Filomat 30:4 (2016), 905–912
DOI 10.2298/FIL1604905K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Identities Relating to Degenerate Bernoulli Polynomials

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Abstract. Recently, Carlitz degenerate Bernoulli numbers and polynomials have been studied by several authors (see [3, 4]). In this paper, we consider new degenerate Bernoulli numbers and polynomials, different from Carlitz degenerate Bernoulli numbers and polynomials, and give some formulae and identities related to these numbers and polynomials.

1. Introduction

The ordinary Bernoulli numbers are defined by

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1}$$

with the usual convention about replacing B^n by B_n.

The Bernoulli polynomials are defined by

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad \text{(see [1–20])}. \tag{2}$$

From (1) and (2), we note that

$$\begin{aligned} &\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \\ &= \frac{t}{e^t - 1} e^{xt} \\ &= \left(\frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} e^{at} \right) e^{xt} \end{aligned} \tag{3}$$

2010 *Mathematics Subject Classification*. 11B75, 11B83, 11S80
Keywords. Degenerate Bernoulli numbers and polynomials
Received: 18 June 2015; Accepted: 16 September 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
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$$= \sum_{n=0}^{\infty} \left(d^{n-1} \sum_{a=0}^{d-1} B_n \left(\frac{a+x}{d} \right) \right) \frac{t^n}{n!}.$$

Thus, by (3), we get

$$B_n(x) = d^{n-1} \sum_{a=0}^{d-1} B_n \left(\frac{a+x}{d} \right), \tag{4}$$

where $n \in \mathbb{N} \cup \{0\}$ and $d \in \mathbb{N}$.

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$. The generalized Bernoulli numbers are defined by

$$B_{n,\chi} = d^{n-1} \sum_{a=0}^{d-1} \chi(a) B_n \left(\frac{a}{d} \right), \quad (n \geq 0), \quad \text{(see [12, 18, 20])}. \tag{5}$$

Carlitz introduced the degenerate Bernoulli polynomials given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta(x \mid \lambda) \frac{t^n}{n!}, \quad \text{(see [3, 4])}. \tag{6}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0 \mid \lambda)$ are called the degenerate Bernoulli numbers.

From (6), we note that

$$\lim_{\lambda \to 0} \beta_n(x \mid \lambda) = B_n(x), \quad (n \geq 0). \tag{7}$$

In this paper, we consider new degenerate Bernoulli numbers and polynomials, different from Carlitz degenerate Bernoulli numbers and polynomials, and give some formulae and identities related to these numbers and polynomials.

2. Degenerate Bernoulli Polynomials

Let us consider the new degenerate Bernoulli polynomials as follows:

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{8}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. Note that $\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x)$. From (8), we have

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{1}{\lambda}} - \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} = \frac{\log(1+\lambda t)}{\lambda}. \tag{9}$$

We observe that

$$\frac{1}{\lambda} \log(1+\lambda t) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n+1} t^{n+1}. \tag{10}$$

Thus, by (8), (9) and (10), we get

$$\beta_{n,\lambda}(1) - \beta_{n,\lambda} = \begin{cases} 0 & \text{if } n = 0, \\ (-\lambda)^{n-1}(n-1)! & \text{if } n \geq 1, \end{cases} \quad \beta_{0,\lambda} = 1. \tag{11}$$

From (8), we note that

$$\begin{aligned}
&\log\left(1+\lambda t\right)^{\frac{1}{\lambda}}\left(1+\lambda t\right)^{\frac{x}{\lambda}} \\
&=\left(\left(1+\lambda t\right)^{\frac{1}{\lambda}}-1\right)\left(\sum_{m=0}^{\infty}\beta_{m,\lambda}\left(x\right)\frac{t^m}{m!}\right) \\
&=t\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(1\mid\lambda)_{l+1}}{l+1}\beta_{n-l,\lambda}\left(x\right)\binom{n}{l}\right)\frac{t^n}{n!},
\end{aligned}\tag{12}$$

where

$$(x\mid\lambda)_n = x\left(x-\lambda\right)\cdots\left(x-\lambda\left(n-1\right)\right).$$

It is known that Daehee numbers are given by the generating function

$$\frac{\log\left(1+t\right)}{t}=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}.$$

Now, we observe that

$$\begin{aligned}
&\log\left(1+\lambda t\right)^{\frac{1}{\lambda}}\left(1+\lambda t\right)^{\frac{x}{\lambda}} \\
&=\left(\frac{\log\left(1+\lambda t\right)}{\lambda t}\right)\left(t\left(1+\lambda t\right)^{\frac{x}{\lambda}}\right) \\
&=t\left(\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}D_l\lambda^l\left(x\mid\lambda\right)_{n-l}\right)\frac{t^n}{n!}\right).
\end{aligned}\tag{13}$$

Thus, by (12) and (13), we get

$$\sum_{l=0}^{n}\binom{n}{l}\frac{(1\mid\lambda)_{l+1}}{l+1}\beta_{n-l,\lambda}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}D_l\lambda^l\left(x\mid\lambda\right)_{n-l}.\tag{14}$$

By (8), we easily get

$$\beta_{n,\lambda}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}\left(x\mid\lambda\right)_{n-l},\quad (n\geq 0).\tag{15}$$

Therefore, by (14) and (15), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\frac{(1\mid\lambda)_{l+1}}{l+1}\beta_{n-l,\lambda}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}D_l\lambda^l\left(x\mid\lambda\right)_{n-l},$$

and

$$\beta_{n,\lambda}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}\left(x\mid\lambda\right)_{n-l}.$$

Moreover,

$$\beta_{n,\lambda}\left(1\right)-\beta_{n,\lambda}=\begin{cases}0, & \text{if } n=0,\\ \left(-\lambda\right)^{n-1}\left(n-1\right)! & \text{if } n\geq 1,\quad \beta_{0,\lambda}=1.\end{cases}$$

By (8), we get

$$\begin{aligned}
&\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&=\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{a+x}{\lambda}} \\
&=\frac{1}{d}\left(\frac{\log(1+\lambda t)^{\frac{d}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{a+x}{\lambda}}\right) \\
&=\frac{1}{d}\sum_{a=0}^{d-1}\left(\sum_{n=0}^{\infty}\beta_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right)d^n\frac{t^n}{n!}\right) \\
&=\sum_{n=0}^{\infty}\left\{d^{n-1}\sum_{a=0}^{d-1}\beta_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{16}$$

Thus, by (16), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}(x)=d^{n-1}\sum_{a=0}^{d-1}\beta_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right).$$

It is not difficult to show that

$$\begin{aligned}
&\frac{\log(1+\lambda t)}{\lambda}\sum_{l=0}^{n-1}(1+\lambda t)^{\frac{l}{\lambda}} \\
&=\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{n}{\lambda}}-\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \\
&=\sum_{m=0}^{\infty}\{\beta_{m,\lambda}(n)-\beta_{m,\lambda}\}\frac{t^m}{m!} \\
&=t\sum_{m=0}^{\infty}\left\{\frac{\beta_{m+1,\lambda}(n)-\beta_{m+1,\lambda}}{m+1}\right\}\frac{t^m}{m!}.
\end{aligned} \tag{17}$$

Thus we get

$$\begin{aligned}
&\frac{\log(1+\lambda t)}{\lambda}\sum_{l=0}^{n-1}(1+\lambda t)^{\frac{l}{\lambda}} \\
&=t\left(\frac{\log(1+\lambda t)}{\lambda t}\right)\left(\sum_{l=0}^{n-1}(1+\lambda t)^{\frac{l}{\lambda}}\right) \\
&=\left(t\sum_{k=0}^{\infty}\frac{D_k\lambda^k}{k!}t^k\right)\left(\sum_{m=0}^{\infty}\left(\sum_{l=0}^{n-1}(l\mid\lambda)_m\right)\frac{t^m}{m!}\right) \\
&=t\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}D_i\lambda^i\sum_{l=0}^{n-1}(l\mid\lambda)_{k-i}\right)\frac{t^k}{k!}.
\end{aligned} \tag{18}$$

From (17) and (18), we have

$$\frac{\beta_{k+1,\lambda}(n)-\beta_{k+1,\lambda}}{k+1}=\sum_{l=0}^{n-1}\left(\sum_{i=0}^{k}\binom{k}{i}D_i\lambda^i(l\mid\lambda)_{k-i}\right). \tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 2.3. *For $n\geq 1$ and $k\geq 0$, we have*

$$\frac{1}{k+1}\left\{\beta_{k+1,\lambda}(n)-\beta_{k+1,\lambda}\right\}=\sum_{l=0}^{n-1}\left(\sum_{i=0}^{k}\binom{k}{i}D_i\lambda^i(l\mid\lambda)_{k-i}\right).$$

Replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (3), we get

$$\begin{aligned}&\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &=\sum_{n=0}^{\infty}B_n(x)\lambda^{-n}\frac{1}{n!}(\log(1+\lambda t))^n \\ &=\sum_{m=0}^{\infty}B_m(x)\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!} \\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m(x)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{20}$$

where $S_1(n,m)$ is the Stirling number of the first kind.

On the other hand,

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{21}$$

Therefore, by (20) and (21), we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, we have*

$$\beta_{n,\lambda}(x)=\sum_{m=0}^{n}B_m(x)\lambda^{n-m}S_1(n,m).$$

Replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (8), we have

$$\begin{aligned}\frac{t}{e^t-1}e^{xt}&=\sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{1}{m!}\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^m \\ &=\sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!} \\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta_{m,\lambda}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{22}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

Thus, by (22), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$B_n(x) = \sum_{m=0}^{n} \beta_{m,\lambda}(x)\lambda^{n-m} S_2(n,m).$$

For $d \in \mathbb{N}$, let χ be a Dirichlet character with conductor d. Then, we define the generalized degenerate Bernoulli numbers attached to χ:

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}\chi(a)(1+\lambda t)^{\frac{a}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,\chi,\lambda}\frac{t^n}{n!}. \tag{23}$$

From (8) and (23), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_{n,\chi,\lambda}\frac{t^n}{n!} &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}\chi(a)(1+\lambda t)^{\frac{a}{\lambda}} \\ &= \frac{1}{d}\sum_{a=0}^{d-1}\chi(a)\frac{\log(1+\lambda t)^{\frac{d}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}(1+\lambda t)^{\frac{a}{\lambda}} \\ &= \frac{1}{d}\sum_{a=0}^{d-1}\chi(a)\sum_{n=0}^{\infty}\beta_{n,\frac{\lambda}{d}}\left(\frac{a}{d}\right)\frac{d^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(d^{n-1}\sum_{a=0}^{d-1}\chi(a)\beta_{n,\frac{\lambda}{d}}\left(\frac{a}{d}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{24}$$

Therefore, by (24), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, $d \in \mathbb{N}$, we have*

$$\beta_{n,\chi,\lambda} = d^{n-1}\sum_{a=0}^{d-1}\chi(a)\beta_{n,\frac{\lambda}{d}}\left(\frac{a}{d}\right).$$

3. Further Remark

Let p be a fixed prime number. Throughout this section, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. In Section 2, we introduced the degenerate Bernoulli polynomials given by the generating function

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}.$$

Let d be a positive integer with $(d,p)=1$. Then we set

$$\begin{aligned} X &= \varprojlim_{N}\left(\mathbb{Z}/dp^N\mathbb{Z}\right); \\ a + dp^N\mathbb{Z}_p &= \left\{x \in X \mid x \equiv a \pmod{dp^N}\right\}; \\ X^* &= \bigcup_{\substack{0<a<dp \\ p\nmid a}}\left(a + dp\mathbb{Z}_p\right). \end{aligned}$$

We shall usually take $0 \leq a < dp^N$ when we write $a + dp^N\mathbb{Z}_p$. Now, we will use Theorem 2.2 to prove a p-adic distribution result.

Theorem 3.1. *For $k \geq 0$, let $\mu_{k,\beta}$ be defined by*

$$\mu_{k,\beta}^{(\lambda)}\left(a + dp^N\mathbb{Z}_p\right) = \left(dp^N\right)^{k-1} \beta_{k,\frac{\lambda}{dp^N}}\left(\frac{a}{dp^N}\right). \tag{25}$$

Then $\mu_{k,\beta}^{(\lambda)}$ extends to a $\mathbb{C}_p$-valued distribution on compact open sets $U \subset X$.

Proof. It suffices to show that

$$\begin{aligned}
&\sum_{i=0}^{p-1} \mu_{k,\beta}^{(\lambda)}\left(a + idp^N + dp^{N+1}\mathbb{Z}_p\right) \\
&= \left(dp^{N+1}\right)^{k-1} \sum_{i=0}^{p-1} \beta_{k,\frac{\lambda}{dp^{N+1}}}\left(\frac{a + idp^N}{dp^{N+1}}\right) \\
&= \left(dp^N\right)^{k-1} p^{k-1} \sum_{i=0}^{p-1} \beta_{k,\frac{\frac{\lambda}{dp^N}}{p}}\left(\frac{\frac{a}{dp^N} + i}{p}\right) \\
&= \left(dp^N\right)^{k-1} \beta_{k,\frac{\lambda}{dp^N}}\left(\frac{a}{dp^N}\right) \\
&= \mu_{k,\beta}^{(\lambda)}\left(a + dp^N\mathbb{Z}_p\right).
\end{aligned}$$

□

The locally constant function χ can be integrated against the distribution $\mu_{k,\beta}$ defined by (25), and the result is

$$\begin{aligned}
&\int_X \chi(x)\, d\mu_{k,\beta}^{(\lambda)}(x) \qquad (26) \\
&= \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} \chi(x)\, \mu_{k,\beta}^{(\lambda)}\left(x + dp^N\mathbb{Z}_p\right) \\
&= \lim_{N\to\infty} \left(dp^N\right)^{k-1} \sum_{x=0}^{dp^N-1} \chi(x)\, \beta_{k,\frac{\lambda}{dp^N}}\left(\frac{x}{dp^N}\right) \\
&= \beta_{k,\chi,\lambda}.
\end{aligned}$$

From (26), we have

$$\int_X \chi(x)\, d\mu_{k,\beta}(\lambda)(x) = \beta_{k,\chi,\lambda}, \quad (k \geq 0).$$

References

[1] Açikgöz, M., Erdal, D., and Araci, S., A new approach to q-Bernoulli numbers and q-Bernoulli polynomials related to q-Bernstein polynomials, *Adv. Difference Equ.* (2010), Art. ID 951764, 9.

[2] Bayad, A. and Kim, T., Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials, *Russ. J. Math. Phys.* **18** (2011), no. 2, 133–143.

[3] Carlitz, L., A degenerate Staudt-Clausen theorem, *Arch. Math. (Basel)* **7** (1956), 28–33.

[4] ______, *Degenerate Stirling, Bernoulli and Eulerian numbers*, *Utilitas Math.* **15** (1979), 51–88.

[5] Dere, R. and Simsek, Y., Applications of umbral algebra to some special polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)* **22** (2012), no. 3, 433–438.

[6] Ding, D. and Yang, J., Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)* **20** (2010), no. 1, 7–21.

[7] Dolgy, D. V., Kim, D. S., Kim, T., and Mansour, T., Barnes-type degenerate Euler polynomials, *Appl. Math. Comput.* **261** (2015), 388–396.

[8] He, Y. and Zhang, W., A convolution formula for Bernoulli polynomials, *Ars Combin.* **108** (2013), 97–104.

[9] Kim, T., An analogue of Bernoulli numbers and their congruences, *Rep. Fac. Sci. Engrg. Saga Univ. Math.* **22** (1994), no. 2, 21–26.

[10] ———, q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients, *Russ. J. Math. Phys.* **15** (2008), no. 1, 51–57.

[11] ———, Barnes' type multiple degenerate Bernoulli and Euler polynomials, *Appl. Math. Comput.* **258** (2015), 556–564.

[12] Kim, T. and Adiga, C., Sums of products of generalized Bernoulli numbers, *Int. Math. J.* **5** (2004), no. 1, 1–7.

[13] Kudo, A., A congruence of generalized Bernoulli number for the character of the first kind, *Adv. Stud. Contemp. Math. (Pusan)* **2** (2000), 1–8.

[14] Lim, D. and Do, Y., Some identities of Barnes-type special polynomials, *Adv. Difference Equ.* (2015), 2015:42, 12.

[15] Luo, Q.-M. and Qi, F., Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)* **7** (2003), no. 1, 11–18.

[16] Ozden, H., p-adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Comput.* **218** (2011), no. 3, 970–973.

[17] Shiratani, K., Kummer's congruence for generalized Bernoulli numbers and its application, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **26** (1972), 119–138.

[18] Simsek, Y., Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions, *Adv. Stud. Contemp. Math. (Kyungshang)* **16** (2008), no. 2, 251–278.

[19] Wang, N. L., Some identities involving generalized Bernoulli numbers, *J. Inn. Mong. Norm. Univ. Nat. Sci.* **43** (2014), no. 4, 403–407.

[20] Washington, L. C., Introduction to cyclotomic fields, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997.