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A Few Remarks on Bounded Operators on Topological Vector Spaces

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Abstract. We give a few observations on different types of bounded operators on a topological vector space X and their relations with compact operators on X. In particular, we investigate when these bounded operators coincide with compact operators. We also consider similar types of bounded bilinear mappings between topological vector spaces. Some properties of tensor product operators between locally convex spaces are established. In the last part of the paper we deal with operators on topological Riesz spaces.

1. Introduction

Throughout the paper, all topological vector spaces are over the scalar field $\mathbb{K}$ which is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A neighborhood U of 0 in a topological vector space will be simply called a zero neighborhood.

Let X be a topological vector space. A subset A of X is *bounded* if for each zero neighborhood U in X there is a scalar λ such that $A \subseteq \lambda U$. X is said to be *locally bounded* if there is a bounded neighborhood of $0 \in X$. In the literature one can find two different notions of bounded operators. In [11, 13, 15] there is the definition which is a definition of nb-bounded operators below, while in [7] one can find the definition which is a definition of bb-bounded operators; the following terminology is from [16]. A linear operator T on X is said to be:

- nb-*bounded* if there exists a zero neighborhood $U \subseteq X$ such that $T(U)$ is a bounded subset of X;
- bb-*bounded* if T mappings bounded sets into bounded sets.

The class of all nb-bounded operators on X is denoted by $\mathsf{B}_{\mathsf{n}}(X)$ and is equipped with the topology of uniform convergence on some zero neighborhood. A net (S_α) of nb-bounded operators is said to converge to zero uniformly on a zero neighborhood $U \subseteq X$ if for each zero neighborhood V, there is an α_0 with $S_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$.

The class of all bb-bounded operators on X is denoted by $\mathsf{B}_{\mathsf{b}}(X)$ and is endowed with the topology of uniform convergence on bounded sets. Note that a net (S_α) of bb-bounded operators converges to zero

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uniformly on a bounded set $B \subseteq X$ if for each zero neighborhood V, there is an index α_0 with $S_\alpha(B) \subseteq V$ for all $\alpha \geq \alpha_0$.

Also, the class of all continuous operators on X is denoted by $\mathsf{B_c}(X)$ and is assigned to the topology of equicontinuous convergence. A net(S_α) of continuous operators is convergent to the zero operator equicontinuously if for each zero neighborhood $V \subseteq X$ there exists a zero neighborhood U such that for every $\varepsilon > 0$ there is an α_0 with the property $S_\alpha(U) \subseteq \varepsilon V$ for any $\alpha \geq \alpha_0$.

It is easy to see that

$$\mathsf{B_n}(X) \subseteq \mathsf{B_c}(X) \subseteq \mathsf{B_b}(X).$$

Also note that the above inclusions become equalities when X is locally bounded; see [16, 18] for more details concerning these operators. Recall that a topological vector space X has the *Heine-Borel property* if every closed and bounded subset of X is compact.

By $X \otimes Y$, we mean the algebraic tensor product space. If X and Y are locally convex spaces, then the symbol $X \otimes_\pi Y$ will be used for the algebraic tensor product space endowed with the projective tensor product topology. For a review about projective tensor product of locally convex spaces and the related notions, we refer the reader to [15, Chapter III; 5, 6]; for a topological flavour of topological vector spaces and the relevant aspects, one may consult [13, Chapters I, II, III]. Note that the symbol $\mathrm{co}(A \otimes B)$ denotes the *convex hull* of $A \otimes B$.

Some notions will be given in the course of exposition in the beginning of a section.

The paper is organized in the following way. In Section 2 we further investigate bounded linear operators on topological vector spaces. In Section 3 we introduce bounded bilinear mappings between topological vector spaces and consider some their properties. Section 4 is devoted to a topological approach to the notions of central and order bounded below operators defined on a topological Riesz space. With an appropriate topology, we extend some known results for central and order bounded below operators on a Banach lattice, to the topological Riesz space setting.

2. Linear Mappings

Let X be a normed space, $\mathsf{K}(X)$ be the space of all compact operators on X, and $\mathsf{B}(X)$ be the collection of all bounded linear operators. From the equality $\|T(x)\| \leq \|T\|\|x\|$, it follows that X is a topological $\mathsf{B}(X)$-module, where the module multiplication is given via the formula $(T, x) \mapsto T(x)$, for each linear operator T and each $x \in X$. On the other hand, it is known that $\mathsf{K}(X)$ is a closed subspace of $\mathsf{B}(X)$. It is also of interest to investigate situations in which $\mathsf{K}(X)$ and $\mathsf{B}(X)$ are the same. So, it is natural to see if these results can be generalized to ordinary topological vector spaces and to different classes of bounded and compact operators on them.

In [16], two different notions for compact operators on a topological vector space have been introduced. A linear operator T on a topological vector space X is said to be:

- n-*compact* if there is a zero neighborhood $U \subseteq X$ for which $T(U)$ is relatively compact (which means that its closure is compact);
- b-*compact* if for each bounded set $B \subseteq X$, $T(B)$ is relatively compact.

We use the notations $\mathsf{K_n}(X)$ and $\mathsf{K_b}(X)$ for the set of all n-compact linear operators and the set of all b-compact linear operators on X, respectively. It is easy to see that $\mathsf{K_n}(X)$ is a two-sided ideal of $\mathsf{B_n}(X)$ and $\mathsf{K_b}(X)$ is a right ideal of $\mathsf{B_b}(X)$.

In this section we investigate some relations between bounded linear operators and compact ones. For more details about bounded and compact operators on topological vector spaces and the related notions see [7, 15, 16, 18].

In [18], it has been proved that each class of bounded linear operators on a topological vector space X, with respect to the appropriate topology, forms a topological algebra. In this section we show that X is a topological A-module, where A is one of the topological algebras $\mathsf{B_n}(X)$, $\mathsf{B_c}(X)$, and $\mathsf{B_b}(X)$, respectively, and the module multiplication is given via $(T, x) \to T(x)$, for every linear operator T and every $x \in X$. Let us point out that a topological vector space X is a *topological A-module*, where A is a topological algebra over the same field, provided that the module multiplication is continuous as a mapping from $A \times X$, equipped with product topology, into X.

Proposition 2.1. *The module multiplication in* $\mathsf{B}_\mathsf{n}(X)$ *is continuous with respect to the topology of uniform convergence on some zero neighborhood.*

Proof. Let (x_α) be a net in X which is convergent to zero and (T_α) be a net in $\mathsf{B}_\mathsf{n}(X)$ converging to zero uniformly on some zero neighborhood $U \subseteq X$. There is an α_0 such that $x_\alpha \in U$ for each $\alpha \geq \alpha_0$. If V is an arbitrary zero neighborhood in X, Then, there exists an index α_1 with $T_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_1$, so that for sufficiently large α, we have

$$T_\alpha(x_\alpha) \subseteq T_\alpha(U) \subseteq V.$$

□

Proposition 2.2. *The module multiplication in* $\mathsf{B}_\mathsf{c}(X)$ *is continuous with respect to the equicontinuous convergence topology.*

Proof. Let (x_α) be a net in X which is convergent to zero and (T_α) be a net of continuous operators converging to zero equicontinuously. Suppose V is an arbitrary zero neighborhood in X. There exist a zero neighborhood $U \subseteq X$ and an index α_0 with $T_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$. Choose an α_1 such that $x_\alpha \in U$ for every $\alpha \geq \alpha_1$. Thus, for sufficiently large α, we conclude

$$T_\alpha(x_\alpha) \subseteq T_\alpha(U) \subseteq V.$$

□

Proposition 2.3. *The module multiplication in* $\mathsf{B}_\mathsf{b}(X)$ *is sequentially continuous with respect to the topology of uniform convergence on bounded sets.*

Proof. Let (x_n) be a sequence in X which is convergent to zero and (T_n) be a sequence of bb-bounded operators converging to zero uniformly on bounded sets. Note that $E = \{x_n : n \in \mathbb{N}\}$ is bounded in X. Suppose V is an arbitrary zero neighborhood in X. There exists an n_0 such that $T_n(E) \subseteq V$ for any $n > n_0$, so that we have

$$T_n(x_n) \subseteq T_n(E) \subseteq V.$$

□

Question 2.4. *Is the module multiplication in* $\mathsf{B}_\mathsf{b}(X)$ *continuous, in general?*

Note that for a normed space X, $\mathsf{K}(X) = \mathsf{B}(X)$ if and only if X is finite dimensional. In this step, we consider some situations where a class of compact linear operators coincides with the corresponding class of bounded operators.

Proposition 2.5. $\mathsf{K}_\mathsf{b}(X) = \mathsf{B}_\mathsf{b}(X)$ *if and only if X has the Heine-Borel property.*

Proof. $\mathsf{K}_\mathsf{b}(X) = \mathsf{B}_\mathsf{b}(X) \Leftrightarrow I \in \mathsf{K}_\mathsf{b}(X) \Leftrightarrow X$ has the Heine-Borel property, where I denotes the identity operator on X. □

Remark 2.6. Note that when X has the Heine-Borel property, then $\mathsf{K}_\mathsf{n}(X) = \mathsf{B}_\mathsf{n}(X)$. Suppose, for a topological vector space X, $\mathsf{K}_\mathsf{n}(X) = \mathsf{B}_\mathsf{n}(X)$. We consider two cases. First, assume X is locally bounded. Then,
$I \in \mathsf{B}_\mathsf{n}(X) \Rightarrow I \in \mathsf{K}_\mathsf{n}(X) \Rightarrow X$ is locally compact $\Rightarrow X$ is finite dimensional.
The second case, when X is not locally bounded. Then,
$I \notin \mathsf{B}_\mathsf{n}(X) \Rightarrow I \notin \mathsf{K}_\mathsf{n}(X) \Rightarrow X$ is not locally compact $\Rightarrow X$ is infinite dimensional.

Let (X_n) be a sequence of topological vector spaces in which, every X_n has the Heine-Borel property. Put $X = \prod_{n=1}^{\infty} X_n$, with the product topology. It is known that X is a topological vector space. In the following, we establish that each X_n has the Heine-Borel property if and only if so has X.

Theorem 2.7. *Let $X = \prod_{n=1}^{\infty} X_n$, with the product topology; then X has the Heine-Borel property if and only if each X_n has this property, as well.*

Proof. First, assume that each X_n has the Heine-Borel property. We claim that if $B \subseteq X$ is bounded, then there exist bounded subsets $B_i \subseteq X_i$ such that $B \subseteq \prod_{i=1}^{\infty} B_i$. Put

$$B_i = \{x \in X_i : (y_1, \dots, y_{i-1}, x, y_{i+1}, \dots) \in B, y_j \in X_j\}.$$

Each B_i is bounded in X_i. Let W_i be a zero neighborhood in X_i and put

$$W = X_1 \times \dots X_{i-1} \times W_i \times X_{i+1} \times \dots.$$

Since W is a zero neighborhood in X, there exists a positive scalar α such that $B \subseteq \alpha W$, so that $B_i \subseteq \alpha W_i$. Also, it is easy to see that $B \subseteq \prod_{i=1}^{\infty} B_i$. Therefore,

$$\overline{B} \subseteq \overline{\prod_{i=1}^{\infty} B_i} = \prod_{i=1}^{\infty} \overline{B_i},$$

so that we conclude $\overline{B}$ is also compact, i.e. that X has the Heine-Borel property.

For the converse, suppose X has the Heine-Borel property. Choose a bounded set $B_n \subseteq X_n$. Put

$$B = \{0\} \times \dots \times \{0\} \times B_n \times \{0\} \times \dots.$$

It is an easy job to see that B is bounded in X, so that $\overline{B}$ is compact. By using Tychonoff's theorem, we conclude that $\overline{B_n}$ is compact and this would complete our claim. □

Collecting results of Theorem 3.5, Proposition 2.5, and Remark 2.6, we have the following.

Corollary 2.8. *Let (X_n) be a sequence of topological vector spaces, in which, each X_n has the Heine-Borel property. Put $X = \prod_{n=1}^{\infty} X_n$, with the product topology. Then, $\mathsf{K_b}(X) = \mathsf{B_b}(X)$ and $\mathsf{K_n}(X) = \mathsf{B_n}(X)$.*

Remark 2.9. Note that when X has the Heine-Borel property, $\mathsf{K_n}(X)$ need not be equal to $\mathsf{K_b}(X)$. For example, consider the identity operator on $\mathbb{R}^{\mathbb{N}}$. Indeed, it is b-compact but it fails to be n-compact; nevertheless, X has the Heine-Borel property.

Compact operators are not closed in the topologies induced by the corresponding class of bounded linear operators. To see this, consider the following examples.

Example 2.10. $\mathsf{K_n}(X)$ is not a closed subspace of $\mathsf{B_n}(X)$, in general. Let X be c_{00}, the space of all real null sequences, with the uniform norm topology. Suppose that T_n is the linear operator defined by

$$T_n(x_1, x_2, \dots, x_n, \dots) = (x_1, \frac{1}{2}x_2, \dots, \frac{1}{n}x_n, 0, \dots).$$

It is easy to see that each T_n is n-compact. Also, (T_n) converges uniformly on $N_1^{(0)}$, the open unit ball of X with center zero, to the linear operator T defined by $T(x_1, x_2, \dots) = (x_1, \frac{1}{2}x_2, \dots)$. For, if $\varepsilon > 0$ is arbitrary, there is an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. So, for each $n > n_0$, $(T_n - T)(N_1^{(0)}) \subseteq N_\varepsilon^{(0)}$. Now, it is not difficult to see that T is nb-bounded but it is not an n-compact linear operator.

Example 2.11. $\mathsf{K_b}(X)$ fails to be closed in $\mathsf{B_b}(X)$, in general. Let X be c_0, the space of all vanishing sequences, with the coordinate-wise topology. Let P_n be the projection on the first n-components. By using the Tychonoff's theorem, we can conclude that for each $n \in \mathbb{N}$, P_n is b-compact. Also, (P_n) converges uniformly on bounded sets to the identity operator I. We show that I is not b-compact. Suppose that B is the sequence (a_n) defined by $a_n = (1, 1, \dots, 1, 0, 0, \dots)$ in which 1 is appeared n times. B is a Cauchy sequence in c_0, so that it is bounded. Also note that $\overline{B} = B$. Now, if $I \in \mathsf{K_b}(c_0)$, then B should be compact. Since B is not complete, this is impossible.

3. Bilinear Mappings

The notion of a jointly continuous bilinear mapping between topological vector spaces has been studied widely, for example, see [13–15] for more information. In particular, when we deal with the normed spaces framework, these mappings carry bounded sets (with respect to the product topology) to bounded sets. On the other hand, tensor products are a fruitful and handy tool in converting a bilinear mapping to a linear operator in any setting; for example, the projective tensor product for normed spaces and the Fremlin projective tensor products for vector lattices and Banach lattices (see [8, 9] for ample information). In a topological vector space setting, we can consider two different non-equivalent ways to define a bounded bilinear mapping. It turns out that these aspects of boundedness are in a sense "intermediate" notions of a jointly continuous one. On the other hand, different types of bounded linear operators between topological vector spaces and some of their properties have been investigated (see [16, 18]). In this section, by using the concept of projective tensor product between locally convex spaces, we show that, in a sense, different notions of a bounded bilinear mapping coincide with different aspects of a bounded operator. We prove that for two bounded linear operators, the tensor product operator also has the same boundedness property, as well.

Definition 3.1. Let X, Y, and Z be topological vector spaces. A bilinear mapping $\sigma : X \times Y \to Z$ is said to be:

(*i*) n-*bounded* if there exist some zero neighborhoods $U \subseteq X$ and $V \subseteq Y$ such that $\sigma(U \times V)$ is bounded in Z;

(*ii*) b-*bounded* if for any bounded sets $B_1 \subseteq X$ and $B_2 \subseteq Y$, $\sigma(B_1 \times B_2)$ is bounded in Z.

We first show that these concepts of bounded bilinear mappings are not equivalent.

Example 3.2. Let $X = \mathbb{R}^{\mathbb{N}}$ be the space of all real sequences with the Tychonoff product topology. Consider the bilinear mapping $\sigma : X \times X \to X$ defined by $\sigma(x, y) = xy$ where $x = (x_i)$, $y = (y_i)$ and the product is pointwise. It is easily verified that σ is b-bounded; but since X is not locally bounded, it can not be an n-bounded bilinear mapping.

It is not difficult to see that every n-bounded bilinear mapping is jointly continuous and every jointly continuous bilinear mapping is b-bounded, so that these concepts of bounded bilinear mappings are related to jointly continuous bilinear mappings. Note that a b-bounded bilinear mapping need not be jointly continuous, even separately continuous; by a separately continuous bilinear mapping, we mean one which is continuous in each of its components. Consider the following example (which is actually originally an exercise from [13, Chapter I, Exercise 13]; we will give a proof for it for the sake of completeness).

Example 3.3. Let X be the space $C[0, 1]$, consisting of all real continuous functions on $[0, 1]$. Suppose τ_1 is the topology generated by the seminorms $p_x(f) = |f(x)|$, for each $x \in [0, 1]$, and τ_2 is the topology induced by the metric defined via the formulae

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

Consider the bilinear mapping $\sigma : (X, \tau_1) \times (X, \tau_1) \to (X, \tau_2)$ defined by $\sigma(f, g) = fg$. It is easy to show that σ is a b-bounded bilinear mapping. But is it not even separately continuous; for example the mapping $g = 1_X$, the identity operator from (X, τ_1) into (X, τ_2), is not continuous. To see this, suppose

$$V = \{f \in X : d(f, 0) < \frac{1}{2}\}.$$

V is a zero neighborhood in (X, τ_2). If the identity operator is continuous, there should be a zero neighborhood $U \subseteq (X, \tau_1)$ with $U \subseteq V$. Therefore, there are $\{x_1, \ldots, x_n\}$ in $[0, 1]$ and $\varepsilon > 0$ such that

$$U = \{f \in X, |f(x_i)| < \varepsilon, i = 1, \ldots, n\}.$$

For each subinterval $[x_i, x_{i+1}]$, consider positive reals α_i and α_{i+1} such that $x_i < \alpha_i < \alpha_{i+1} < x_{i+1}$. For an $n \in \mathbb{N}$, Define,

$$f_i(x) = \begin{cases} \frac{n(x-x_i)}{\alpha_i - x_i}, & \text{if } x_i \leq x \leq \alpha_i, \\ n, & \text{if } \alpha_i \leq x \leq \alpha_{i+1}, \\ \frac{n(x-x_{i+1})}{\alpha_{i+1}-x_{i+1}}, & \text{if } \alpha_{i+1} \leq x \leq x_{i+1}. \end{cases}$$

Now consider the continuous function f on $[0,1]$ defined by $f_i's$. Obviously, $f \in U$. Put $\beta = \sum_{i=1}^{n} (\alpha_{i+1} - \alpha_i)$. We can choose $n \in \mathbb{N}$ and β in such a way that $\frac{\beta n}{n+1} > \frac{1}{2}$. Thus,

$$\int_0^1 \frac{|f(x)|}{1+|f(x)|} dx > \frac{\beta n}{n+1} > \frac{1}{2}.$$

This completes the claim.

In what follows, by using the concept of the projective tensor product of locally convex spaces, we are going to show that these concepts of bounded bilinear mappings are, in fact, the different types of bounded operators defined on a locally convex topological vector space. Recall that if U and V are zero neighborhoods for locally convex spaces X and Y, respectively, then $\mathrm{co}(U \otimes V)$ is a typical zero neighborhood for the locally convex space $X \otimes_\pi Y$.

Proposition 3.4. *Let X, Y and Z be locally convex vector spaces and $\theta : X \times Y \to X \otimes_\pi Y$ be the canonical bilinear mapping. If $\varphi : X \times Y \to Z$ is an* n*-bounded bilinear mapping, there exists an* nb*-bounded operator $T : X \otimes_\pi Y \to Z$ such that $T \circ \theta = \varphi$.*

Proof. By [15, III.6.1], there is a linear mapping $T : X \otimes_\pi Y \to Z$ such that $T \circ \theta = \varphi$. Therefore, it is enough to show that T is nb-bounded. Since φ is n-bounded, there are zero neighborhoods $U \subseteq X$ and $V \subseteq Y$ such that $\varphi(U \times V)$ is bounded in Z. Let W be an arbitrary zero neighborhood in Z. There is $r > 0$ with $\varphi(U \times V) \subseteq rW$. It is not hard to show that $T(U \otimes V) \subseteq rW$, so that $T(\mathrm{co}(U \otimes V)) \subseteq rW$. But, by the fact mentioned before this proposition, $\mathrm{co}(U \otimes V)$ is a zero neighborhood in the space $X \otimes_\pi Y$. This completes the proof. □

Proposition 3.5. *Let X, Y and Z be locally convex vector spaces and $\theta : X \times Y \to X \otimes_\pi Y$ be the canonical bilinear mapping. If $\varphi : X \times Y \to Z$ is a* b*-bounded bilinear mapping, there exists a* bb*-bounded operator $T : X \otimes_\pi Y \to Z$ such that $T \circ \theta = \varphi$.*

Proof. As in the proof of the previous theorem, the existence of the linear mapping $T : X \otimes_\pi Y \to Z$ such that $T \circ \theta = \varphi$ follows by [15, III.6.1]. We prove that the linear mapping T is bb-bounded. Consider a bounded set $B \subseteq X \otimes_\pi Y$. There exist bounded sets $B_1 \subseteq X$ and $B_2 \subseteq Y$ such that $B \subseteq B_1 \otimes B_2$. To see this, put

$$B_1 = \{x \in X, \exists\, y \in Y, \text{such that } x \otimes y \in B\},$$

$$B_2 = \{y \in Y, \exists\, x \in X, \text{such that } x \otimes y \in B\}.$$

It is not difficult to see that B_1 and B_2 are bounded in X and Y, respectively, and $B \subseteq B_1 \otimes B_2$. Also, since θ is jointly continuous, $B_1 \otimes B_2$ is also bounded in $X \otimes_\pi Y$. Thus, from the inclusion

$$T(B) \subseteq T(B_1 \otimes B_2) = T \circ \theta(B_1 \times B_2) = \varphi(B_1 \times B_2)$$

and using the fact that φ is a b-bounded bilinear mapping, it follows that T is a bb-bounded linear operator. This concludes the claim and completes the proof of the proposition. □

Remark 3.6. Note that the similar result for jointly continuous bilinear mappings between locally convex spaces is known and commonly can be found in the contexts concerning topological vector spaces (see for example [15, III.6]).

We are going now to investigate whether or not the tensor product of two operators preserves different kinds of bounded operators between topological vector spaces. The response is affirmative. Recall that for vector spaces X, Y, Z, and W, and linear operators $T : X \to Y$, $S : Z \to W$, by the tensor product of T and S, we mean the unique linear operator $T \otimes S : X \otimes Z \to Y \otimes W$ defined via the formulae

$$(T \otimes S)(x \otimes z) = T(x) \otimes S(z);$$

one may consult [14] for a comprehensive study regarding the tensor product operators.

Theorem 3.7. *Let X, Y, Z, and W be locally convex spaces, and $T : X \to Y$ and $S : Z \to W$ be* nb*-bounded linear operators. Then the tensor product operator $T \otimes S : X \otimes_\pi Z \to Y \otimes_\pi W$ is* nb*-bounded.*

Proof. Let $U \subseteq X$ and $V \subseteq Z$ be two zero neighborhoods such that $T(U)$ and $S(V)$ are bounded subsets of Y and W, respectively. Let $O_1 \subseteq Y$ and $O_2 \subseteq W$ be two arbitrary zero neighborhoods. There exist positive reals α and β with $T(U) \subseteq \alpha O_1$ and $S(V) \subseteq \beta O_2$. Then

$$(T \otimes S)(U \otimes V) = T(U) \otimes S(V) \subseteq \alpha\beta(O_1 \otimes O_2) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2),$$

so that $(T \otimes S)(\mathrm{co}(U \otimes V)) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2)$. This is the desired result. □

Theorem 3.8. *Suppose X, Y, Z, and W are locally convex spaces, and $T : X \to Y$ and $S : Z \to W$ are* bb*-bounded linear operators. Then the tensor product operator $T \otimes S : X \otimes_\pi Z \to Y \otimes_\pi W$ is also* bb*-bounded.*

Proof. Fix a bounded set $B \subseteq X \otimes_\pi Z$. By the argument used in Proposition 3.5, there are bounded sets $B_1 \subseteq X$ and $B_2 \subseteq Z$ with $B \subseteq B_1 \otimes B_2$. Let $O_1 \subseteq Y$ and $O_2 \subseteq W$ be two arbitrary zero neighborhoods. There are positive reals α and β such that $T(B_1) \subseteq \alpha O_1$ and $S(B_2) \subseteq \beta O_2$. Therefore,

$$(T \otimes S)(B) \subseteq (T \otimes S)(B_1 \otimes B_2) = T(B_1) \otimes S(B_2) \subseteq \alpha\beta(O_1 \otimes O_2) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2),$$

hence $(T \otimes S)(B) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2)$, as required. □

Theorem 3.9. *Suppose X, Y, Z, and W are locally convex spaces, and $T : X \to Y$ and $S : Z \to W$ are continuous linear operators. Then the tensor product operator $T \otimes S : X \otimes_\pi Z \to Y \otimes_\pi W$ is jointly continuous.*

Proof. Let $O_1 \subseteq Y$ and $O_2 \subseteq Z$ be two arbitrary zero neighborhoods. There exist zero neighborhoods $U \subseteq X$ and $V \subseteq Z$ such that $T(U) \subseteq O_1$ and $S(V) \subseteq O_2$. It follows

$$(T \otimes S)(U \otimes V) = T(U) \otimes S(V) \subseteq (O_1 \otimes O_2) \subseteq \mathrm{co}(O_1 \otimes O_2),$$

so that $(T \otimes S)(\mathrm{co}(U \otimes V)) \subseteq \mathrm{co}(O_1 \otimes O_2)$, as claimed. □

4. Operators in Topological Riesz Spaces

In this section we give a topological approach to the notions of central operators and order bounded below operators defined on a topological Riesz space. With an appropriate topology, we extend to topological Riesz spaces some known results for these operators on Banach lattices.

We recall some concepts and terminology. A *Riesz space* (or *vector lattice*) is an ordered real vector space X which is also a lattice. For $x \in X$ one defines $x^+ = x \vee 0$ (the *positive part* of x), $x^- = (-x) \vee 0$ (the *negative part* of x), and $|x| = x \vee (-x) = x^+ + x^-$ (the *absolute value* or *modulus* of x).

A subset S of a Riesz space X is said to be *solid* if $y \in S$, $x \in X$, and $|x| \leq |y|$ imply $x \in S$. A *topological Riesz space* is a Riesz space which is at the same time a (ordered) topological vector space. By a *locally solid Riesz space* we mean a topological Riesz space with a locally solid topology. A *Banach lattice* is a Riesz space which is also a Banach space, where the norm is a lattice norm.

For more information on topological Riesz spaces and related notions, and also for terminology used in this section, we refer the reader to [1–4, 10].

4.1. Central operators

We recall that a linear operator T on a Riesz space X is called *central* if there exists a positive real number γ such that for each $x \in X$, we have $|T(x)| \leq \gamma|x|$ (see [5, 6, 17], where there are also interesting results concerning this class of operators). In [17], Wickstead showed that if X is a Banach lattice, then $\mathsf{Z}(X)$, the space of all central operators on X (called the *center* of X), is a unital Banach algebra with respect to the norm given by

$$\|T\| = \inf\{\lambda \geq 0 : |T| \leq \lambda I\},$$

for each $T \in \mathsf{Z}(X)$, where I denotes the identity operator on X.

We are going to generalize this result to central operators on a locally solid Riesz space X endowed with the *τ-topology*: a net (T_α) of central operators converges to zero in the τ-topology if for each $\varepsilon > 0$ there is an α_0 such that $|T_\alpha(x)| \leq \varepsilon|x|$, for each $\alpha \geq \alpha_0$ and for each $x \in X$. It is easy to see that $\mathsf{Z}(X)$ is a unital algebra. We show that $\mathsf{Z}(X)$ is in fact a topological algebra.

Proposition 4.1. *The operations of addition, scalar multiplication and product are continuous in $\mathsf{Z}(X)$ with respect to the τ-topology.*

Proof. Suppose (T_α) and (S_α) are two nets of central operators which are convergent to zero in the τ-topology. Let $\varepsilon > 0$ be given. There are indices α_0 and α_1 such that $|T_\alpha(x)| \leq \frac{\varepsilon}{2}|x|$ for each $\alpha \geq \alpha_0$ and $x \in X$, and $|S_\alpha(x)| \leq \frac{\varepsilon}{2}|x|$ for each $\alpha \geq \alpha_1$ and $x \in X$. Choose α_2 with $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. Then for each $\alpha \geq \alpha_2$ and for each $x \in X$, we have,

$$|(T_\alpha + S_\alpha)(x)| \leq |T_\alpha(x)| + |S_\alpha(x)| \leq \varepsilon|x|.$$

Now, we show the continuity of the scalar multiplication. Suppose (γ_α) is a net of reals which converges to zero. Without loss of generality, we may assume that $|\gamma_\alpha| \leq 1$ for each α. Therefore, for each $x \in X$ we have

$$|\gamma_\alpha||T_\alpha(x)| \leq |\gamma_\alpha|\varepsilon|x| \leq \varepsilon|x|,$$

for all $\alpha \geq \alpha_0$.
For continuity of the product, we have for $x \in X$

$$|T_\alpha(S_\alpha(x))| \leq |T_\alpha||S_\alpha(x)| \leq |T_\alpha|(\varepsilon|x|) = |T_\alpha(\varepsilon|x|)| \leq \varepsilon^2|x|,$$

so that $|T_\alpha(S_\alpha(x))| \leq \varepsilon^2|x|$. Note that by [5, 6], for a central operator T on a vector lattice X, the modulus of T, $|T|$, exists and satisfies $|T|(|x|) = |T(x)|$, for any $x \in X$. □

Proposition 4.2. *Suppose (T_α) is a net of central operators on a topological Riesz space X converging (in the τ-topology) to a linear operator T. Then, T is also central.*

Proof. There is an α_0 such that for each $\alpha \geq \alpha_0$ and for each $x \in X$, we have $|(T_\alpha - T)(x)| \leq |x|$. Fix an $\alpha \geq \alpha_0$. There exists a positive real γ_α such that $|T_\alpha(x)| \leq \gamma_\alpha|x|$. So, we have

$$|T(x)| \leq |T_\alpha(x)| + |x| \leq \gamma_\alpha|x| + |x| = (\gamma_\alpha + 1)|x|.$$

Therefore, T is also a central operator. □

Proposition 4.3. *Let X be a complete locally solid Riesz space. Then $\mathsf{Z}(X)$ is also complete with respect to the τ-topology.*

Proof. Let (T_α) be a Cauchy net in $\mathsf{Z}(X)$ and V be an arbitrary zero neighborhood in X. Fix $x_0 \in X$. Choose $\delta > 0$ such that $\delta|x_0| \in V$. There exists an α_0 such that $|(T_\alpha - T_\beta)(x)| \leq \delta|x|$ for each $\alpha \geq \alpha_0$ and for each $\beta \geq \alpha_0$. So, we conclude that $(T_\alpha - T_\beta)(x_0) \in V$. This means that $(T_\alpha(x_0))$ is a Cauchy net in X. Therefore, it is convergent. Put $T(x) = \lim T_\alpha(x)$. Since this convergence holds in $\mathsf{Z}(X)$, by Proposition 4.2, T is also central. □

Collecting the results of Propositions 4.1, 4.2 and 4.3, we have the following

Theorem 4.4. *Let X be a complete locally solid Riesz space. Then, $\mathsf{Z}(X)$ is a complete unital topological algebra.*

In addition, we have continuity of the lattice operations (defined via formulas of [5, 6]) with respect to the assumed topology, which is proved in the following theorem; in other words $(\mathsf{Z}(X), \tau)$ is a locally solid Riesz space.

Theorem 4.5. *The lattice operations of $\mathsf{Z}(X)$ are continuous with respect to the assumed topology.*

Proof. By [5, 6], the supremum and the infimum operations in $\mathsf{Z}(X)$ are given by the formulas

$$(T \vee S)(x) = T(x) \vee S(x) \text{ and } (T \wedge S)(x) = T(x) \wedge S(x), \ \ T, S \in \mathsf{Z}(X), x \in X^+.$$

Let (T_α) and (S_α) be two nets of central operators which are convergent to operators T and S in the τ-topology, respectively. Let $\varepsilon > 0$ be arbitrary. There are some α_0 and α_1 such that for each $x \in X$, we have $|(T_\alpha - T)(x)| \le \frac{\varepsilon}{2}|x|$ for each $\alpha \ge \alpha_0$ and $|(S_\alpha - S)(x)| \le \frac{\varepsilon}{2}|x|$ for each $\alpha \ge \alpha_1$. Pick an α_2 with $\alpha_2 \ge \alpha_0$ and $\alpha_2 \ge \alpha_1$. Then for each $\alpha \ge \alpha_2$, by using the Birkhoff's inequality (for example, see [3]), for each $x \in X^+ = \{y \in X : y \ge 0\}$,

$$|(T_\alpha \vee S_\alpha)(x) - (T \vee S)(x)| = |(T_\alpha \vee S_\alpha)(x) - (T_\alpha \vee S)(x) + (T_\alpha \vee S)(x) - (T \vee S)(x)|$$

$$\le |T_\alpha(x) \vee S_\alpha(x) - T_\alpha(x) \vee S(x)| + |T_\alpha(x) \vee S(x) - T(x) \vee S(x)|$$

$$\le |(T_\alpha - T)(x)| + |(S_\alpha - S)(x)| \le \varepsilon x.$$

If x is not positive, by using $x = x^+ - x^-$, we get

$$|(T_\alpha \vee S_\alpha)(x) - (T \vee S)(x)| \le 2\varepsilon|x|$$

for sufficiently large α. Since, the lattice operations in a Riesz space can be obtained via each other, we conclude that all of them are continuous in $\mathsf{Z}(X)$ with respect to the τ-topology. □

4.2. Order bounded below operators

In this subsection we present a new approach to order bounded below operators on a topological Riesz space. The concept of a bounded below operator on a Banach space has been studied widely (for example, see Section 2.1 in [1]), and the norm of the operator was used essentially in its definition. In a Riesz space we have the concepts order and modulus which help us to have a different vision to these operators. On the other hand, when we deal with a topological Riesz space and an operator on it, the concept "neighborhoods" enables us to consider a topological view for an order bounded below operator. With the topology introduced in the previous subsection we extend some results of [1, Section 2.1] to topological Riesz spaces.

A linear operator T on a topological Riesz space X is said to be *order bounded below* if $|T(x)| \ge \gamma|x|$ for some positive real number γ and each $x \in X$. The class of all order bounded below operators on a topological Riesz space X is denoted by $\mathsf{O}(X)$. We consider the τ-topology on it. It is easy to see that every order bounded below operator is one-to-one; also, every central order bounded below operator on a locally solid Riesz space has a closed range.

In the following lemma we show that if a linear operator is sufficiently close to an order bounded below operator with respect to the τ-topology, then it is also order bounded below. So, the set of all central operators on a topological Riesz space X which are also order bounded below is an open subset of $\mathsf{Z}(X)$.

Lemma 4.6. *If a linear operator S on a topological Riesz space X is sufficiently close to an order bounded below operator T with respect to the τ-topology, then S is also order bounded below.*

Proof. Since T is an order bounded below operator on X there is a positive real number γ such that $|T(x)| \geq \gamma|x|$ for each $x \in X$. Choose a linear operator S such that $|(T-S)(x)| \leq \frac{\gamma}{2}|x|$. Then, for each $x \in X$, we have

$$|Sx| = |(S-T)(x) + T(x)| \geq \gamma|x| - \frac{\gamma}{2}|x| = \frac{\gamma}{2}|x|,$$

so that $|S(x)| \geq \frac{\gamma}{2}|x|$, which completes the proof. □

Now, we assert the main theorem of this subsection which is an extension of Theorem 2.9 from [1] to a locally solid Riesz space.

Theorem 4.7. *Let X be a complete locally solid Riesz space and T a continuous order bounded below operator on X. If a net (T_α) of surjective operators converges in the τ-topology to T, then T is also surjective.*

Proof. Fix a constant $\gamma > 0$ such that $|T(x)| \geq 2\gamma|x|$ for each $x \in X$. There is an α_0 with $|(T_\alpha - T)(x)| \leq \gamma|x|$ for each $x \in X$ and for each $\alpha \geq \alpha_0$, so that $|T_\alpha(x)| \geq \gamma|x|$. Pick $y \in X$. To show that T is surjective, we have to show that there exists $x \in X$ such that $T(x) = y$. There is a net $\{x_\alpha\}$ in X such that $T_\alpha(x_\alpha) = y$. From $\gamma|x_\alpha| \leq |T_\alpha(x_\alpha)| = |y|$, it follows that $|x_\alpha| \leq \frac{1}{\gamma}|y|$.

We claim that $\{x_\alpha\}$ is a Cauchy net in X. Let V be an arbitrary zero neighborhood in X. Choose $\delta > 0$ such that $\delta\frac{1}{\gamma}|y| \in V$. Then for sufficiently large α and β we have

$$2\gamma|x_\alpha - x_\beta| \leq |T(x_\alpha) - T(x_\beta)| \leq |T(x_\alpha) - y| + |y - T(x_\beta)|$$
$$= |T(x_\alpha) - T_\alpha(x_\alpha)| + |T_\beta(x_\beta) - T(x_\beta)| \leq \frac{\delta}{2}|x_\alpha| + \frac{\delta}{2}|x_\beta| \leq \delta\frac{1}{\gamma}|y|.$$

So, we conclude that $\{x_\alpha\}$ is really a Cauchy net in X.

Suppose that (x_α) converges to x. Thus, it follows that $(T(x_\alpha))$ converges to $T(x)$. On the other hand, from the equality

$$|T(x_\alpha) - y| = |T(x_\alpha) - T_\alpha(x_\alpha)| \leq \delta|x_\alpha| \leq \delta\frac{1}{\gamma}|y|$$

it follows that $(T(x_\alpha))$ converges to $y \in X$, hence $T(x) = y$. This complete the proof. □

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