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On the Spaces of Nörlund Almost Null and Nörlund Almost Convergent Sequences

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Abstract. In this article, the sequence spaces $f_0(N^t)$ and $f(N^t)$ are introduced as the domain of Nörlund mean in the spaces f_0 and f of almost null and almost convergent sequences which are isomorphic to the spaces f_0 and f, respectively, and some inclusion relations are given. Additionally, alpha, beta and gamma duals of the sequence spaces $f_0(N^t)$ and $f(N^t)$ are determined. Finally, the classes $(\lambda(N^t) : \mu)$ and $(\mu : \lambda(N^t))$ of matrix transformations are characterized for given sequence spaces λ and μ together with two Steinhaus type results.

1. Introduction

We denote the space of all complex valued sequences by ω. Each subspace of ω is called as a *sequence space*, as well. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. By ℓ_1, ℓ_p, cs, cs_0 and bs, we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where $1 < p < \infty$.

A linear topological space λ is called a *K-space* if each of the maps $p_k : \lambda \to \mathbb{C}$ defined by $p_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. A K-space λ is called an *FK-space* if λ is a complete linear metric space. If an FK-space has a normable topology then it is called a *BK-space*, (cf. Choudhary and Nanda [4]). The alpha, beta and gamma duals of a sequence space λ are respectively defined by

$$\lambda^\alpha := \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1 \ \text{ for all } \ x = (x_k) \in \lambda\},$$

$$\lambda^\beta := \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \ \text{ for all } \ x = (x_k) \in \lambda\},$$

$$\lambda^\gamma := \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \ \text{ for all } \ x = (x_k) \in \lambda\}.$$

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Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every $x = (x_k) \in \lambda$ the A-transform $Ax = \{(Ax)_n\}$ of x exists and belongs to μ; where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \ \text{ for each } \ n \in \mathbb{N}. \tag{1}$$

By $(\lambda : \mu)$, we mean the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$, i.e., $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$, and we have Ax belongs to μ for all $x \in \lambda$, where A_n denotes the sequence in the n-th row of A. A sequence x is said to be *A-summable to l* if Ax converges to l and is called as the *A-limit of x*.

If a normed sequence space λ contains a sequence (b_n) with the following property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0$$

then (b_n) is called a Schauder basis for λ. The series $\sum_{k=0}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_{k=0}^{\infty} \alpha_k b_k$.

Let λ be *FK*-space. If $\phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have *AK* property, where e^k is a sequence whose k^{th} term is 1 and the other terms are 0 for each $k \in \mathbb{N}$, and $\phi = span\{e^k\}$. If ϕ is dense in λ, then λ is called *AD-space*, thus *AK* implies *AD*.

The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}$$

which is also a sequence space.

The spaces $\ell_\infty(N^t)$ and $\ell_p(N^t)$ consisting of all sequences whose Nörlund transforms are in the spaces ℓ_∞ and ℓ_p with $1 \le p < \infty$ were worked by Wang [19], respectively. Additionally, the inverse of Nörlund matrix and some multiplication theorems for Nörlund mean were given by Mears in [13] and [12].

Let (t_k) be a nonnegative real sequence with $t_0 > 0$ and $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$, as follows;

$$a^t_{nk} := \begin{cases} \frac{t_{n-k}}{T_n} & , \quad 0 \le k \le n, \\ \\ 0 & , \quad k > n \end{cases} \tag{2}$$

for all $k, n \in \mathbb{N}$. It is known that the Nörlund matrix N^t is a Teoplitz matrix if and only if $t_n/T_n \to 0$, as $n \to \infty$. Furthermore, if we take $t = e = (1, 1, 1, \ldots)$, then the Nörlund matrix N^t is reduced to the Cesàro mean C_1 of order one and if we choose $t_n = A_n^{r-1}$ for every $n \in \mathbb{N}$, then the Nörlund mean N^t corresponds to the Cesàro mean C_r of order r, where $r > -1$ and

$$A_n^r := \begin{cases} \frac{(r+1)(r+2)\cdots(r+n)}{n!} & , \quad n = 1, 2, 3, \ldots, \\ \\ 1 & , \quad n = 0. \end{cases}$$

Let $t_0 = D_0 = 1$ and define D_n for $n \in \{1, 2, 3, \ldots\}$ by

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}. \tag{3}$$

Then, the inverse matrix $U^t = (u^t_{nk})$ of Nörlund matrix N^t was defined by Mears in [13], as follows

$$u^t_{nk} := \begin{cases} (-1)^{n-k} D_{n-k} T_k & , \quad 0 \le k \le n, \\ \\ 0 & , \quad k > n \end{cases} \tag{4}$$

for all $k, n \in \mathbb{N}$.

The shift operator P is defined on ω by $(Px)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞ as a nonnegative linear functional such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit l if all Banach limits of x are l, and is denoted by $f - \lim x_k = l$. Lorentz [10] proved that $f - \lim x_k = l$ if and only if $\sum_{k=0}^{m} x_{n+k}/(m+1) \to l$, as $m \to \infty$, uniformly in n. It is known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By f_0 and f, we denote the spaces of all almost null and all almost convergent sequences, that is,

$$\begin{aligned} f_0 &:= \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\}, \\ f &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = l \text{ uniformly in } n \right\}. \end{aligned}$$

In this paper we construct the new sequence spaces $f_0(N^t)$ and $f(N^t)$ of non-absolute type as the domain of Nörlund mean N^t with respect to the sequence $t = (t_k)$ in the spaces f_0 and f of almost null and almost convergent sequences. We give some inclusion relations and determine the alpha, beta and gamma duals of the spaces $f_0(N^t)$ and $f(N^t)$. Finally, we characterize some classes of matrix transformations related to the new sequence space $f(N^t)$ and give two Steinhaus type results.

2. The Spaces of Nörlund Almost Null and Nörlund Almost Convergent Sequences

We introduce the spaces $f_0(N^t)$ and $f(N^t)$ as the set of all N^t-almost null and N^t-almost convergent sequences, respectively, that is,

$$\begin{aligned} f_0(N^t) &:= \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} \sum_{k=0}^{n+j} \frac{t_{n+j-k}}{T_{n+j}} x_k = 0 \textit{ uniformly in } n \right\}, \\ f(N^t) &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \frac{1}{m+1} \sum_{j=0}^{m} \sum_{k=0}^{n+j} \frac{t_{n+j-k}}{T_{n+j}} x_k = l \textit{ uniformly in } n \right\}. \end{aligned}$$

By $c_0(N^t)$ and $c(N^t)$ which are investigated in a separate paper, we denote the spaces of all sequences whose N^t-transforms are in the classical sequence spaces c_0 and c, respectively.

We define the sequence $y = (y_k)$ by the N^t-transform of a sequence $x = (x_k) \in \omega$, that is,

$$y_k = (N^t x)_k = \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \ \text{ for all } \ k \in \mathbb{N}. \tag{5}$$

Therefore, by applying U^t to the sequence y in (5) we obtain that

$$x_k = (U^t y)_k = \sum_{j=0}^{k} (-1)^{k-j} D_{k-j} T_j y_j \ \text{ for all } \ k \in \mathbb{N}. \tag{6}$$

Theorem 2.1. *The spaces $f_0(N^t)$ and $f(N^t)$ are BK-spaces with the norm given by*

$$\|x\|_{f(N^t)} = \|N^t x\|_f = \sup_{m,n\in\mathbb{N}} |t_{mn}(N^t x)|,$$

where

$$t_{mn}(N^t x) = \frac{1}{m+1}\sum_{j=0}^{m}\frac{1}{T_{n+j}}\sum_{k=0}^{n+j} t_{n+j-k}x_k \text{ for all } m,n \in \mathbb{N}.$$

Proof. Since f_0 and f are BK-spaces with the norm $\|\cdot\|_\infty$ (see Boos [2, Example 7.3.2 (b)]) and N^t is a triangle matrix, Theorem 4.3.2 of Wilansky [20, p. 61] gives that $f_0(N^t)$ and $f(N^t)$ are BK-spaces with respect to the norm $\|\cdot\|_{f(N^t)}$. □

Let λ denotes any of the spaces f_0 or f. With the notation of (5), since the transformation $T : \lambda(N^t) \to \lambda$ defined by $x \mapsto y = Tx = N^t x$ is a norm preserving linear bijection, we have the following:

Corollary 2.2. *The sequence spaces $f_0(N^t)$ and $f(N^t)$ are linearly norm isomorphic to the spaces f_0 and f, respectively, that is, $f_0(N^t) \cong f_0$ and $f(N^t) \cong f$.*

Now, we can mention on the existence of the Schauder bases of the spaces $f_0(N^t)$ and $f(N^t)$. It is known from Corollary 3.3 of Başar and Kirişçi [1] that the Banach space f has no Schauder basis. It is also known from Theorem 2.3 of Jarrah and Malkowsky [6] that the domain λ_A of a matrix A in a normed sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle. Combining these facts, one can immediately conclude that both of the spaces $f_0(N^t)$ and $f(N^t)$ have no Schauder basis.

Theorem 2.3. *The following statements hold:*

(i) The inclusion $f_0(N^t) \subset f(N^t)$ is strict.
(ii) The inclusion $c(N^t) \subset f(N^t)$ is strict.
(iii) The sequence spaces $f(N^t)$ and ℓ_∞ are overlap, but neither of them contains the other.

Proof. (i) If we take a sequence $x = (x_k)$ in the space $f_0(N^t)$, then we have $N^t x \in f_0$ and since the inclusion $f_0 \subset f$ trivially holds, then we have $N^t x \in f$ which gives us $x \in f(N^t)$. Thus, the inclusion $f_0(N^t) \subset f(N^t)$ holds.

Now, we consider a sequence $x = (x_k)$ in the space $f(N^t)$ but is not in the space $f_0(N^t)$, that is, we show that $f(N^t)\setminus f_0(N^t)$ is not empty. Consider the sequence $x = e$. Then, $N^t e = e \in f\setminus f_0$, we have $x \in f(N^t)\setminus f_0(N^t)$. Hence, the inclusion $f_0(N^t) \subset f(N^t)$ strictly holds.

(ii) If $x \in c(N^t)$, then we have $N^t x \in c$ and the inclusion $c \subset f$ is well-known, so that $N^t x \in f$, i.e., $x \in f(N^t)$. Hence, the inclusion $c(N^t) \subset f(N^t)$ holds.

Now, we should show that the set $f(N^t)\backslash c(N^t)$ is not empty. For this, we consider the sequence $x = (x_k)$ defined by

$$x_k = \sum_{j=0}^{k}(-1)^k D_{k-j}T_j \text{ for all } k \in \mathbb{N}. \tag{7}$$

Therefore, we obtain that

$$(N^t x)_n = \sum_{k=0}^{n}\frac{t_{n-k}}{T_n}\sum_{j=0}^{k}(-1)^k D_{k-j}T_j = (-1)^n \text{ for all } n \in \mathbb{N}.$$

Then, it is clear that $N^t x$ is not in c but,

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k+n}\frac{t_{k+n-j}}{T_{k+n}}\sum_{j=0}^{k}(-1)^k D_{k-j}T_j &= \lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}(-1)^{n+k}\\ &= \lim_{m\to\infty}\frac{(-1)^n}{m+1}\left[\frac{1+(-1)^m}{2}\right]\\ &= 0 \text{ uniformly in } n.\end{aligned}$$

That is to say that $N^t x \in f$. Thus, the set $f(N^t)\backslash c(N^t)$ is not empty.

(iii) To prove this, first we prove that $f(N^t)$ and ℓ_∞ are not disjoint. If we take $z = e$ then, since $N^t z = e \in f$, $z \in f(N^t)$. Furthermore, it is trivial that $z \in \ell_\infty$. This shows that there exists at least one point belonging to both $f(N^t)$ and ℓ_∞, as was asserted.

Now, we prove that the sequence spaces $f(N^t)$ and ℓ_∞ do not include each other. Let us consider the sequence $x = (x_k)$ defined by (7). Then, since $N^t x = \{(-1)^n\} \in f$, $x \in f(N^t)$ but $x \notin \ell_\infty$. Hence, $x \in f(N^t)\backslash\ell_\infty$.

Now, we consider the sequence $s = U^t v$ with $v = (0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots)$, where the sequence v is defined by Miller and Orhan [14], and belongs to the set $\ell_\infty \setminus f$ and the blocks of 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10. Then, it is clear that $s \in \ell_\infty$ but is not in the space $f(N^t)$. This shows that the spaces $f(N^t)$ and ℓ_∞ do not include each other.

This completes the proof. □

3. The Alpha, Beta and Gamma Duals of the Spaces $f_0(N^T)$ and $F(N^T)$

In the present section, we determine the alpha, beta and gamma duals of the spaces $f_0(N^t)$ and $f(N^t)$.

We start with quoting the following two lemmas whose some parts related with the characterization of matrix transformations on/in the space f and are needed in the rest of the study. Here and after, we denote the collection of all finite subset of $\mathbb{N}$ by $\mathcal{F}$.

Lemma 3.1. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (f_0 : \ell_1) = (f : \ell_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}}\sum_{n=0}^{\infty}\left|\sum_{k\in K}a_{nk}\right| < \infty. \tag{8}$$

(ii) $A \in (\ell_\infty : \ell_\infty) = (f : \ell_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{\infty}|a_{nk}| < \infty. \tag{9}$$

(iii) (cf. Sıddıqi [17]) $A \in (f : c) = (f_0 : c)$ *if and only if (9) holds and*

$$\exists\alpha_k \in \mathbb{C} \text{ such that } \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{10}$$

$$\exists\alpha \in \mathbb{C} \text{ such that } \lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = \alpha, \tag{11}$$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|\Delta(a_{nk} - \alpha_k)| = 0. \tag{12}$$

Here and after, Δ denotes the forward difference matrix, i.e., $\Delta(a_{nk} - \alpha_k) = a_{n,k+1} - \alpha_{k+1} - (a_{nk} - \alpha_k)$ for all $n, k \in \mathbb{N}$.

(iv) $A \in (\ell_\infty : c)$ *if and only if (10) holds, and*

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{nk} - \alpha_k| = 0 \text{ uniformly in } n. \tag{13}$$

Lemma 3.2. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) (cf. Duran [5]) *$A \in (\ell_\infty : f)$ if and only if (9) holds and*

$$\exists \alpha_k \in \mathbb{C} \text{ such that } f-\lim a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{14}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \left| \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} - \alpha_k \right| = 0 \text{ uniformly in } n. \tag{15}$$

(ii) (cf. Duran [5]) *$A \in (f : f)$ if and only if (9) and (14) hold, and*

$$\exists \alpha \in \mathbb{C} \text{ such that } f-\lim \sum_{k=0}^{\infty} a_{nk} = \alpha, \tag{16}$$

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} \left| \Delta \left(\frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} - \alpha_k \right) \right| = 0 \text{ uniformly in } n. \tag{17}$$

(iii) (cf. King [8]) *$A \in (c : f)$ if and only if (9), (14) and (16) hold.*

(iv) (cf. Nanda [15]) *$A \in (\ell_p : f)$ if and only if (14) holds, and*

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} |a_{nk}|^q < \infty, \quad (1 \le p < \infty), \tag{18}$$

$$\sup_{k,n\in\mathbb{N}} |a_{nk}| < \infty, \quad (0 < p < 1). \tag{19}$$

Theorem 3.3. *The alpha dual of the spaces $f_0(N^t)$ and $f(N^t)$ is the set*

$$d_1^t := \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k\in K} (-1)^{n-k} D_{n-k} T_k a_n \right| < \infty \right\}.$$

Proof. Let us define the matrix $B = (b_{nk}^t)$ via $a = (a_k) \in \omega$ by

$$b_{nk}^t := \begin{cases} (-1)^{n-k} D_{n-k} T_k a_n & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases} \tag{20}$$

for all $k, n \in \mathbb{N}$. Since the relation (6) holds, it is immediate that

$$a_n x_n = \sum_{k=0}^{n} (-1)^{n-k} D_{n-k} T_k a_n y_k = (By)_n \text{ for all } n \in \mathbb{N}. \tag{21}$$

By the relation (21), we read that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in f_0(N^t)$ or $\in f(N^t)$ if and only if $By \in \ell_1$ whenever $y = (y_k) \in f_0$ or $\in f$. This leads to the fact that $a \in [f_0(N^t)]^\alpha = [f(N^t)]^\alpha$ if and only if $B \in (f_0 : \ell_1) = (f : \ell_1)$. Therefore, we derive by Part (i) of Lemma 3.1 that

$$\sup_{K\in\mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k\in K} (-1)^{n-k} D_{n-k} T_k a_n \right| < \infty. \tag{22}$$

This means that the alpha dual of the spaces $f_0(N^t)$ and $f(N^t)$ is the set d_1^t, as desired. □

Theorem 3.4. *Define the set d_2^t by*

$$d_2^t := \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k a_j \right| < \infty \right\}. \tag{23}$$

Then, $[f_0(N^t)]^\beta = [f(N^t)]^\beta = d_2^t \cap cs$.

Proof. Let $x = (x_k)$ be any sequence in the space $f_0(N^t)$ or $f(N^t)$. Then, one can immediately observe that $a = (a_k) \in [f_0(N^t)]^\beta = [f(N^t)]^\beta$ if and only if the matrix $B = (b^t_{nk})$ is in the class $(f : cs)$ of infinite matrices, where b^t_{nk} is defined by (20). This is equivalent to the fact that $a = (a_k) \in [f_0(N^t)]^\beta = [f(N^t)]^\beta$ if and only if the matrix $E = (e^t_{nk}) = SB$ is in the class $(f_0 : c) = (f : c)$ of infinite matrices, where $S = (s_{nk})$ denotes the usual summation matrix and

$$e^t_{nk} = \begin{cases} \sum_{j=k}^{n}(-1)^{j-k}D_{j-k}T_k a_j & , \quad 0 \le k \le n-1, \\ a_n T_n & , \quad k = n, \\ 0 & , \quad k > n \end{cases} \tag{24}$$

for all $k, n \in \mathbb{N}$. In this situation, we derive from (9) and (10) with e^t_{nk} instead of a_{nk} that the following two conditions are satisfied:

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \sum_{j=k}^{n} (-1)^{j-k} D_{j-k} T_k a_j \right| < \infty,$$
$$(a_k) \in cs.$$

This step leads us to the desired consequence that the beta dual of the spaces $f_0(N^t)$ and $f(N^t)$ is the set $d^t_2 \cap cs$. □

Theorem 3.5. *The gamma dual of the spaces $f_0(N^t)$ and $f(N^t)$ is the set d^t_2.*

Proof. This is similar to the proof of Theorem 3.4 with Part (ii) of Lemma 3.1 instead of Part (iii) of Lemma 3.1. By avoiding the repetition of the similar statements, we omit the detail. □

4. Matrix Transformations Related to the Space *F*(N^T)

Let λ denotes any of the classical sequence spaces ℓ_∞, c, c_0 and ℓ_p or any of the sequence spaces f_0 and f. Then, the domain λ_{N^t} is called as *Nörlund sequence space*. Therefore, since $\lambda_{N^t} \cong \lambda$ it is trivial with the notation (5) that the two sided implication "$x \in \lambda_{N^t}$ if and only if $y \in \lambda$" holds.

For the sake of brevity in notation, we shall also write here and after that

$$d_{nk} = \sum_{j=0}^{n} \frac{t_{n-j}}{T_n} b_{jk} \text{ and } b(n,k) = \sum_{j=0}^{n} b_{jk} \tag{25}$$

for all $k, n \in \mathbb{N}$.

Following Yeşilkayagil and Başar [21], we shall employ the concept of the pair of summability matrices (shortly PSM) defined by a relation between two infinite matrices such that one of them applied to the sequences in a Nörlund space and the other one applied to the sequences in a space which is isomorphic to the Nörlund space. We also give a basic theorem related to the PSM. Therefore, we characterize the classes $(\lambda(N^t) : \mu)$ and $(\mu : \lambda(N^t))$ of infinite matrices. Here and after, we suppose that λ and μ are given two sequence spaces.

Now, we may focus on the PSM. Let us suppose that the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ transform the sequences $x = (x_k)$ and $y = (y_k)$ which are connected with the relation (5) to the sequences $u = (u_n)$ and $v = (v_n)$, respectively, i.e.,

$$u_n = \left(Ax\right)_n = \sum_k a_{nk} x_k \text{ for each } n \in \mathbb{N}, \tag{26}$$

$$v_n = \left(By\right)_n = \sum_k b_{nk} y_k \text{ for each } n \in \mathbb{N}. \tag{27}$$

It is clear here that the method B is applied to the N^t-transform of the sequence $x = (x_k)$ while the method A is directly applied to the terms of the sequence $x = (x_k)$.

Let us assume that the matrix product BN^t exists. We say in this situation that the matrices A and B in (26), (27) are the *PSM* if u_n is reduced to v_n (or equivalently v_n is reduced to u_n) under the application of the formal summation by parts. This leads us to the fact that BN^t exists and is equal to A and $(BN^t)x = B(N^t x)$ formally holds, if one side exists. Therefore, we have the relation

$$a_{nk} = \sum_{j=k}^{\infty} \frac{t_{j-k}}{T_j} b_{nj} \text{ or equivalently } b_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k a_{nj} \text{ for all } k, n \in \mathbb{N}. \tag{28}$$

By taking into account the relation (5) one can derive that

$$\sum_{k=0}^{m} b_{nk} y_k = \sum_{k=0}^{m} b_{nk} \left(\frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \right) = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{t_{j-k}}{T_j} b_{nj} x_k \tag{29}$$

for all $m, n \in \mathbb{N}$. Therefore, we obtain by (29) as $m \to \infty$ that v_n reduces to u_n, as follows:

$$v_n = \sum_k b_{nk} y_k = \sum_k b_{nk} \left(\frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \right) = \sum_k \sum_{j=k}^{\infty} \frac{t_{j-k}}{T_j} b_{nj} x_k = u_n.$$

But, the order of summation may not be reversed. So, the methods A and B are not necessarily equivalent.

Theorem 4.1. *Let the elements of the matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation (28). Then, $A \in (\lambda(N^t) : \mu)$ if and only if $B \in (\lambda : \mu)$.*

Proof. Let $A = (a_{nk})$ and $B = (b_{nk})$ be a PSM.

Suppose that $A \in (\lambda(N^t) : \mu)$. Then, Ax exists and belongs to μ for all $x \in \lambda(N^t)$. Therefore, we have the following equality derived from mth partial sum of the series $\sum_{k=0}^{\infty} a_{nk} x_k$ with the relation (6):

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \sum_{j=k}^{m} (-1)^{j-k} D_{j-k} T_k a_{nj} y_k \tag{30}$$

for all $m, n \in \mathbb{N}$. Then, we have from (30) by letting $m \to \infty$ that $Ax = By$. Therefore, it is immediate that $By \in \mu$ whenever $y \in \lambda$; i.e., $B \in (\lambda : \mu)$.

Conversely, suppose that $B \in (\lambda : \mu)$. Then, By exists and belongs to μ for all $y \in \lambda$. Therefore, by letting $m \to \infty$ in (29), we get $By = Ax$ which gives the desired fact that $A \in (\lambda(N^t) : \mu)$. □

By interchanging the spaces λ_{N^t} and λ with the space μ, we have

Theorem 4.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $C = (c_{nk})$ are connected with the relation*

$$c_{nk} = \sum_{j=0}^{n} \frac{t_{n-j}}{T_n} a_{jk} \text{ for all } k, n \in \mathbb{N}. \tag{31}$$

Then, $A \in (\mu : \lambda(N^t))$ if and only if $C \in (\mu : \lambda)$.

Proof. Let us take any $s = (s_k) \in \mu$ and consider the following equality with (31) that

$$\sum_{k=0}^{m} c_{nk} s_k = \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{t_{n-j}}{T_n} a_{jk} s_k \text{ for all } m, n \in \mathbb{N},$$

which yields as $m \to \infty$ that $(Cs)_n = \{N^t(As)\}_n$ for all $n \in \mathbb{N}$. Now, we immediately deduce from here that $As \in \lambda(N^t)$ whenever $s \in \mu$ if and only if $Cs \in \lambda$ whenever $s \in \mu$.

This step completes the proof. □

Of course, Theorems 4.1 and 4.2 have several consequences depending on the choice of the sequence spaces λ and μ. By Theorem 4.1, the necessary and sufficient conditions for $A \in (\lambda(N^t) : \mu)$ may be derived by replacing the elements of A by those of the elements of $B = AU^t$, where the necessary and sufficient conditions on the matrix B are read from the concerning results in the existing literature. Since Theorems 4.1 and 4.2 are respectively related with the matrix transformations on the Nörlund sequence spaces and into the Nörlund sequence spaces, the characterizations of the matrix mappings between the Nörlund sequence spaces may be derived by combining Theorems 4.1 and 4.2. Now, we may quote our results on the characterization of some matrix classes concerning with the Nörlund sequence spaces.

The rest of the paper, we assume that the elements of the matrices $A = (a_{nk})$, $B = (b_{nk})$, $C = (c_{nk})$ and $D = (d_{nk})$ are connected with the relations (25), (28) and (31); and $q = p/(p-1)$ when $1 \leq p < \infty$.

Corollary 4.3. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (f(N^t) : \ell_\infty)$ *if and only if (9) holds with* b_{nk} *instead of* a_{nk}.
(ii) $A \in (f(N^t) : f)$ *if and only if (9), (14), (16) and (17) hold with* b_{nk} *instead of* a_{nk}.
(iii) $A \in (f(N^t) : f; p)$ *if and only if (9), (14), (16) and (17) hold with* b_{nk} *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.
(iv) $A \in (f(N^t) : c)$ *if and only if (9)-(12) hold with* b_{nk} *instead of* a_{nk}.
(v) $A \in (f(N^t) : c; p)$ *if and only if (9)-(12) hold with* b_{nk} *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.
(vi) $A \in (f(N^t) : c_0)$ *if and only if (9)-(12) hold with* b_{nk} *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 0$.
(vii) $A \in (\ell_\infty(N^t) : f)$ *if and only if (9), (14) and (15) hold with* b_{nk} *instead of* a_{nk}.
(viii) $A \in (\ell_\infty(N^t) : c)$ *if and only if (10) and (13) hold with* b_{nk} *instead of* a_{nk}.
(ix) $A \in (\ell_p(N^t) : f)$ *if and only if (14), (18) and (19) hold with* b_{nk} *instead of* a_{nk}.

Corollary 4.4. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (f(N^t) : \ell_\infty(N^t))$ *if and only if (9) holds with* d_{nk} *instead of* a_{nk}.
(ii) $A \in (f(N^t) : f(N^t))$ *if and only if (9), (14), (16) and (17) hold with* d_{nk} *instead of* a_{nk}.
(iii) $A \in (f(N^t) : f(N^t); p)$ *if and only if (9), (14), (16) and (17) hold with* d_{nk} *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.
(iv) $A \in (f(N^t) : c(N^t))$ *if and only if (9)-(12) hold with* d_{nk} *instead of* a_{nk}.
(v) $A \in (f(N^t) : c(N^t); p)$ *if and only if (9)-(12) hold with* d_{nk} *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.
(vi) $A \in (f(N^t) : c_0(N^t))$ *if and only if (9)-(12) hold with* d_{nk} *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 0$.
(vii) $A \in (\ell_\infty(N^t) : f(N^t))$ *if and only if (9), (14) and (15) hold with* d_{nk} *instead of* a_{nk}.
(viii) $A \in (\ell_p(N^t) : f(N^t))$ *if and only if (14), (18) and (19) hold with* d_{nk} *instead of* a_{nk}.

Corollary 4.5. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (f(N^t) : bs)$ *if and only if (9) holds with* $b(n,k)$ *instead of* a_{nk}.
(ii) $A \in (f(N^t) : fs)$ *if and only if (9), (14), (16) and (17) hold with* $b(n,k)$ *instead of* a_{nk}.
(iii) $A \in (f(N^t) : fs; p)$ *if and only if (9), (14), (16) and (17) hold with* $b(n,k)$ *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.
(iv) $A \in (f(N^t) : cs)$ *if and only if (9)-(12) hold with* $b(n,k)$ *instead of* a_{nk}.
(v) $A \in (f(N^t) : cs; p)$ *if and only if (9)-(12) hold with* $b(n,k)$ *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.
(vi) $A \in (f(N^t) : cs_0)$ *if and only if (9)-(12) hold with* $b(n,k)$ *instead of* a_{nk}, *and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 0$.
(vii) $A \in (\ell_\infty(N^t) : fs)$ *if and only if (9), (14) and (15) hold with* $b(n,k)$ *instead of* a_{nk}.
(viii) $A \in (\ell_p(N^t) : fs)$ *if and only if (14), (18) and (19) hold with* $b(n,k)$ *instead of* a_{nk}.

Corollary 4.6. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (\ell_\infty : f(N^t))$ *if and only if (9), (14) and (15) hold with* c_{nk} *instead of* a_{nk}.
(ii) $A \in (f : f(N^t))$ *if and only if (9), (14), (16) and (17) hold with* c_{nk} *instead of* a_{nk}.

(iii) $A \in (f : f(N^t); p)$ *if and only if (9), (14), (16) and (17) hold with* c_{nk} *instead of* a_{nk}*, and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.

(iv) $A \in (c : f(N^t)) = (c_0 : f(N^t))$ *if and only if (9), (14) and (16) hold with* c_{nk} *instead of* a_{nk}.

(v) $A \in (c : f(N^t); p)$ *if and only if (9), (14) and (16) hold with* c_{nk} *instead of* a_{nk}*, and* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$, $\alpha = 1$.

(vi) $A \in (f : \ell_\infty(N^t))$ *if and only if (9) holds with* c_{nk} *instead of* a_{nk}.

Finally, we mention about Steinhaus type theorems which were formulated by Maddox [11], as follows: Let λ and μ be any two sequence spaces having some notion of limit or sum, and $(\lambda : \mu; p)$ denotes the class of regular matrices and ν also be any sequence space such that $\nu \supset \lambda$. Then, a result of the form $(\lambda : \mu; p) \cap (\nu : \mu) = \emptyset$, is called a *theorem of Steinhaus type*.

Now, we can give the following theorem including two Steinhaus type conclusions.

Theorem 4.7. *The following statements hold:*

(i) The classes $(f(N^t) : c; p)$ *and* $(\ell_\infty(N^t) : c)$ *are disjoint.*

(ii) The classes $(f(N^t) : f; p)$ *and* $(\ell_\infty(N^t) : f)$ *are disjoint.*

Proof. (i) Suppose conversely that the classes $(f(N^t) : c; p)$ and $(\ell_\infty(N^t) : c)$ are not disjoint. Then, there exists at least one infinite matrix A satisfying the conditions of Parts (v) and (viii) of Corollary 4.3. Therefore, we derive by using the condition (13) with $\alpha_k = 0$ and with f_{nk} instead of a_{nk} that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \left| f_{nk} \right| = 0. \tag{32}$$

Nevertheless, from Part (v) of Corollary 4.3 with $\alpha = 1$ and with f_{nk} instead of a_{nk} we have

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} f_{nk} = 1$$

which contradicts (32). This completes the proof of Part (i).

(ii) This is similar to the proof of Part (i) of the present theorem with Parts (iii) and (vii) instead of Parts (v) and (viii) of Corollary 4.3, respectively. So, we leave the details. □

5. Conclusion

Lorentz [10] introduced the concept of almost convergence in 1948. Başar and Kirişçi [1] determined the beta and gamma duals of the spaces f, fs and $\widehat{f}$, and proved some basic results on the space f and characterized the class of matrix transformations on the space $\widehat{f}$ into any given sequence space, where fs denotes the space of almost convergent series.

Kayaduman and Şengönül investigated the spaces $\overline{f}_0$ and $\overline{f}$ that consist of all sequences whose Cesàro mean of order one transforms are in the spaces f_0 and f in [7], respectively. Şengönül and Kayaduman [16] defined the spaces $\widehat{f_0}$ and $\widehat{f}$ as the domain of Riesz mean in the sequence spaces f_0 and f. They also showed that the spaces $\widehat{f_0}$ and $\widehat{f}$ are linearly isomorphic to the sequence spaces f_0 and f, respectively. After computing the beta and gamma duals of $\widehat{f_0}$ and $\widehat{f}$ they characterized the classes $(\widehat{f} : \mu)$ and $(\mu : \widehat{f})$ of infinite matrices and they determined some core theorems related to the space $\widehat{f}$. Recently, Sönmez [18] worked the domain $f(B)$ of the triple band matrix $B(r, s, t)$ in the space f.

Candan [3] established the sequence spaces $f(\widetilde{B})$ and $f_0(\widetilde{B})$ consisting of all sequences $x = (x_k) \in \omega$ such that $(r_k x_k + s_{k-1} x_{k-1}) \in f_0$ or $\in f$ as the domain of the double sequential band matrix $\widetilde{B}(\widetilde{r}, \widetilde{s})$ in the sequence spaces f_0 and f. In this study, Candan determined the beta and gamma duals of the spaces $f_0(\widetilde{B})$ and $f(\widetilde{B})$, and also gave some inclusion theorems related with the spaces $f_0(\widetilde{B})$ and $f(\widetilde{B})$. Finally, he has recently characterized the classes $(f(\widetilde{B}) : \mu)$ and $(\mu : f(\widetilde{B}))$ of infinite matrices.

Kirişci [9] studied the domains $(f_0)_{E^r}$ and f_{E^r} of the Euler means of order r in the spaces f_0 and f, respectively. Yeşilkayagil and Başar [22] have presented the domains $A_\lambda(f_0)$ and $A_\lambda(f)$ of the matrix A_λ in the spaces f_0 and f, respectively, and they have established some inclusion relations deal with the concerning sequence spaces. Finally, they computed the alpha, beta and gamma duals of the sequence spaces $A_\lambda(f_0)$ and $A_\lambda(f)$ and gave the characterization of the classes $(A_\lambda(f) : \mu)$ and $(\mu : A_\lambda(f_0))$ of infinite matrices.

Since in the special case $t = e$, the Nörlund mean N^t is reduced to the Cesàro mean C_1 of order one; our corresponding results are much more general than those given by Kayaduman and Şengönül, in [7]. Although the domain of certain triangle matrices in the spaces f_0 and f is studied, the investigation of the domain of the Nörlund mean N^t in the same sequence spaces was open. So, the main results of the present paper fill up the gap in the existing literature.

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