) = 0$.

Case I: Firstly, we assume that $c \neq 0$ and so $div(B) = 0$. Since $div(A) = 0$ always holds, the equation (57) reduces to the following form:

$$b(\nabla_U A)(Y) + c[(\nabla_U B)(Y) + (\nabla_V A)(Y)] = 0. \tag{59}$$

Then, summing the equations (49) and (59), using (54) and $c \neq 0$, we get $(\nabla_V A)(Y) = 2(\nabla_Y A)(V)$. In view of the last equation, (55) yields

$$b(\nabla_Z A)(V) + c(\nabla_V B)(Z) = 0. \tag{60}$$

Thus, it follows from (56) and (60) and as $c \neq 0$, we obtain $\nabla_U U + \nabla_V V = 0$.

Case II: In this case, we assume $c = 0$. Then, by (48) (as $b \neq 0$),

$$(\nabla_Z A)(X) = (\nabla_X A)(Z). \tag{61}$$

Putting $X = U$ in (47) and using the equation (61) and $c = 0$, we obtain for all $X, Z \in TM$,

$$(\nabla_Z A)(X) = g(\nabla_Z U, X) = 0. \tag{62}$$

Thus, the generator U is a parallel vector field and so for all $X \in TM$, we have $\nabla_X U = 0$. Also, putting $X = U$, we get $\nabla_U U = 0$, i.e., the integral curves of U are geodesics. Moreover, from (46) and (62), the Ricci tensor satisfies the condition $\nabla S = 0$, i.e., this manifold becomes Ricci symmetric. Additionally, taking second covariant derivative of (62) and using Ricci identity, we get for all $X, Y, Z \in TM$, $R(U, X, Y, Z) = 0$. Thus, contracting the last equation over X and Y and as $c = 0$, we get $S(U, Z) = (a + b)A(Z) = 0$ and so we obtain $a + b = 0$.

As a consequence, we can state the following theorem.

Theorem 3.2. *Let (M^n, g), $(n > 2)$ be a pseudo-projectively conservative $G(QE)_n$ whose associated scalars are constants and the scalars α and β appeared in $\bar{P}$ satisfies the relation $\alpha + \beta \neq 0$. Then, the followings hold:*

(1) The generator V is orthogonal to $\nabla_U U$ and the generator U is orthogonal to $\nabla_V V$.
(2) The 1-form A is divergence-free.
(3) If the 1-form B has non-zero divergence, (or $\nabla_U U + \nabla_V V \neq 0$), then
   (i) (M^n, g) is a $(QE)_n$ in which sum of the associated scalar functions is zero.
   (ii) the main generator U is a parallel vector field.
   (iii) the integral curves of the generator vector field U are geodesics.
   (iv) (M^n, g) is Ricci symmetric.

Remark 3.3. ([18]) Let M be a complete $n \geqslant 2$ dimensional Riemannian manifold admitting a special concircular vector field ρ satisfying

$$\nabla_\mu \nabla_\lambda \rho = (-l\rho + m) g_{\mu\lambda}, \tag{63}$$

for some scalars μ and λ. Then, if $l = m = 0$, then M is the direct product $M^* \times I$ of an $(n-1)$-dimensional complete Riemannian manifold M^* with straight line I.

This remark is a special case of De Rham's decomposition theorem, [5]. Thus, in view of the above remark, the following corollary is obtained:

Corollary 3.4. *Let (M^n, g), $(n > 2)$ be a pseudo-projectively conservative $G(QE)_n$ whose associated scalars are constants and the scalars α and β appeared in $\bar{P}$ satisfies the relation $\alpha + \beta \neq 0$. If the main generator of this manifold is gradient and $divB \neq 0$, then this manifold is the direct product of the form $M^* \times I$, where M^* is an $(n-1)$-dimensional complete Riemannian manifold and I is the straight line.*

4. Existence of a 4-Dimensional Generalized Quasi Einstein Manifold

In general relativity and cosmology, the purpose of studying various types of semi-Riemannian manifolds is to represent the different phases in the evolution of the universe. The evolution of the universe to its present state can be divided into three phases: *The initial phase* is just after the Big-Bang when the effects of both viscosity and heat flux were quite pronounced. In *the intermediate phase*, the effect of viscosity was no longer significant but the heat flux was still not negligible. *The final phase* extends to the present state of the universe when both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid [12].

Thus, the importance of quasi Einstein and generalized quasi Einstein manifolds lies in the fact that these semi-Riemannian manifolds represent the second and the third phase respectively in the evolution of the universe. Additionally, a semi-Riemannian $G(QE)_4$ is related to the study of general relativistic fluid spacetime admitting heat flux, [15]. Because of all these reasons, in this section, we prove the existence of a generalized quasi Einstein manifold with non-zero and non-constant scalar curvature by constructing a non trivial example.

Now, we shall consider a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (e^{2x^4})(dx^1)^2 + (x^4)^4[(dx^2)^2 + (dx^3)^2] + (dx^4)^2, \tag{64}$$

where $-1-\sqrt{7} < x^4 < 1-\sqrt{3}$ or $-1+\sqrt{7} < x^4 < 1+\sqrt{3}$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the only non vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{14} = 1,\ \ \Gamma^4_{11} = -e^{2x^4},\ \ \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{x^4},\ \ \Gamma^4_{22} = \Gamma^4_{33} = -2(x^4)^3, \tag{65}$$

$$R_{1221} = R_{1331} = 2e^{2x^4}(x^4)^3,\ \ R_{1441} = e^{2(x^4)}, \tag{66}$$

$$R_{2332} = 4(x^4)^6,\ \ R_{2442} = R_{3443} = 2(x^4)^2, \tag{67}$$

$$R_{11} = \frac{4e^{2(x^4)}}{(x^4)} + e^{2(x^4)},\ \ R_{22} = R_{33} = 2(x^4)^2[(x^4)+3],\ \ R_{44} = 1 + \frac{4}{(x^4)^2} \tag{68}$$

and the components which can be obtained from these by symmetry properties. Also it can be shown that the scalar curvature is

$$r = \frac{16}{(x^4)^2} + \frac{8}{(x^4)} + 2, \tag{69}$$

which is non zero and non constant.

Let us now define associated scalar functions as

$$a = \frac{6}{(x^4)^2} + \frac{2}{(x^4)},\quad b = 2 - \frac{8}{(x^4)^2},\quad c = (\frac{4}{(x^4)^2} - 1)tan(2\lambda) \tag{70}$$

and the 1-forms

$$A_i(x) = \begin{cases} e^{(x^4)}sin(\lambda) & \text{if } i = 1 \\ 0 & \text{if } i = 2,3 \\ cos(\lambda) & \text{if } i = 4 \end{cases} \tag{71}$$

and

$$B_i(x) = \begin{cases} e^{(x^4)}cos(\lambda) & \text{if } i = 1 \\ 0 & \text{if } i = 2,3 \\ -sin(\lambda) & \text{if } i = 4. \end{cases} \tag{72}$$

Here λ is some non-zero function of (x^4) satisfying the conditions

$$sin^2(\lambda) = \frac{(x^4)^2 + 2(x^4) - 6}{4(x^4) - 4}\ \ and\ \ cos^2(\lambda) = \frac{(x^4)^2 - 2(x^4) - 2}{-4(x^4) + 4}.$$

Then, we can show that

1. $R_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1,$
2. $R_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2,$

3. $R_{33} = ag_{33} + bA_3A_3 + 2cA_3B_3$,
4. $R_{44} = ag_{44} + bA_4A_4 + 2cA_4B_4$.

Since all the cases other than (1)-(4) are trivial, we obtain

$$R_{ij} = ag_{ij} + bA_iA_j + c(A_iB_j + A_jB_i)\ , \quad for \ \ i, j = 1, 2, 3, 4. \tag{73}$$

Moreover, we find

$$g^{ij}A_iA_j = 1\ ,\ \ g^{ij}B_iB_j = 1\ ,\ and\ g^{ij}A_iB_j = 0 \tag{74}$$

and so

$$r = 4a + b = \frac{16}{(x^4)^2} + \frac{8}{(x^4)} + 2. \tag{75}$$

Therefore, this proves that the manifold under consideration is a generalized quasi Einstein manifold with non-zero and non-constant scalar curvature.

Hence we can state:

Theorem 4.1. *Let* $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 :\ x^4 \in (-1-\sqrt{7}, 1-\sqrt{3}) \bigcup (-1+\sqrt{7}, 1+\sqrt{3})\}$ *be an open subset of* $\mathbb{R}^4$ *endowed with the Riemannian metric given by*

$$ds^2 = g_{ij}dx^idx^j = (e^{2x^4})(dx^1)^2 + (x^4)^4[(dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

where x^1, x^2, x^3, x^4 *are the standard coordinates of* $\mathbb{R}^4$*. Then* (M^4, g) *is a generalized quasi Einstein manifold with non zero and non constant scalar curvature* $r = \frac{16}{(x^4)^2} + \frac{8}{(x^4)} + 2$.

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