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Stability Analysis of Neutral Linear Fractional System with Distributed Delays

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Abstract. The aim of the present work is to study the initial value problem for neutral linear fractional differential system with distributed delays in incommensurate case. Furthermore, in the autonomous case with derivatives in the Riemann-Liouville or Caputo sense we establish that if all roots of the introduced characteristic equation have negative real parts, then the zero solution is globally asymptotically stable. The proposed condition coincides with the conditions which guaranty the same result in the particular case of system with constant delays.

1. Introduction

In the last two decades the fractional calculus has been intensively investigated by reason of their applicability in several science fields as rheology, viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas et al. [5], Kiryakova [6], Podlubny [9] and Das [1] and the references therein. The survey from Li, Zhang [7] gives a good overview of the contributions to the stability theory of fractional differential equations. Stability results are received in the noteworthy works of Deng, Li, Lu [2] and Qian, Li, Agarwal, Wong [10] for fractional system with constant delays, and in [11] for fractional system with distributed delays. It must be mentioned that the first detailed study of the linear delay differential equations and system with distributed delays (fundamental theory, stability, oscillation behavior, etc.) was done by A. D. Myshkis in his fundamental monograph [8]. The theory for fractional equations and systems with distributed delays, and especially in the case of neutral systems as in the integer case, is generally speaking more difficult at least technically but not only. In [12] it is studied fractional linear autonomous system of neutral type with constant delays. Our paper extends and improves the results obtained in [12] for the case of distributed delays.

The aim of this work is to clear the existence and the uniqueness of the solution of the initial value problem for linear incommensurate fractional differential system with distributed delays in the cases of Riemann-Liouville and Caputo derivatives. For the autonomous case we generalize the result obtained in [12] and prove the classical result that if all roots of an analogue of the characteristic equation introduced in [7] have negative real parts, then the zero solution of the considered homogeneous linear fractional differential system with distributed delay is asymptotically stable.

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2. Preliminaries

As is known, there are many different definitions of the fractional derivative, all of which generalize the usual integer order derivative. Below we recall the definitions of Riemann-Liouville and Caputo fractional derivatives as well as some of their basic properties.

Let us denote by $L_1^{loc}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f:\mathbb{R}\to\mathbb{R}$. Then for each $a\in\mathbb{R}$ and $f\in L_1^{loc}(\mathbb{R},\mathbb{R})$ the left-sided fractional integral operator of order $\alpha>0,\alpha\in\mathbb{R}$ is defined by

$$(D_{a+}^{-\alpha}f)(t)=(I_{a+}^{\alpha}f)(t)=\frac{1}{\Gamma(\alpha)}\int\limits_a^t (t-s)^{\alpha-1}f(s)ds,\ \ t>a$$

$$(D_{0+}^{0}f)(t)=(I_{0+}^{0}f)(t)=f(t)$$

and the corresponding left side Riemann-Liouville fractional derivative by

$${}_{RL}D_{a+}^{\alpha}f(t)=\left(\frac{d}{dt}\right)^n(I_{a+}^{n-\alpha}f)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int\limits_a^t(t-s)^{n-\alpha-1}f(s)ds,$$

where $t>a, n=[\alpha]+1,\alpha\notin\mathbb{N}$ and $n=\alpha,\alpha\in\mathbb{N}$. Note some properties of the Riemann-Liouville fractional derivative, where with Id we denote the identity operator:

(j) ${}_{RL}D_{a+}^{\alpha}D_{a+}^{-\alpha}=Id,\ \ \alpha>0;$

(jj) $D_{a+}^{-\alpha}{}_{RL}D_{a+}^{\alpha}f(t)=f(t)-\sum\limits_{k=1}^{n}\frac{[{}_{RL}D_{a+}^{\alpha-k}f(t)](a)}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k};$

(jjj) ${}_{RL}D_{0+}^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)t^{\gamma-\alpha}}{\Gamma(\gamma+1-\alpha)},\ \ \alpha>0,\gamma>-1,t>0.$

The Caputo fractional left side derivative ${}_{C}D_{a+}^{\alpha}$ is defined by the equality

$${}_{C}D_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_a^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

where $t>a, n=[\alpha]+1,\alpha\notin\mathbb{N}$ and $n=\alpha,\alpha\in\mathbb{N}$. The next formula clears the close connection between the Caputo and the Riemann-Liouville derivatives:

$${}_{C}D_{a+}^{\alpha}f(t)={}_{RL}D_{a+}^{\alpha}[f(s)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(s-a)^k](t)$$

Note some properties of the Caputo fractional derivative:

(jv) $D_{a+}^{-\alpha}{}_{C}D_{a+}^{\alpha}f(t)=f(t)-\sum\limits_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(t-a)^k;$

(v) ${}_{RL}D_{a+}^{\alpha}f(t)=\sum\limits_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}+{}_{C}D_{a+}^{\alpha}f(t).$

The Laplace transform of the Riemann-Liouville fractional derivative ${}_{RL}D_{0+}^{\alpha}f(t)$ is

(vj) $(\mathfrak{L}{}_{RL}D_{0+}^{\alpha}f)(p)=\int\limits_0^{\infty}e^{-pt}\,{}_{RL}D_{0+}^{\alpha}f(t)dt=p^{\alpha}\hat{f}(p)-\sum\limits_{k=0}^{n-1}p^k[{}_{RL}D_{0+}^{\alpha-k-1}f(t)]_{t=0}$

and the Laplace transform of the Caputo fractional derivative ${}_{C}D_{a+}^{\alpha}f(t)$ is

(vjj) $(\mathfrak{L}{}_{C}D_{0+}^{\alpha}f)(p)=\int\limits_0^{\infty}e^{-pt}\,{}_{C}D_{0+}^{\alpha}f(t)dt=p^{\alpha}\hat{f}(p)-\sum\limits_{k=0}^{n-1}p^{\alpha-k-1}f^{(k)}(0),\ \ n-1\le\alpha<n,$

where $(\mathfrak{L}f)(p)=\int\limits_0^{\infty}e^{-pt}f(t)dt=\hat{f}(p), n\in\mathbb{N}.$

3. Equivalence between the Initial Value Problem and the Volterra Integral Equation

Consider the neutral linear delayed system of incommensurate type with distributed delay

$$D_{0+}^{\alpha_k}[x_k(t) - \sum_{j=1}^{n} \int_{-t}^{0} x_j(t+\theta) dv_k^j(t,\theta)] = \sum_{j=1}^{n} \int_{-\sigma}^{0} x_j(t+\theta) d_\theta u_k^j(t,\theta) + f_k(t) \quad k = 1, 2, \dots, n \tag{3.1}$$

where $D_{0+}^{\alpha_k}$ denotes either ${}_{RL}D_{0+}^{\alpha_k}$ (the Riemann-Liouville fractional derivative) or ${}_{C}D_{0+}^{\alpha_k}$ (the Caputo fractional derivative), $\alpha_k \in (0,1), \tau, \sigma \in \mathbb{R}_{0+} = (0, \infty)$.

Introduce the following notations:

$$X(t) = (x_1(t), ..., x_n(t))^T, \ \ |X(t)| = \sum_{k=1}^{n} |x_t(t)|, \ \ F(t) = (f_1(t), \dots, f_n(t))^T$$

$${}_{RL}D_{0+}^{\alpha} X(t) = ({}_{RL}D_{0+}^{\alpha_1} x_1(t), \dots, {}_{RL}D_{0+}^{\alpha_n} x_n(t))^T, \ \ {}_{C}D_{0+}^{\alpha} X(t) = ({}_{C}D_{0+}^{\alpha_1} x_1(t), \dots, {}_{C}D_{0+}^{\alpha_n} x_n(t))^T,$$

for each $t \in \mathbb{R}_+ = [0, \infty)$. For $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}, W(t,\theta) = \{w_j^i(t,\theta)\}_{i,j=1}^n$ we denote $|W(t,\theta)| = \sum_{k,j=1}^{n} |w_k^j(t,\theta)|$. With $\mathfrak{C}$ we denote the Banach space of the initial vector functions

$$\mathfrak{C} = \{\Phi : [-h, 0] \to \mathbb{R}^n \ \mid \ \Phi(t) = (\phi_1(t), \dots, \phi_n(t))^T, \phi_k \in C([-h,0], \mathbb{R}), 1 \le k \le n, h = max(\sigma, \tau)\}$$

with the norm $\|\Phi\| = \sup_{t\in[-\sigma,0]} \sum_{k=1}^{n} |\phi_k(t)| = \sup_{t\in[-\sigma,0]} |\Phi(t)|$ and by $BV[a,b]$ the linear space of functions $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ with bounded variation in θ on $[a,b] \subset \mathbb{R}, a \le b$ for every $t \in \mathbb{R}_+$, where $Var_{[a,b]} W(t,\cdot) = \sum_{k,j=1}^{n} Var_{[a,b]} w_k^j(t,\cdot)$.

We say that for the kernels $U, V : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ the conditions (S) are fulfilled if the following conditions hold:

(S1) The functions $(t,\theta) \to U(t,\theta)$ and $(t,\theta) \to V(t,\theta)$ are measurable on $(t,\theta) \in \mathbb{R}\times\mathbb{R}$ and normalized so that $U(t,\theta) = 0, V(t,\theta) = 0$ for $\theta \ge 0$ and $U(t,\theta) = U(t,-\sigma)$ for $\theta \le -\sigma$ and $V(t,\theta) = V(t,-\tau)$ for $\theta \le -\tau$.

(S2) For each $t \in \mathbb{R}_+$ the functions $U(t,\theta)$ and $V(t,\theta)$ are continuous from the left in θ on $(-\sigma, 0)$ and $(-\tau, 0)$ respectively.

(S3) $U(t,\cdot) \in BV[-h, 0]$ and there exists a function $z \in L_1^{loc}(\mathbb{R}, \mathbb{R}_+)$ such that $Var_{[-\sigma,0]} U(t,\cdot) \le z(t)$.

(S4) $V(t,\cdot) \in BV[-h, 0]$ and is uniformly nonatomic at zero (see [4]), i.e. for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $Var_{[-\delta,0]} V(t,\cdot) = \sum_{k,j=1}^{n} Var_{[-\delta,0]} v_k^j(t,\cdot) < \epsilon$.

(S5) For each $t \in \mathbb{R}_+$ the following relations hold: $\int_{-\sigma}^{0} |U(t,\theta) - U(t^*,\theta)| d\theta \to 0$, $\int_{-\tau}^{0} |V(t,\theta) - V(t^*,\theta)| d\theta \to 0$ when $t^* \to t$.

(S6) The Lebesgue decomposition of the kernel $V(t,\theta)$ has no singular part and has the form:

$$V(t,\theta) = \aleph(t,\theta) + \int_{-\tau}^{\theta} B(t,s) ds, \tag{3.2}$$

where $t \in \mathbb{R}_+, \theta \in [-\tau, 0], \aleph(t,\theta) = \{a_k^j(t) H(\theta + \tau_k^j(t))\}_{k,j=1}^n, B(t,\theta) = \{b_k^j(t,\theta)\}_{k,j=1}^n$ and $H(t)$ is the Heaviside function.

(S7) $T \in C(\mathbb{R}_+, \mathbb{R}_+^n), A \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^n)$, where $T(t) = \{\tau_k^j(t)\}_{k,j=1}^n$, $A(t) = \{a_k^j(t)\}_{k,j=1}^n$ is locally bounded and $B(t,\theta)$ is continuous in respect of t for each $\theta \in [-\tau, 0]$.

First we will study the system (3.1) when the derivatives are in Riemann-Liouville sense and consider the Cauchy problem for (3.1) under the initial conditions

$$D_{0+}^{\alpha_k-1}x_k(t) = \phi_k(t), t \in [-h, 0], \Phi \in \mathfrak{C}, 1 \le k \le n. \tag{3.3}$$

Taking into account Lemma 3.2 in [5], the initial value problem IVP (3.1), (3.3) can be rewritten in the form

$$D_{0+}^{\alpha}(X(t) - \int_{-\tau}^{0}[d_\theta V(t,\theta)]X(t+\theta)) = \int_{-\sigma}^{0}[d_\theta U(t,\theta)]X(t+\theta) + F(t), \tag{3.4}$$

$$X_\alpha(t) = \Phi(t), t \in [-h, 0], \Phi \in \mathfrak{C}, \tag{3.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $X_\alpha(t) = (\Gamma(\alpha_1)t^{1-\alpha_1}x_1(t), \ldots, \Gamma(\alpha_n)t^{1-\alpha_n}x_n(t))^T$.

Let $J \subset \mathbb{R}$ be an arbitrary interval and denote with $\ell(J, \mathbb{R}^n)$ the real linear space of all Lebesgue measurable functions $G = (g_1, \ldots, g_n)^T : J \to \mathbb{R}^n$.

Definition 3.1. A function $G \in \ell(J, \mathbb{R}^n)$ is said to be α-continuous at $t_0 \in J$ if there exists an $\alpha = (\alpha_1, ..., \alpha_n)$ such that the functions $f_k(t) := |t - t_0|^{\alpha_k} g_k(t)$ are continuous at $t_0 \in J, 1 \le k \le n$.

Denote by $C_M^\alpha(C_\infty^\alpha)$ the real linear space of all α-continuous at zero functions $G : J_M \to \mathbb{R}^n (G : J_\infty \to \mathbb{R}^n)$ for which the functions $f_k, 1 \le k \le n$ in the representation given in Definition 3.1 are continuous in $J_M = [-h, M](J_\infty = [-h, \infty))$.

Definition 3.2. The vector function $X(t) = (x_1(t), ..., x_n(t))^T$ is a solution of the IVP (3.1), (3.3) in $J_\infty(J_M)$ if $X \in C_\infty^\alpha(C_M^\alpha)$ satisfies the system (3.1) for $t \in \mathbb{R}_{0+}$ $(t \in (0, M])$ and the initial condition (3.3) for $t \in [-h, 0]$.

Consider the system

$$\begin{aligned} x_k(t) = &\frac{\phi_k(0)t^{\alpha_k-1}}{\Gamma(\alpha_k)} + \sum_{j=1}^{n}\int_{-\tau}^{0} x_j(t+\theta)dv_k^j(t,\theta) + \\ &+ \frac{1}{\Gamma(\alpha_k)}[\int_0^t (t-s)^{\alpha_k-1}(\sum_{j=1}^{n}\int_{-\sigma}^{0} x_j(s+\theta)d_\theta u_k^j(s,\theta))ds + \int_0^t (t-s)^{\alpha_k-1}f_k(s)ds] \end{aligned} \tag{3.6}$$

$k = 1, 2, \ldots, n$.

Lemma 3.3. *Let the following conditions hold:*

1. *Conditions* (*S*) *hold.*
2. $F \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^n)$ *be locally bounded.*
3. $\tau_k^j(0) > 0, 1 \le k, j \le n.$

Then every solution $X \in C_\infty^\alpha$ *of IVP* (3.1), (3.3) *is a solution of the system* (3.6) *and satisfies the condition*(3.5) *and vice versa.*

Proof. Let $X \in C_\infty^\alpha$be an arbitrary solution of the IVP (3.1), (3.3). Then applying for $k = 1, 2, \ldots, n$ the operator $D_{0+}^{-\alpha_k}$ on the both sides of (3.1) taking into account (3.2) and (jj) we obtain the equality

$$x_k(t) - \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\theta) dv_k^j(t,\theta) - \frac{t^{\alpha_k-1}}{\Gamma(\alpha_k)} \left([{}_{RL}D_{0+}^{\alpha_k-1} x_k(t)](0) - [{}_{RL}D_{0+}^{\alpha_k-1} \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\theta) dv_k^j(t,\theta)](0) \right) = \tag{3.7}$$
$$= \frac{1}{\Gamma(\alpha_k)} [\int_0^t (t-s)^{\alpha_k-1} (\sum_{j=1}^{n} \int_{-\sigma}^{0} x_j(s+\theta) d_\theta u_k^j(s,\theta)) ds + \int_0^t (t-s)^{\alpha_k-1} f_k(s) ds]$$

Applying Lemma 3.2 in [5] to the third and the fourth addend in the left side of (3.7) and taking into account conditions (S6) and (S7) we receive that

$$\frac{t^{\alpha_k-1}}{\Gamma(\alpha_k)} [{}_{RL}D_{0+}^{\alpha_k-1} x_k(t)](0) = t^{\alpha_k-1} \lim_{t\to 0+} t^{1-\alpha_k} x_k(t) = t^{\alpha_k-1} \phi_k(0) \tag{3.8}$$

and

$$\frac{t^{\alpha_k-1}}{\Gamma(\alpha_k)} [{}_{RL}D_{0+}^{\alpha_k-1} \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\theta) dv_k^j(t,\theta)](0) = t^{\alpha_k-1} \lim_{t\to 0+} t^{1-\alpha_k} \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\theta) dv_k^j(t,\theta) = \tag{3.9}$$
$$= t^{\alpha_k-1} \lim_{t\to 0+} t^{1-\alpha_k} \sum_{j=1}^{n} (a_k^j(t) x_j(t-\tau_k^j(t)) + \int_{-\tau}^{0} b_k^j(t,\theta) x_j(t+\theta) d\theta)$$

From conditions (S) and condition 3 of the lemma it follows that

$$\lim_{t\to 0+} t^{1-\alpha_k} \sum_{j=1}^{n} (a_k^j(t) x_j(t-\tau_k^j(t)) + \int_{-\tau}^{0} b_k^j(t,\theta) x_j(t+\theta) d\theta) = 0. \tag{3.10}$$

Then from (3.7) it follows that the solution $X \in C_\infty^\alpha$ of IVP (3.1), (3.3) is a solution of the system (3.6) and satisfies the condition (3.5).

Let $X \in C_\infty^\alpha$ be a solution of the system (3.6) for $t \in \mathbb{R}_{0+}$ which satisfies the condition (3.5). Taking into account (j) and (jjj) we conclude that every such solution is a solution of the IVP (3.1), (3.3). □

Remark 3.4. Note that if the Lebesgue decompositions of the kernels $U(t,\theta)$ and $V(t,\theta)$ have no singular and absolutely continuous part and the delays are finite number and constants as in the case considered in [12] then the condition 3 of Lemma 3.3 is ultimately fulfilled.

Corollary 3.5. *Let the conditions of Lemma 3.3 hold. Then the condition $\phi_k(0) = 0, 1 \le k \le n$ is necessary for the IVP* (3.1), (3.3) *to have a solution $X \in C(J_\infty, \mathbb{R}^n)$.*

Proof. Let suppose that there exists a solution of IVP(3.1), (3.3) such that $X \in C(J_\infty, \mathbb{R}^n)$. For arbitrary fixed $t_0 \in \mathbb{R}_{0+}$ we denote $M(t_0) = \max_{1\le k\le n}(\max(\sup_{t\in[0,t_0]} |x_k(t)|, \sup_{t\in[0,t_0]} |f(t)_k|))$. Then for each $t \in (0, t_0]$ in virtue of Lemma 3.3 for the third addend from (3.6) we obtain the estimation

$$|\frac{1}{\Gamma(\alpha_k)} \sum_{j=1}^{n} \int_0^t [(t-s)^{\alpha_k-1} \int_{-\sigma}^{0} x_j(s+\theta) d_\theta u_k^j(s,\theta)] ds + \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k-1} f_k(s) ds| \le \tag{3.11}$$
$$\le \frac{M(t_0)}{\alpha_k \Gamma(\alpha_k)} \left(\sup_{t\in[0,t_0]} Var_{[-\sigma,0]} U(t,\cdot) + 1 \right) t^{\alpha_k}$$

Lemma 1 in [8] implies that the second addend from (3.6) has a finite limit for $t \to 0+$. Then from (3.6) and (3.11) it follows that $\lim_{t\to 0+} x_k(t) < \infty$ if $\phi_k(0) = 0, 1 \le k \le n$. □

4. Existence and Uniqueness of the Solutions of the Initial Value Problem

Let $\Phi \in \mathfrak{C}$ be an arbitrary initial function and for each $M > 0$ define the set

$$\Omega_M^{\Phi} = \{G : J_M \to \mathbb{R}^n \mid G \in C_M^{\alpha}, G_{\alpha}(t) = \Phi(t), t \in [-h, 0]\},$$

where $G_{\alpha}(t) = (\Gamma(\alpha_1)t^{1-\alpha_1}g_1(t), \dots, \Gamma(\alpha_n)t^{1-\alpha_n}g_n(t))^T$.

Introduce for $G, G^* \in \Omega_M^{\Phi}$ the metric function $d_M : \Omega_M^{\Phi} \times \Omega_M^{\Phi} \to \mathbb{R}_+$ with

$$d_M(G, G^*) = \sup_{t \in J_M} \sum_{k=1}^{n} |t|^{1-\alpha_k}|g_k(t) - g_k^*(t)|.$$

Since every $G \in \Omega_M^{\Phi}$ is α-continuous at zero, then the metric function d_M takes only finite values and therefore the set Ω_M^{Φ} equipped with d_M is a complete metric space.

Introduce for each $G = (g_1, ..., g_n)^T \in \Omega_M^{\Phi}$ the operator $\mathfrak{R}$ with $\mathfrak{R}G(t) = (\mathfrak{R}_1 g_1(t), ..., \mathfrak{R}_n g_n(t))$, where

$$\begin{aligned}\mathfrak{R}_k g_k(t) = {} & \frac{\phi_k(0)t^{\alpha_k - 1}}{\Gamma(\alpha_k)} + \sum_{j=1}^{n} \int_{-\tau}^{0} g_j(t+\theta)dv_k^j(t,\theta) + \\ & + \frac{1}{\Gamma(\alpha_k)}[\int_0^t (t-s)^{\alpha_k-1}(\sum_{j=1}^{n}\int_{-\sigma}^{0} g_j(s+\theta)d_{\theta}u_k^j(s,\theta))ds + \int_0^t (t-s)^{\alpha_k-1}f_k(s)ds]\end{aligned} \tag{4.1}$$

for $t > 0$ and $D_{0+}^{\alpha_k-1}\mathfrak{R}_k g_k(t) = \phi_k(t)$ for $t \in [-h, 0]$.

Theorem 4.1. *Let the conditions of Lemma 3.3 hold. Then there exists $M > 0$ such that the IVP* (3.1), (3.3) *has a unique solution $X \in C_M^{\alpha}$.*

Proof. Since $F \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^n)$ and is locally bounded we can conclude that the third addend in the right side of (4.1) is a continuous function for $t \in \mathbb{R}_{0+}$. From conditions (S) and Lemma 1 in [8] it follows that $\sum_{j=1}^{n}\int_{-\tau}^{0} g_j(t+\theta)dv_k^j(t,\theta)$ is a continuous function for $t \in \mathbb{R}_{0+}$ and each $1 \le j,k \le n$. Taking into account that from (4.1) and Lemma 3.2 in [5] it follows that $(\mathfrak{R}G)_{\alpha}(t)|_{t=0} = \Phi(0)$ holds, we can conclude that the operator $\mathfrak{R}$ maps Ω_M^{Φ} into Ω_M^{Φ}.

Let $G, G^* \in \Omega_M^{\Phi}$ be arbitrary. Then from (3.7) for $0 < t < \tau$ we have the estimation

$$\begin{aligned}|\mathfrak{R}_k g_k(t) - \mathfrak{R}_k g_k^*(t)| \le {} & \sum_{j=1}^{n} |\int_{-\sigma}^{0} (g_j(t+\theta) - g_j^*(t+\theta))d_{\theta}v_k^j(t,\theta)| + \\ & + \frac{1}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1}\sum_{j=1}^{n}|\int_{-\sigma}^{0}(g_j(s+\theta) - g_j^*(s+\theta))d_{\theta}u_k^j(s,\theta)|ds \le \\ & \le \sum_{j=1}^{n} |a_k^j(t)(g_j(t-\tau_k^j(t) - g_j^*(t-\tau_k^j(t))))| + \int_{-t}^{0} |b_k^j(t,\theta)|\,|g_j(t+\theta) - g_j^*(t+\theta)|d\theta + \\ & + \frac{\sup_{t\in J_M} Var_{s\in[-\sigma,0]}U(t,s)}{\Gamma(\alpha_k)} \sup_{t \in J_M}|(g_k(t) - g_k^*(t))| \int_0^t (t-s)^{\alpha_k-1}ds\end{aligned} \tag{4.2}$$

Conditions (S) and condition 3 of Lemma 3.3 implies that there exists point $t^* \in (0, \tau)$ and constant $B^* > 0$, such that for $t \in [-h, t^*]$we have that $t - \tau_k^j(t) < 0$ and therefore from (4.2) follows the inequality

$$|\mathfrak{R}_k g_k(t) - \mathfrak{R}_k g_k^*(t)| \leq (tB^* + t^{\alpha_k} \frac{\sup_{t\in J_M} Var_{s\in[-\sigma,0]} U(t,s)}{\alpha_k \Gamma(\alpha_k)}) \sup_{t\in J_M} |(g_k(t) - g_k^*(t))| \tag{4.3}$$

Let $0 < q < 1$ be arbitrary. Then there exists $M(q) \in (0, t^*]$ such that for $t \in [-h, M]$ from (4.3) it follows $d_M(\mathfrak{R}G, \mathfrak{R}G^*) \leq q d_M(G, G^*)$ and therefore the operator $\mathfrak{R}$ is contractive in Ω_M^Φ. □

From Theorem 4.1 it follows that any solution of the IVP (3.1), (3.3) is unique on each interval where this solution does exists. Let assume that there exist two solutions X^1, X^2 of the IVP (3.1), (3.3), with intervals of existence J_{M^1} and $J_{M^2}, M^1 < M^2$. Then it is simply to see that $X^1(t) = X^2(t), t \in J_{M^1}$ and therefore $X^2(t)$ is a continuations of $X^1(t)$.

Corollary 4.2. *Let the conditions of Theorem 4.1 hold. Then the IVP* (3.1), (3.3) *has a unique solution* $X \in C_\infty^\alpha$.

Proof. Let $\Phi \in \mathfrak{C}$ be an arbitrary fixed initial function. Then according to Theorem 4.1 there exists $M_0 > 0$ such that the IVP (3.1), (3.3) has a unique solution $X_\Phi^0 \in C_{M_0}^\alpha$.

Without loss of generality we can assume that $0 < M_0 < h$. Let consider an auxiliary IVP for the system (3.6) with the initial condition

$$X(t) = X_\Phi^0(t), t \in [M_0 - h, M_0]. \tag{4.4}$$

Then there exists $M_1 > M_0$ such that the IVP (3.6), (4.4) has a unique solution in the interval $[M_0, M_1]$ which is α-continuous at zero, continuous for $t \in [M_0 - h, M_1] \setminus \{0\}$ and satisfies the initial condition (4.4).

As in the proof of Theorem 4.1, for each $M_1 > M_0$ we define the metric space

$$\Omega_{M_1}^\Phi = \{G : [M_0 - h, M_1] \to \mathbb{R}^n \mid G \in C([M_0, M_1], \mathbb{R}^n), G(t) = X_\Phi^0(t), t \in [M_0 - h, M_0]\}$$

with metric function $d_{M_1} : \Omega_{M_1}^\Phi \times \Omega_{M_1}^\Phi \to \mathbb{R}_+$ with

$$d_{M_1}(G, G^*) = \sup_{t\in[M_0-h,M_1]} \sum_{k=1}^n |t|^{1-\alpha_k} |g_k(t) - g_k^*(t)|.$$

For each $G = (g_1, ..., g_n)^T \in \Omega_{M_1}^\Phi$ define the operator $\mathfrak{R}G(t) = (\mathfrak{R}_1 g_1(t), ..., \mathfrak{R}_n g_n(t))$ with equation (4.1). Further the proof that the operator $\mathfrak{R}$ is contractive when M_1 is small enough is similar to the proof of Theorem 4.1 and will be omitted.

Then using the step method until $M_m < h$ we find a sequence of solutions $X_\Phi^m(t), m = 2, 3, \ldots$ satisfying the system (3.6) for $t \in [M_m, M_{m+1}]$ and the initial condition

$$X(t) = X_\Phi^m(t), t \in [M_m - h, M_m], m = 1, 2, \ldots \tag{4.5}$$

The corresponding metric spaces for each step are defined with

$$\Omega_{M_m}^\Phi = \{G : [M_{m-1} - h, M_m] \to \mathbb{R}^n \mid G \in C([M_{m-1}, M_m], \mathbb{R}^n), G(t) = X_\Phi^{m-1}(t), t \in [M_{m-1} - h, M_{m-1}]\}$$

with metric function

$$d_{M_m}(G, G^*) = \sup_{t\in[M_{m-1}-h,M_m]} \sum_{k=1}^n |t|^{1-\alpha_k} |g_k(t) - g_k^*(t)|.$$

and corresponding operator $\mathfrak{R}G(t) = (\mathfrak{R}_1 g_1(t), ..., \mathfrak{R}_n g_n(t))$ acting in $\Omega_{M_m}^\Phi$ and defined by (4.1).

If there exists a number $m_0 \geq 1$ for which $M_{m_0} \geq h$ then the function $X_\Phi^{m_0}(t)$ will be continuous at teast in the interval $(M_{m_0} - h, M_{m_0}]$. Then the function $X_\Phi^{m_0+1}(t)$ will be continuous in $[M_{m_0}, M_{m_0+1}]$ and can be used as continuous initial function for the IVP (3.6), (4.5) for $m = m_0 + 2$. The existence of a unique solution for the IVP (3.6), (4.5) defined in the interval $[M_{m_0+1} - h, \infty)$ is well known result and thus in this case the statement of Corollary 4.2 is proved.

Let assume that $M_m < h, m = 1, 2, \dots$ and denote $\sup_{m \in \mathbb{N}} M_m = t^{max} \leq h$. If we suppose that $M_{m^*} = h$ for some $m^* \in \mathbb{N}$ then $M_{m^*+1} > t^{max}$ which is impossible.

Let introduce partial ordering by inclusion in the set of the definition intervals of all solutions of the IVP (3.6), (3.5) and the IVP (3.6), (4.4) for each initial interval $[M - h, M], M \in (0, t^{max})$. From Zorn's lemma it follows that there exists a maximal solution $X^{max}(t)$ of the IVP (3.1), (3.3) defined in an open interval $t \in (0, t^{max})$ which is a continuation of all other solutions of the IVP (3.6), (3.5) and the IVP (3.6), (4.4) for each initial interval $[M - h, M], M \in (0, t^{max})$. From Theorem 4.1 and (3.6) passing $t \to t^{max}$ it follows that the equation (3.6) holds for $t = t^{max}$, which is impossible. Thus $t^{max} = \infty$. □

Consider the system (3.1) in the case when the fractional derivatives are in Caputo sense with the initial condition

$$X(t) = \Phi(t), t \in [-h, 0], \Phi \in \mathfrak{C} \tag{4.6}$$

It is not difficult using (jv) to see that if the conditions (S1) - (S5) hold then every solution $X \in C(J_\infty, \mathbb{R}^n)$ of the IVP (3.1), (4.6) (with derivatives in the Caputo sense) is a continuous solution on $\mathbb{R}_+$ of the system

$$\begin{aligned} x_k(t) = \phi_k(0) &+ \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\theta) dv_k^j(t,\theta) + \\ &+ \frac{1}{\Gamma(\alpha_k)} [\int_0^t (t-s)^{\alpha_k - 1} (\sum_{j=a}^{n} \int_{-\sigma}^{0} x_j(s+\theta) d_\theta u_k^j(s,\theta)) ds + \int_0^t (t-s)^{\alpha_k - 1} f_k(s) ds] \end{aligned} \tag{4.7}$$

for $k = 1, 2, \dots, n$ satisfying the initial condition (4.6) for $t \in [-h, 0]$ and vice versa - every continuous solution on $\mathbb{R}_+$ of (4.7) satisfying the initial condition (4.6) for $t \in [-h, 0]$ is a solution of IVP (3.1), (4.6).

Theorem 4.3. *Let the following conditions hold:*

1. *The conditions (S1) - (S5) hold.*
2. *$F \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^n)$ and is locally bounded.*

Then the IVP (3.1), (4.6)*has a unique solution $X \in C(J_\infty, \mathbb{R}^n)$.*

The proof is fully analogical of the proofs of Theorem 4.1 and Corollary 4.2 and will be omitted.

Corollary 4.4. *Let the conditions of Theorem 4.3 hold. Then the necessary and sufficient condition the IVP* (3.1), (3.3) *(with RL derivatives) to have a unique solution $X \in C(J_\infty, \mathbb{R}^n)$ is $\phi_k(0) = 0, 1 \leq k \leq n$.*

Proof. Obviously the systems (3.6) and (4.7) coincides when $\phi_k(0) = 0, 1 \leq k \leq n$ and therefore the statement of Corollary 4.4 follows from Corollary 3.5 and Theorem 4.3. □

5. Stability Analysis of the Autonomous Linear Fractional Differential System

Consider the IVP for the autonomous case of system (3.1)

$$D_{0+}^{\alpha}(X(t) - \int_{-\tau}^{0} [dV(\theta)] X(t+\theta)) = \int_{-\sigma}^{0} [dU(\theta)] X(t+\theta)) \tag{5.1}$$

with the initial conditions (3.3).

Definition 5.1. The zero solution of the system (5.1), $t \in \mathbb{R}_+$ is said to be:

(a) Stable (uniformly) if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for every initial function $\Phi \in \mathfrak{C}$ with $\|\Phi\| < \delta$ the corresponding solution $X(t)$ satisfies for each $t \in \mathbb{R}_+$ the inequality $|X(t)| \leq \epsilon$.

(b) Locally asymptotically stable if there is a $\Delta > 0$ such that for every initial function $\Phi \in \mathfrak{C}$ with $\|\Phi\| < \Delta$ for the corresponding solution $X(t)$ we have that $\lim\limits_{t\to\infty} |X(t)| = 0$.

(c) Globally asymptotically stable (GAS) if for every initial function $\Phi \in \mathfrak{C}$ for the corresponding solution $X(t)$ we have that $\lim\limits_{t\to\infty} |X(t)| = 0$.

For the system (5.1) we call the matrix valued function $G(p)$

$$G(p) = \begin{pmatrix} p^{\alpha_1}[1 - \int\limits_{-\tau}^{0} e^{p\theta} dv_1^1(\theta)] - \int\limits_{-\sigma}^{0} e^{p\theta} du_1^1(\theta) & -[p^{\alpha_1} \int\limits_{-\tau}^{0} e^{p\theta} dv_1^2(\theta) + \int\limits_{-\sigma}^{0} e^{p\theta} du_1^2(\theta)] & \dots & -[p^{\alpha_1} \int\limits_{-\tau}^{0} e^{p\theta} dv_1^n(\theta) + \int\limits_{-\sigma}^{0} e^{p\theta} du_1^n(\theta)] \\ -[p^{\alpha_2} \int\limits_{-\tau}^{0} e^{p\theta} dv_2^1(\theta) + \int\limits_{-\sigma}^{0} e^{p\theta} du_2^1(\theta)] & p^{\alpha_2}[1 - \int\limits_{-\tau}^{0} e^{p\theta} dv_2^2(\theta)] - \int\limits_{-\sigma}^{0} e^{p\theta} du_2^2(\theta) & \dots & -[p^{\alpha_2} \int\limits_{-\tau}^{0} e^{p\theta} dv_2^n(\theta) + \int\limits_{-\sigma}^{0} e^{p\theta} du_2^n(\theta)] \\ \dots & & & \\ -[p^{\alpha_n} \int\limits_{-\tau}^{0} e^{p\theta} dv_n^1(\theta) + \int\limits_{-\sigma}^{0} e^{p\theta} du_n^1(\theta)] & -[p^{\alpha_n} \int\limits_{-\tau}^{0} e^{p\theta} dv_n^2(\theta) + \int\limits_{-\sigma}^{0} e^{p\theta} du_n^2(\theta)] & \dots & p^{\alpha_n}[1 - \int\limits_{-\tau}^{0} e^{p\theta} dv_n^n(\theta)] - \int\limits_{-\sigma}^{0} e^{p\theta} du_n^n(\theta) \end{pmatrix} \tag{5.2}$$

characteristic matrix and the equation

$$det(G(p)) = 0 \tag{5.3}$$

characteristic equation.

The next Theorem 5.2 establishes the dependence of the stability of the zero solution of system (5.1) with derivatives in the Riemann-Liouville sense from the distribution of the roots of the characteristic equation.

Theorem 5.2. *Let the following conditions be fulfilled:*

1. *The conditions (S) hold.*
2. *All roots of the characteristic equation* (5.3) *have negative real parts.*

Then the zero solution of the system (5.1) *with derivatives in the Riemann-Liouville sense is globally asymptotically stable.*

Proof. Let denote with $\hat{X}(p) = (\hat{x_1}(p), ..., \hat{x_n}(p))^T, p \in \mathbb{C}$ the Laplace-transform for the function $X(t)$. Applying the Laplace transform on the left side of (5.1), taking into account (vj), (3.2) and Lemma 3.2 in [5] we obtain for each $1 \leq k \leq n$

$$\begin{aligned} \mathfrak{L}_{RL} D_{0+}^{\alpha_k}[x_k(t) - \sum_{j=1}^{n} \int\limits_{-\tau}^{0} x_j(t+\theta) dv_k^j(\theta)](p) = p^{\alpha_k} \hat{x_k}(p) - p^{\alpha_k} \int\limits_{0}^{\infty} e^{-pt} (\sum_{j=1}^{n} \int\limits_{-\tau}^{0} x_j(t+s) dv_k^j(s)) dt - \\ - \Gamma(\alpha_k)[\phi_k(0) - \lim_{t\to 0+} t^{1-\alpha_k} \sum_{j=1}^{n} a_k^j x_j(t - \tau_k^j) - \lim_{t\to 0+} t^{1-\alpha_k} \sum_{j=1}^{n} \int\limits_{-\tau}^{0} b_k^j(\theta) x_j(t+\theta) d\theta] \end{aligned} \tag{5.4}$$

Taking into account (3.3) and the Fubini-Tonelli theorem we obtain

$$\begin{aligned} \int\limits_{0}^{\infty} e^{-pt} (\sum_{j=1}^{n} \int\limits_{-\tau}^{0} x_j(t+\theta) dv_k^j(\theta)) dt = \sum_{j=1}^{n} \int\limits_{-\tau}^{0} (\int\limits_{0}^{\infty} e^{-pt} x_j(t+\theta) dt) dv_k^j(\theta) = \sum_{j=1}^{n} \int\limits_{-\tau}^{0} (\int\limits_{\theta}^{\infty} e^{-p(s-\theta)} x_j(s) ds) dv_k^j(\theta) = \\ = \sum_{j=1}^{n} \int\limits_{-\tau}^{0} (\int\limits_{\theta}^{0} e^{-p(s-\theta)} D_{0+}^{1-\alpha_j} \phi_j(s) ds) dv_k^j(\theta) + \sum_{j=1}^{n} \int\limits_{-\tau}^{0} e^{p\theta} (\int\limits_{0}^{\infty} e^{-ps} x_j(s) ds) dv_k^j(\theta) = \\ = \sum_{j=1}^{n} \int\limits_{-\tau}^{0} (\int\limits_{\theta}^{0} e^{-p(s-\theta)} D_{0+}^{1-\alpha_j} \phi_j(s) ds) dv_k^j(\theta) + \sum_{j=1}^{n} \hat{x}_j(p) \int\limits_{-\tau}^{0} e^{p\theta} dv_k^j(\theta) \end{aligned} \tag{5.5}$$

From conditions (S) and condition 3 of the Lemma 3.3 it follows that

$$\lim_{t\to 0+} t^{1-\alpha_k}\sum_{j=1}^{n} a_k^j x_j(t-\tau_k^j) - \lim_{t\to 0+} t^{1-\alpha_k}\sum_{j=1}^{n}\int_{-\tau}^{0} b_k^j(\theta)x_j(t+\theta)d\theta = 0 \tag{5.6}$$

Applying the Laplace transform on the right side of (5.1), the same way as in (5.3) we obtain that

$$\sum_{j=1}^{n}\int_{-\sigma}^{0}(\int_{0}^{\infty} e^{-pt}x_j(t+\theta)dt)dv_k^j(\theta) = \sum_{j=1}^{n}\int_{-\sigma}^{0}(\int_{\theta}^{0} e^{-p(s-\theta)}D_{0+}^{1-\alpha_j}\phi_j(s)ds)du_k^j(s) + \sum_{j=1}^{n}\hat{x}_j(p)\int_{-\sigma}^{0} e^{ps}du_k^j(s). \tag{5.7}$$

Then from (5.1) taking into account (5.4) - (5.7) we obtain the system

$$G(p)\hat{X}(p) = r(p) \tag{5.8}$$

where $G(p)$ is the characteristic matrix given with (5.2), $r(p) = (r_1(p),\dots,r_n(p))^T$, and for each $1 \le k \le n$, we have

$$r_k(p) = p^{\alpha_k}\sum_{j=1}^{n}\int_{-\tau}^{0}(\int_{\theta}^{0} e^{-p(s-\theta)}D_{0+}^{1-\alpha_j}\phi_j(s)ds)dv_k^j(\theta) + \sum_{j=1}^{n}\int_{-\sigma}^{0}(\int_{\theta}^{0} e^{-p(s-\theta)}D_{0+}^{1-\alpha_j}\phi_j(s)ds)du_k^j(\theta) + \phi_k(0)\Gamma(\alpha_k).$$

Multiplying both sides of (5.8) with $p \in \mathbb{C}$ we obtain that

$$G(p)p\hat{X}(p) = pb(p) \tag{5.9}$$

Condition 2 of the theorem implies that for $Rep \ge 0$ the function $p\hat{X}(p)$ is a unique solution of the system (5.9). The final value theorem of the Laplace transform [3] implies that for each $k, 1 \le k \le n$ we have that $\lim_{t\to\infty} x_k(t) = \lim_{p\to 0, Rep\ge 0} p\hat{x}_k(p) = 0$ □

Remark 5.3. According the knowledge of the authors the result of Theorem 5.2 is new even in the important particular case of finite number constant delays.

Consider the system (5.1) in the case when $D_{0+}^{\alpha_k}$ are fractional derivatives in the Caputo sense ${}_C D_{0+}^{\alpha_k}$, $\alpha_k \in (0,1), 1 \le k \le n$, with the initial condition (3.9). It is not difficult to see that if the conditions of Theorem 5.2 hold, then the statements of the theorem are still true.

Theorem 5.4. *Let the following conditions be fulfilled:*

1. *The conditions (S1) - (S5) hold.*
2. *All roots of the characteristic equation* (5.3) *have negative real parts.*

Then the zero solution of system (5.1) *with derivatives in Caputo sense is globally asymptotically stable.*

Proof. Indeed as in the proof of Theorem 4.3 applying the Laplace transform on the both sides of (5.1), taking into account (vjj), (3.9) we receive that

$$\mathfrak{L}D_{0+}^{\alpha_k}[x_k(t) - \sum_{j=1}^{n}\int_{-\tau}^{0} x_j(t+\theta)dv_k^j(\theta)](p) =$$

$$= p^{\alpha_k}\hat{x}_k(p) - p^{\alpha_k}\int_{0}^{\infty} e^{-pt}(\sum_{j=1}^{n}\int_{-\tau}^{0} x_j(t+\theta)dv_k^j(\theta))dt - p^{\alpha_k-1}[\phi_k(0) - \sum_{j=1}^{n}\int_{-\tau}^{0} x_j(\theta)dv_k^j(\theta)]$$

and then in the same way as in Theorem 5.2 from (5.1) we obtain the system

$$G(p)\hat{X}(p) = r^*(p), \tag{5.10}$$

where the matrix $G(p)$ is the characteristic matrix given with (5.2), $r^*(p) = (r_1^*(p), \dots, r_n^*(p))^T$,

$$r_k^*(p) = p^{\alpha_k} \sum_{j=1}^{n} \int_{-\tau}^{0} (\int_{\theta}^{0} e^{-p(s-\theta)} \phi_j(s) ds) dv_k^j(\theta) + \sum_{j=1}^{n} \int_{-\sigma}^{0} (\int_{\theta}^{0} e^{-p(s-\theta)} \phi_j(s) ds) du_k^j(\theta) + p^{\alpha_k - 1}[\phi_k(0) - \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(\theta) dv_k^j(\theta)]$$

for each k, $1 \le k \le n$. Further the proof is similar as the proof of Theorem 5.2. □

Remark 5.5. Note that the conditions (S) are ultimately fulfilled in the case considered in [12], when the kernels have the form $V(\theta) = \{v_j^i(\theta)\}_{i,j=1}^n = \{a_{ij} H(\theta - \tau_{ij})\}_{i,j=1}^n$ and $U(\theta) = \{u_j^i(\theta)\}_{i,j=1}^n = \{c_{ij} H(\theta - \sigma_{ij})\}_{i,j=1}^n$, $a_{ij}, c_{ij} \in \mathbb{R}$, $\sigma_{ij} \in [0, \sigma]$, $\tau_{ij} \in [0, \tau]$, $1 \le i, j \le n$, i.e. the kernels are autonomous and the Lebesgue decompositions have no absolutely continuous and singular part and include only finite number of jumps.

Hence the statement of Theorem 5.4 generalized the result obtained in [12] for neutral systems with left side derivatives in the Caputo sense in the particular case of finite number constant delays.

Moreover, if $U(\theta)$ has no jumps and singular part, then we have stability criteria for linear system of fractional integro-differential equations.

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