



Revised Concavity Method and Application to Klein-Gordon Equation

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Abstract. A revised version of the concavity method of Levine, based on a new ordinary differential inequality, is proposed. Necessary and sufficient condition for nonexistence of global solutions of the inequality is proved. As an application, finite time blow up of the solution to Klein-Gordon equation with arbitrary positive initial energy is obtained under very general structural conditions.

1. Introduction

The concavity method of Levine [8–10] is one of the most powerful methods for proving finite time blow up of the solutions to nonlinear dispersive equations. The main idea of Levine is to replace the investigations of the equation with the study of ordinary differential inequality

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq 0, \quad t \geq 0, \quad \gamma > 1, \quad (1)$$

where $\Psi(t)$ is some nonnegative functional of the solution to the original dispersive equation. Later on, inequality (1) was extended by Kalantarov and Ladyzhenskaya [3] to

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -2\delta\Psi(t)\Psi'(t) - \mu\Psi^2(t), \quad t \geq 0, \quad \gamma > 1, \quad \delta \geq 0, \quad \mu \geq 0, \quad (2)$$

while Straughan [12] and Korpusov [4] generalized (1) to

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq -\beta\Psi(t), \quad t \geq 0, \quad \gamma > 1, \quad \beta > 0. \quad (3)$$

Under special choice of the initial data $\Psi(0)$, $\Psi'(0)$ and some additional assumption of δ , μ in (2) and β in (3) the solutions to (1), (2) and (3) blow up for a finite time.

As an application of inequalities (1)–(3) nonexistence of global solutions with arbitrary positive initial energy is proved for nonlinear wave equation [2, 12], nonlinear Klein-Gordon equation [4, 13, 16], generalized Boussinesq equation [6, 14] and others [1, 5]. When the initial energy is small positive one (subcritical), then the finite time blow up of the solutions is investigated by means of the potential well method. In this case the proof is based on the sign invariance of some functionals and the concavity method of Levine, i.e. inequality (1), see for more details [11, 15] and the references therein.

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The careful analysis of the reduction of the nonlinear dispersive equations to inequality (3) shows that an additional term $\alpha\Psi^2(t)$ always appears in the rhs of (3), i.e. the following inequality holds

$$\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) \geq \alpha\Psi^2(t) - \beta\Psi(t), \quad t \geq 0, \quad \gamma > 1, \quad \alpha > 0, \quad \beta > 0. \tag{4}$$

The aim of this paper is to prove a necessary and sufficient condition for finite time blow up of the solution to (4). Moreover, we find more general sufficient assumptions on $\Psi(0)$, $\Psi'(0)$ and β guaranteeing nonexistence of global solutions to (4). As a consequence we obtain very general conditions for finite time blow up of the solutions to Cauchy problem for Klein-Gordon equation

$$u_{tt} - \Delta u + u = f(u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{5}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n; \quad u_0(x) \in H^1(\mathbb{R}^n), \quad u_1(x) \in L^2(\mathbb{R}^n). \tag{6}$$

The nonlinear term $f(u)$ has one of the following forms:

$$f(u) = \sum_{k=1}^l a_k |u|^{p_k-1} u - \sum_{j=1}^s b_j |u|^{q_j-1} u, \quad \text{or} \tag{7}$$

$$f(u) = a_1 |u|^{p_1} + \sum_{k=2}^l a_k |u|^{p_k-1} u - \sum_{j=1}^s b_j |u|^{q_j-1} u,$$

where the constants a_k, p_k ($k = 1, 2, \dots, l$) and b_j, q_j ($j = 1, 2, \dots, s$) fulfill the conditions

$$\begin{aligned} a_1 > 0, \quad a_k \geq 0, \quad b_j \geq 0 \quad \text{for } k = 2, \dots, l, \quad j = 1, \dots, s, \\ 1 < q_s < q_{s-1} < \dots < q_1 < p_1 < p_2 < \dots < p_l, \\ p_l < \infty \quad \text{for } n = 1, 2; \quad p_l < \frac{n+2}{n-2} \quad \text{for } n \geq 3. \end{aligned}$$

The paper is organized in the following way. In Section 2 the main results concerning nonexistence of global solutions to inequality (4) are formulated and proved. Section 3 deals with the application of these results to proving finite time blow up of the solutions to Klein-Gordon equation (5)-(7).

2. Main Results

In the following theorem we summarize the known results for finite time blow up of the solutions to inequality (4). Let us note that for $\alpha = \beta = 0$ inequalities (4) transforms into inequality (1), while for $\alpha = 0$ and $\beta > 0$ – into inequality (3).

Theorem 2.1. *Suppose $\Psi(t)$, $t \geq b$, is a nonnegative, twice-differentiable function satisfying inequality (4) and*

$$\Psi(b) > 0, \quad \Psi'(b) > 0, \quad b = \text{const} \geq 0.$$

(i) ([3, 8]) *If $\alpha = \beta = 0$, $\gamma > 1$, then $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^* < \infty$, where*

$$t^* \leq t_1^* = b + \frac{\Psi(b)}{(\gamma - 1)\Psi'(b)}; \tag{8}$$

(ii) ([4, 12]) *If $\alpha = 0$, $\gamma > 1$ and*

$$0 < \beta < \frac{(2\gamma - 1)\Psi'^2(b)}{2\Psi(b)}, \tag{9}$$

then $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^ < \infty$, where*

$$t^* \leq t_2^* = b + \frac{\Psi(b)}{(\gamma - 1)\sqrt{\Psi'^2(b) - \frac{2\beta}{2\gamma-1}\Psi(b)}};$$

(iii) ([7]) If $\alpha > 0, \gamma > 1$ and

$$0 < \beta < \frac{(2\gamma - 1) \Psi'^2(b)}{2 \Psi(b)} + \alpha \Psi(b), \tag{10}$$

then $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^* < \infty$, where

$$t^* \leq t_3^* = \begin{cases} b + \frac{\Psi(b)}{(\gamma - 1)\Psi'(b)} & \text{if } \alpha \Psi(b) - \beta > 0; \\ b + \frac{\Psi(b)}{(\gamma - 1)\sqrt{\Psi'^2(b) - \frac{2\beta}{2\gamma-1}\Psi(b) + \frac{2\alpha}{2\gamma-1}\Psi^2(b)}} & \text{if } \alpha \Psi(b) - \beta \leq 0. \end{cases}$$

Assumptions (9) and (10) for nonexistence of global solutions to (3) and (4) are sufficient but not necessary ones. The specific structure of the inequality (4) allows us to formulate a necessary and sufficient condition for finite time blow up of the solution to (4). Let us mention that a similar to Theorem 2.2 result is announced in [1].

Theorem 2.2 (Necessary and sufficient condition). *Suppose $\Psi(t), t \geq 0$, is a nonnegative, twice-differentiable function satisfying inequality (4). Then $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^* < \infty$ if and only if there exists a constant $b \geq 0$ such that*

$$\Psi'(b) > 0, \tag{11}$$

$$\alpha \Psi(t) - \beta > 0 \text{ for every } t \geq b. \tag{12}$$

Proof. Assume that (11) and (12) hold. Then $\Psi(t)$ satisfies inequality (1) for $t \geq b$ and from Theorem 2.1 (i) it follows that $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^* \leq t_1^* < \infty$, where t_1^* is defined in (8).

Conversely, suppose that $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^* < \infty$. Then there exists a constant $b \in [0, t^*)$ such that conditions (11) and (12) for $t = b$ are satisfied, i.e.

$$\Psi'(b) > 0, \quad \alpha \Psi(b) - \beta > 0. \tag{13}$$

In order to prove condition (12) for every $t \in [b, t^*)$ we will show that function $\Psi(t)$ is strictly increasing. Indeed, for $z(t) = \Psi^{1-\gamma}(t)$ we have the identities

$$z'(t) = (1 - \gamma)\Psi^{-\gamma}(t)\Psi'(t), \quad z''(t) = (1 - \gamma)\Psi^{-1-\gamma}(t) (\Psi''(t)\Psi(t) - \gamma\Psi'^2(t)). \tag{14}$$

From (4), (13) and (14) it follows that $z(t)$ satisfies the inequalities:

$$\begin{aligned} z''(t) &\leq -(\gamma - 1) (\alpha z(t) - \beta z^{\frac{\gamma}{\gamma-1}}(t)), \quad t \in [b, t^*), \\ z(b) &> 0, \quad z'(b) < 0. \end{aligned} \tag{15}$$

Suppose by contradiction that there exists $t_0 \in (b, t^*)$ such that $z'(t) < 0$ for $t \in [b, t_0]$ and $z'(t_0) = 0$. Since $z(t)$ is a strictly decreasing function for $t \in [b, t_0]$ from (13) we get

$$\alpha - \beta z^{\frac{1}{\gamma-1}}(t) \geq \alpha - \beta z^{\frac{1}{\gamma-1}}(b) > 0 \quad \text{for } t \in [b, t_0].$$

Hence from (15) we have $z''(t) < 0$ for $t \in [b, t_0]$, i.e. $z'(t)$ is strictly decreasing in $t \in [b, t_0]$ and we obtain the following impossible chain of inequalities

$$0 > z'(b) > z'(t_0) = 0.$$

Thus $z'(t) < 0$ for every $t \in [b, t^*)$, i.e. $\Psi'(t) > 0$ for every $t \in [b, t^*)$. From (13) and the monotonicity of $\Psi(t)$ we get

$$\alpha \Psi(t) - \beta \geq \alpha \Psi(b) - \beta > 0, \quad \text{for every } t \in [b, t^*), \tag{16}$$

which proves (12) and Theorem 2.2. \square

The formulation of Theorem 2.2 is quite abstract and it is not clear when assumptions (11) and (12) are satisfied. In the following theorem we give conditions which guarantee the validity of (11) and (12). As a consequence, we prove finite time blow up of the solution $\Psi(t)$ by means of Theorem 2.2. It is very important to note that these conditions are more general than (9) and (10).

Theorem 2.3 (Main result). *Suppose $\Psi(t), t \geq 0$, is a nonnegative, twice-differentiable function satisfying inequality (4). If*

$$\begin{aligned} &\Psi(0) > 0, \quad \Psi'(0) > 0, \\ &0 < \beta < \frac{(2\gamma - 1) \Psi'^2(0)}{2 \Psi(0)} + \alpha\Psi(0) + \frac{\alpha\Psi(0)}{2\gamma - 2} (1 - A^{2-2\gamma}), \quad A = \frac{\gamma - 1}{\alpha} \frac{\Psi'^2(0)}{\Psi^2(0)} + 1, \end{aligned} \tag{17}$$

then $\Psi(t) \rightarrow \infty$ for $t \rightarrow t^* < \infty$.

Proof. If

$$0 < \beta < \frac{(2\gamma - 1) \Psi'^2(0)}{2 \Psi(0)} + \alpha\Psi(0),$$

then the statement in Theorem 2.3 follows from the proof of [7][Theorem 3.1].

Thus further on we consider the case

$$\frac{(2\gamma - 1) \Psi'^2(0)}{2 \Psi(0)} + \alpha\Psi(0) \leq \beta < \frac{(2\gamma - 1) \Psi'^2(0)}{2 \Psi(0)} + \alpha\Psi(0) + \frac{\alpha\Psi(0)}{2\gamma - 2} (1 - A^{2-2\gamma}). \tag{18}$$

Suppose by contradiction that $\Psi(t)$ is defined for every $t \geq 0$.

The first step in the proof is to show that $\Psi'(t) > 0$ for every $t \geq 0$. Analogously to the proof of Theorem 2.2, the function $z(t) = \Psi^{1-\gamma}(t)$ satisfies the following inequalities:

$$\begin{aligned} &z''(t) \leq -(\gamma - 1) (\alpha z(t) - \beta z^{\frac{\gamma}{\gamma-1}}(t)), \quad t \geq 0, \\ &z(0) > 0, \quad z'(0) < 0. \end{aligned} \tag{19}$$

Assume that $z'(t) < 0$ for $t \in [0, t_1), t_1 > 0$ and $z'(t_1) = 0$. Multiplying (19) by $z'(t) \leq 0$ and integrating from 0 to $t \leq t_1$, we get

$$z'^2(t) \geq -\alpha(\gamma - 1)z^2(t) + \frac{2\beta(\gamma - 1)^2}{2\gamma - 1} z^{\frac{2\gamma-1}{\gamma-1}}(t) + C = P(z^2(t)), \tag{20}$$

where $P(\xi)$ is defined as

$$P(\xi) = -\alpha(\gamma - 1)\xi + \frac{2\beta(\gamma - 1)^2}{2\gamma - 1} \xi^{\frac{2\gamma-1}{2(\gamma-1)}} + C, \tag{21}$$

$$C = z'^2(0) + \alpha(\gamma - 1)z^2(0) - \frac{2\beta(\gamma - 1)^2}{2\gamma - 1} z^{\frac{2\gamma-1}{\gamma-1}}(0) = \frac{2(\gamma - 1)^2}{2\gamma - 1} \Psi^{1-2\gamma}(0) \left[\frac{(2\gamma - 1) \Psi'^2(0)}{2 \Psi(0)} + \frac{\alpha(2\gamma - 1)}{2(\gamma - 1)} \Psi(0) - \beta \right].$$

From (18) it follows that

$$C \geq \frac{\alpha(\gamma - 1)}{2\gamma - 1} \Psi^{2-2\gamma}(0) A^{2-2\gamma} > 0.$$

Now we will prove that $P(\xi)$ takes positive values for every $\xi \in (0, \infty)$, see Fig 1 (a). Indeed, the function $P(\xi)$ is a convex function for $\xi > 0$ and attains its minimum at the point $\xi_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{2(\gamma-1)}{2\gamma-1}}$ because

$$P'(\xi) = (\gamma - 1) \left(-\alpha + \beta \xi^{\frac{1}{2(\gamma-1)}}\right), \quad P'(\xi_1) = 0, \quad P''(\xi) = \frac{\beta}{2} \xi^{\frac{1-2\gamma}{2(\gamma-1)}} > 0 \quad \text{for } \xi > 0.$$

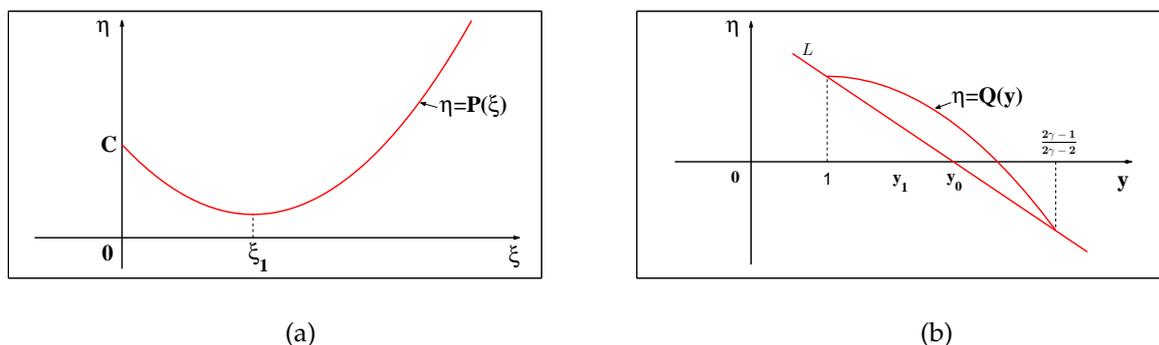


Figure 1: (a): Graph of the function $P(\xi)$ defined in (21); (b): Graph of the function $Q(y)$ defined in (23)

Let us check that $P(\xi_1) > 0$ and hence $P(\xi) \geq P(\xi_1) > 0$ for $\xi > 0$. Using (21) and the definition of A in (17), tedious calculations give us

$$\begin{aligned} P(\xi_1) &= -\alpha(\gamma-1)\left(\frac{\alpha}{\beta}\right)^{2(\gamma-1)} + \frac{2\beta(\gamma-1)^2}{2\gamma-1}\left(\frac{\alpha}{\beta}\right)^{2\gamma-1} + C \\ &= -\frac{(\gamma-1)\beta}{2\gamma-1}\left(\frac{\alpha}{\beta}\right)^{2\gamma-1} + \frac{2(\gamma-1)^2}{2\gamma-1}\Psi^{1-2\gamma}(0)\left[\frac{(2\gamma-1)}{2}\frac{\Psi'(0)}{\Psi(0)} + \frac{\alpha(2\gamma-1)}{2(\gamma-1)}\Psi(0) - \beta\right] \\ &= \frac{\alpha(\gamma-1)}{2\gamma-1}\Psi^{2-2\gamma}(0)\left[-\left(\frac{\alpha}{\beta}\right)^{2\gamma-2}\Psi^{2\gamma-2}(0) + (2\gamma-1)A - 2(\gamma-1)\frac{\beta}{\alpha}\Psi^{-1}(0)\right] \\ &= \frac{\alpha(\gamma-1)}{2\gamma-1}\left(\frac{\alpha}{\beta}\right)^{2\gamma-2}A^{2\gamma-1}\left[-A^{1-2\gamma} + (2\gamma-1)\left(\frac{\beta}{\alpha}\Psi^{-1}(0)A^{-1}\right)^{2\gamma-2} - 2(\gamma-1)\left(\frac{\beta}{\alpha}\Psi^{-1}(0)A^{-1}\right)^{2\gamma-1}\right]. \end{aligned}$$

Thus

$$P(\xi_1) = \frac{\alpha(\gamma-1)}{2\gamma-1}\left(\frac{\alpha}{\beta}\right)^{2\gamma-2}A^{2\gamma-1}Q(y_1), \tag{22}$$

where

$$Q(y) = -A^{1-2\gamma} + (2\gamma-1)y^{2\gamma-2} - 2(\gamma-1)y^{2\gamma-1}, \quad y_1 = \frac{\beta}{\alpha}\Psi^{-1}(0)A^{-1}. \tag{23}$$

We have to prove that $Q(y_1) > 0$. The function $Q(y)$ is a concave one for $y \geq 1$, because

$$Q''(y) = 2(\gamma-1)(2\gamma-1)y^{2\gamma-4}(2\gamma-3-(2\gamma-2)y) \leq -2(\gamma-1)(2\gamma-1)y^{2\gamma-4} < 0.$$

Moreover, at the points $y = 1$ and $y = \frac{2\gamma-1}{2\gamma-2} = 1 + \frac{1}{2\gamma-2} > 1$ the function $Q(y)$ takes values with different signs:

$$Q(1) = -A^{1-2\gamma} + 1 > 0, \quad Q\left(\frac{2\gamma-1}{2\gamma-2}\right) = -A^{1-2\gamma} < 0.$$

Let L be a line that passes through the points $(1, Q(1))$ and $\left(\frac{2\gamma-1}{2\gamma-2}, Q\left(\frac{2\gamma-1}{2\gamma-2}\right)\right)$:

$$L = \{(y, \eta) : \eta - 1 + A^{1-2\gamma} + (2\gamma-2)(y-1) = 0\}.$$

We denote by y_0 the intersection point of the line L with $\{\eta = 0\}$, i.e.

$$y_0 = 1 + \frac{1}{2\gamma-2}(1 - A^{1-2\gamma}).$$

Hence $Q(y) > 0$ for every $y \in [1, y_0)$, see Fig. 1 (b).

Now we will show that y_1 belongs to the interval $(1, y_0]$. From (17) and (18) we get

$$y_1 = \frac{\beta}{\alpha} \Psi^{-1}(0) A^{-1} \geq \left(\frac{(2\gamma - 1) \Psi'^2(0)}{2\alpha \Psi^2(0)} + 1 \right) A^{-1} > A A^{-1} = 1,$$

$$y_1 \leq \left(1 + \frac{(2\gamma - 1) \Psi'^2(0)}{2\alpha \Psi^2(0)} + \frac{1}{2\gamma - 2} (1 - A^{2-2\gamma}) \right) A^{-1} = \left(\frac{2\gamma - 1}{2\gamma - 2} A - \frac{1}{2\gamma - 2} A^{2-2\gamma} \right) A^{-1}$$

$$= \frac{2\gamma - 1}{2\gamma - 2} - \frac{1}{2\gamma - 2} A^{1-2\gamma} = 1 + \frac{1}{2\gamma - 2} (1 - A^{1-2\gamma}) = y_0.$$

Since $Q(y) > 0$ for $y \in [1, y_0]$ and $y_1 \in (1, y_0]$ it follows that $Q(y_1) > 0$. From (22) we get

$$P(\xi_1) = \frac{\alpha(\gamma - 1)}{2\gamma - 1} \left(\frac{\alpha}{\beta} \right)^{2\gamma-2} A^{2\gamma-1} Q(y_1) > 0.$$

Hence

$$P(z^2(t)) \geq P(\xi_1) > 0$$

for every $t \in [0, t_1]$. From (20) for $t = t_1$ we get

$$z^2(t_1) \geq P(z^2(t_1)) \geq P(\xi_1) > 0,$$

which contradicts our assumption that $z'(t_1) = 0$. Thus for every $t \geq 0$ we obtain $z'(t) < 0$ and

$$z'(t) \leq -\sqrt{P(\xi_1)},$$

or equivalently from (14), $\Psi(t)$ is a strictly increasing function and

$$\Psi'(t) \geq \frac{\sqrt{P(\xi_1)}}{\gamma - 1} \Psi^\gamma(t) > \frac{\sqrt{P(\xi_1)}}{\gamma - 1} \Psi^\gamma(0) > 0. \tag{24}$$

Integrating (24) from 0 to t we have

$$\Psi(t) > \Psi(0) + \frac{\sqrt{P(\xi_1)}}{\gamma - 1} \Psi^\gamma(0)t.$$

Hence the following inequality holds:

$$\Psi(t) > \frac{\beta}{\alpha} \quad \text{for } t \geq b, \quad b = \frac{\gamma - 1}{\sqrt{P(\xi_1)}} \left(\frac{\beta}{\alpha} - \Psi(0) \right) \Psi^{-\gamma}(0). \tag{25}$$

From (24) and (25) we obtain that conditions (11) and (12) are satisfied. By means of Theorem 2.2 it follows that $\Psi(t)$ blows up for a finite time t^* , which contradicts our assumption that $\Psi(t)$ is globally defined. Theorem 2.3 is proved. \square

3. Application to Nonlinear Klein-Gordon Equation

We use the following short notations:

$$\|u\| = \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)}, \quad \|u\|_1 = \|u(t, \cdot)\|_{H^1(\mathbb{R}^n)}, \quad (u, v) = \int_{\mathbb{R}^n} u(t, x)v(t, x) dx.$$

Let us recall the well-known local existence result for problem (5)-(7):

Theorem 3.1 (Local existence). *If (6) and (7) hold, then there exists a unique local weak solution*

$$u(t, x) \in C([0, T_m); H^1(\mathbb{R}^n)) \cap C^1([0, T_m); L^2(\mathbb{R}^n)) \cap C^2([0, T_m); H^{-1}(\mathbb{R}^n))$$

to (5)-(7) on a maximal existence time interval $[0, T_m)$, $T_m \leq \infty$. Moreover:

(i) *If*

$$\limsup_{t \uparrow T_m} \|u\|_1 < \infty \quad \text{then} \quad T_m = \infty;$$

(ii) *The weak solution $u(t, x)$ blows up for a finite time if*

$$\limsup_{t \uparrow T_m} \|u\|_1 = \infty \quad \text{for} \quad T_m < \infty;$$

(iii) *For every $t \in [0, T_m)$ the solution $u(t, x)$ satisfies the conservation law*

$$E(0) = E(t) := E(u(t, \cdot)) = \frac{1}{2} (\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2) - \int_{\mathbb{R}^n} \int_0^u f(y) dy dx. \tag{26}$$

Let us mention that for small (subcritical) positive initial energy a complete result for global existence or finite time blow up of the solutions to (5)-(7) is given in [15] by the potential well method.

Below we summarize the results for nonexistence of global solutions to (5)-(7) with arbitrary positive energy.

Theorem 3.2. *Suppose $u(t, x)$ is a weak solution of (5)-(7) in the maximal existence time interval $[0, T_m)$, $T_m \leq \infty$ and*

$$\|u_0\| \neq 0, \quad (u_0, u_1) > 0 \tag{27}$$

(i) ([4]) *If*

$$0 < E(0) < \frac{1}{2} \frac{(u_0, u_1)^2}{\|u_0\|^2},$$

then $u(t, x)$ blows up for a finite time $T_m \leq T_1^ < \infty$, where*

$$T_1^* = \frac{\sqrt{2}}{(p_1 - 1)} \frac{\|u_0\|}{\sqrt{\frac{1}{2} \frac{(u_0, u_1)^2}{\|u_0\|^2} - E(0)}}$$

(ii) ([7]) *If*

$$0 < E(0) < \frac{1}{2} \frac{(p_1 - 1)}{(p_1 + 1)} \|u_0\|^2 + \frac{1}{2} \frac{(u_0, u_1)^2}{\|u_0\|^2},$$

then $u(t, x)$ blows up for a finite time $T_m \leq T_2^ < \infty$, where*

$$T_2^* = \begin{cases} \frac{2}{(p_1 - 1)} \frac{\|u_0\|^2}{(u_0, u_1)} & \text{if } E(0) < \frac{1}{2} \frac{(p_1 - 1)}{(p_1 + 1)} \|u_0\|^2; \\ \frac{\sqrt{2}}{(p_1 - 1)} \frac{\|u_0\|}{\sqrt{\frac{1}{2} \frac{(p_1 - 1)}{(p_1 + 1)} \|u_0\|^2 + \frac{1}{2} \frac{(u_0, u_1)^2}{\|u_0\|^2} - E(0)}} & \text{if } E(0) \geq \frac{1}{2} \frac{(p_1 - 1)}{(p_1 + 1)} \|u_0\|^2. \end{cases}$$

Theorem 3.3 (Main application). *Suppose $u(t, x)$ is a weak solution of (5)-(7) in the maximal existence time interval $[0, T_m)$, $T_m \leq \infty$. If (27) holds and*

$$0 < E(0) \leq \frac{1}{2} \frac{(p_1 - 1)}{(p_1 + 1)} \|u_0\|^2 + \frac{1}{2} \frac{(u_0, u_1)^2}{\|u_0\|^2} + \frac{\|u_0\|^2}{p_1 + 1} \left[1 - \left(1 + \frac{(u_0, u_1)^2}{\|u_0\|^4} \right)^{\frac{1-p_1}{2}} \right] \tag{28}$$

then the weak solution $u(t, x)$ blows up for a finite time $T_m < \infty$.

Proof. By contradiction we suppose that the weak solution $u(t, x)$ is defined for every $t \geq 0$, i.e. $T_m = \infty$. For function $\Psi(t) = \|u\|^2$ we have $\Psi'(t) = 2(u, u_t)$ and

$$\Psi''(t) = 2\|u_t\|^2 + 2(u, u_{tt}) = 2\|u_t\|^2 - 2(\|u\|^2 + \|\nabla u\|^2 - (u, f(u))).$$

By means of conservation law (26), the following equality holds

$$(\|u\|^2 + \|\nabla u\|^2 - (u, f(u))) = (p_1 + 1)E(0) - \frac{(p_1 + 1)}{2}\|u_t\|^2 - \frac{(p_1 - 1)}{2}(\|u\|^2 + \|\nabla u\|^2) - (p_1 + 1)B(t), \tag{29}$$

where $B(t)$ is given by the expression

$$B(t) := B(u(t)) = \sum_{k=2}^l \frac{a_k(p_k - p_1)}{(p_k + 1)(p_1 + 1)} \int_{\mathbb{R}^n} |u|^{p_k+1} dx + \sum_{j=1}^s \frac{b_j(p_1 - q_j)}{(q_j + 1)(p_1 + 1)} \int_{\mathbb{R}^n} |u|^{q_j+1} dx.$$

Since $p_k > p_1 > q_j$ for $k = 2, \dots, l$ and $j = 1, \dots, s$, it follows that for every $t \in [0, T_m)$

$$B(t) \geq 0 \tag{30}$$

and $B(t) \equiv 0$ for $a_k = 0, k = 2, \dots, l$ and $b_j = 0, j = 1, \dots, s$.

From (29), (30) and Hölder inequality we have for function u with $\|u\| \neq 0$ the following estimates:

$$\begin{aligned} \Psi''(t) &= 2\|u_t\|^2 - 2(p_1 + 1)E(0) + (p_1 + 1)\|u_t\|^2 + (p_1 - 1)(\|u\|^2 + \|\nabla u\|^2) + 2(p_1 + 1)B(t) \\ &\geq (p_1 + 3)\|u_t\|^2 - 2(p_1 + 1)E(0) + (p_1 - 1)\|u\|^2 \\ &\geq (p_1 + 3)\frac{(u, u_t)^2}{\|u\|^2} - 2(p_1 + 1)E(0) + (p_1 - 1)\|u\|^2 \end{aligned}$$

Hence $\Psi(t)$ satisfies for $t \geq 0$ the differential inequality

$$\Psi''(t)\Psi(t) - \frac{(p_1 + 3)}{4}\Psi'^2(t) \geq (p_1 - 1)\Psi^2(t) - 2(p_1 + 1)E(0)\Psi(t).$$

If we set $\gamma = \frac{p_1+3}{4} > 1$, $\alpha = (p_1 - 1) > 0$, $\beta = 2(p_1 + 1)E(0) > 0$, then Theorem 2.3 can be applied since $\Psi(0) = \|u_0\|^2 > 0$, $\Psi'(0) = 2(u_0, u_1) > 0$ and condition (17) coincides with condition (28). From Theorem 2.3 it follows that $\Psi(t) \rightarrow \infty$ for $t \rightarrow T_m < \infty$. This contradicts the assumption that $\Psi(t)$ is globally defined. Since $\Psi(t) = \|u\|^2$ then $\|u\|_1 \rightarrow \infty$ for $t \rightarrow T_m < \infty$ and according to Theorem 3.1 (ii) the weak solution $u(t, x)$ blows up for a finite time. Theorem 3.3 is proved. \square

In the following theorem we construct explicitly initial data with arbitrary high energy such that all conditions of Theorem 3.3 are satisfied.

Theorem 3.4. *For every positive constant K there exist infinitely many initial data u_0^K, u_1^K such that $E(u_0^K, u_1^K) = K$ and the existing time for the corresponding to (5)-(7) solution u^K is finite, i.e. $T_m < \infty$.*

Proof. Let $w(x) \in H^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $v(x) \in L^2(\mathbb{R}^n)$ be fixed functions satisfying

$$\|w\|_1 \neq 0, \quad w(x) > 0, \quad \|v\| \neq 0, \quad (w, v) = 0. \tag{31}$$

For example, when w is an even function and v is an odd one, then (31) are satisfied. Let K be a fixed arbitrary positive constant. We construct initial data u_0^K, u_1^K in the following way:

$$u_0^K(x) = qw(x), \quad u_1^K(x) = qw(x) + sv(x), \tag{32}$$

where $q > 0$ and $s > 0$ are constants. Our aim is to choose the constants q and s in (32) such that initial data u_0^K, u_1^K satisfy all conditions of Theorem 3.3 as well as $E(u_0^K, u_1^K) = E(0) = K$.

Straightforward computations give us the following formulas for energy $E(0)$ and different norms of u_0^K and u_1^K in terms of norms of w and v :

$$\|u_0^K\|^2 = q^2\|w\|^2, \quad \|u_1^K\|^2 = q^2\|w\|^2 + s^2\|v\|^2, \quad (u_0^K, u_1^K) = q^2\|w\|^2, \quad \|\nabla u_0^K\|^2 = q^2\|\nabla w\|^2,$$

$$E(u_0^K, u_1^K) = E(0) = S(q) + \frac{s^2}{2}\|v\|^2, \quad \text{where}$$

$$S(q) = q^2\|w\|^2 + \frac{q^2}{2}\|\nabla w\|^2 - \sum_{k=1}^l \frac{a_k}{p_k + 1} q^{p_k+1} \int_{\mathbb{R}^n} w^{p_k+1} dx + \sum_{j=1}^s \frac{b_j}{q_j + 1} q^{q_j+1} \int_{\mathbb{R}^n} w^{q_j+1} dx.$$

From (31) and (32) it follows that initial data u_0^K, u_1^K fulfill conditions in (27) for every $q > 0$ and $s > 0$. Now we have to choose the constants q and s in order to satisfy (28) and $E(u_0^K, u_1^K) = K$. These two assumptions are equivalent to

$$K = S(q) + \frac{s^2}{2}\|v\|^2 < \frac{q^2}{p_1 + 1} \left(p_1 + 1 - 2^{-\frac{p_1-1}{2}} \right) \|w\|^2. \quad (33)$$

Note that for all sufficiently large q the inequality $S(q) < 0$ holds, because the leading term of S with respect to q has a negative sign. Thus we can choose $q = q_*$ sufficiently large so that

$$S(q_*) < 0 \quad \text{and} \quad q_* > \left(\frac{(p_1 + 1)K}{p_1 + 1 - 2^{-\frac{p_1-1}{2}}} \right)^{1/2} \|w\|^{-1},$$

which guarantees that the lhs of (33) is less than the rhs of (33). We fix $s = s_*$ so that $S(q_*) + \frac{s_*^2}{2}\|v\|^2 = K$, i.e., $s_* = (2(K - S(q_*)))^{1/2}\|v\|^{-1}$. Thus initial data (32) with already chosen constants q_* and s_* satisfy all conditions in Theorem 3.3. Moreover, these initial data have arbitrary high positive energy $E(0) = K$. Theorem 3.4 is proved. \square

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