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On A Characterization of Compactness and the Abel-Poisson Summability of Fourier Coefficients In Banach Spaces

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Abstract. In this paper, for an isometric strongly continuous linear representation denoted by α of the topological group of the unit circle in complex Banach space, we study an integral representation for Abel-Poisson mean $A_r^\alpha(x)$ of the Fourier coefficients family of an element x, and it is proved that this family is Abel-Poisson summable to x. Finally, we give some tests which are related to characterizations of relatively compactness of a subset by means of Abel-Poisson operator A_r^α and α.

1. Introduction

Let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the topological group of the unit circle with Euclidean topology and multiplication operation, H be a complex Banach space, α be an isometric strongly continuous linear representation of T in H, $x \in H$ and $\{F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ be the family of Fourier coefficients of x with respect to α.

This paper is organized as follows. In Section 2 and 3, we provide some necessary preliminaries which play an important role for this work. In Section 4, we obtain an integral representation for the r^{th} Abel-Poisson mean $A_r^\alpha(x)$ of the family $\{F_n^\alpha(x)\}$, and using this integral representation, we prove that the family $\{F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ is Abel-Poisson summable to $x \in H$. As it is known that there are many characterizations of compactness in metric spaces, especially normed spaces by sequences in literature. We focus on the family $\{F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ and give some relatively compactness tests for a subset $S \subset H$ in terms of the r^{th} Abel-Poisson operator A_r^α and α.

2. Preliminaries

Let I be a nonempty arbitrary index set and let $\{x_n\}_{n\in I}$ be an indexed family of vectors in H. The summability, absolutely summability of this family and its sum denoted by $x := \sum\limits_{n\in I} x_n$ are of the sense given in ([1],p.218-233;[10],p.340-348).

Definition 2.1. *Let $a, b \in \mathbb{R}$, an indexed family of functions $\{f_n\}_{n\in I}$ defined on $[a, b]$ with values in H and f be a function from $[a, b]$ to H.*

(i) The family $\{f_n\}_{n\in I}$ is said to be pointwise summable on $[a, b]$ if the family $\{f_n(t)\}_{n\in I}$ is summable for each

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$t \in [a,b]$.

(ii) The family $\{f_n\}_{n\in I}$ *is said to be uniformly summable with sum* f *on* $[a,b]$ *if for every* $\varepsilon > 0$ *there exists a finite subset* $I_\varepsilon \subset I$ *such that for every finite subset* F *with* $I_\varepsilon \subset F \subset I$ *and* $\forall t \in [a,b]$, $\|f(t) - \sum\limits_{n\in F} f_n(t)\| < \varepsilon$.

It is clear that if the family $\{f_n\}_{n\in I}$ *is uniformly summable, then it is pointwise summable and* $f(t) = \sum\limits_{n\in I} f_n(t)$ *for every* $t \in [a,b]$.

Proposition 2.1. *Let* $a,b \in \mathbb{R}$ *and* $\{f_n\}_{n\in I}$ *be an indexed family of functions defined on* $[a,b]$ *with values in* H. *If there exists a non-negative summable family* $\{a_n\}_{n\in I} \subset \mathbb{R}$ *such that* $\|f_n(t)\| \le a_n$ *for* $\forall n \in I$ *and* $\forall t \in [a,b]$, *then the family* $\{f_n\}_{n\in I}$ *is uniformly summable.*

Proof. It is easily seen from Proposition 29.18 in [1]and Theorem 5.27 in [10]. □

Proposition 2.2. *Let* $a,b \in \mathbb{R}$, $\{f_n\}_{n\in I}$ *be a uniformly summable indexed family of functions defined on* $[a,b]$ *with values in* H *and* $f = \sum\limits_{n\in I} f_n$. *If* f_n *is continuous for every* $n \in I$, *then* f *is continuous on* $[a,b]$.

Proof. Since the family $\{f_n\}_{n\in I}$ is uniformly summable on $[a,b]$ with sum f, for every $\varepsilon > 0$ there exists a finite subset $I_\varepsilon \subset I$ such that $\|f(t) - \sum\limits_{n\in I_\varepsilon} f_n(t)\| < \dfrac{\varepsilon}{3}$ for all $t \in [a,b]$. Let $t_o \in [a,b]$ be an arbitrary fixed point. Since the finite sum $\sum\limits_{n\in I_\varepsilon} f_n$ is continuous at the point $t_o \in [a,b]$, there exists a $\delta(t_o,\varepsilon) > 0$ such that for $\forall t, 0 \le |t - t_o| < \delta(t_o,\varepsilon)$, we have $\|\sum\limits_{n\in I_\varepsilon} f_n(t) - \sum\limits_{n\in I_\varepsilon} f_n(t_o)\| < \dfrac{\varepsilon}{3}$. Hence for $\forall t, 0 \le |t - t_o| < \delta(t_o,\varepsilon)$, $\|f(t) - f(t_o)\| \le \|f(t) - \sum\limits_{n\in I_\varepsilon} f_n(t)\| + \|\sum\limits_{n\in I_\varepsilon} f_n(t) - \sum\limits_{n\in I_\varepsilon} f_n(t_o)\| + \|\sum\limits_{n\in I_\varepsilon} f_n(t_o) - f(t_o)\| < \varepsilon$. So, f is continuous on $[a,b]$. □

Proposition 2.3. *Let* $a,b \in \mathbb{R}$, $\{f_n\}_{n\in I}$ *be a uniformly summable indexed family of functions defined on* $[a,b]$ *with values in* H *and* $f := \sum\limits_{n\in I} f_n$. *If* f_n *is continuous on* $[a,b]$ *for every* $n \in I$, *then the family* $\{\int\limits_a^b f_n(t)dt\}_{n\in I}$ *is summable and* $\sum\limits_{n\in I} \int\limits_a^b f_n(t)dt = \int\limits_a^b f(t)dt$.

Proof. Since $\{f_n\}_{n\in I}$ is uniformly summable on $[a,b]$ with sum f, for every $\varepsilon > 0$ there exists a finite subset $I_\varepsilon \subset I$ such that $\|f(t) - \sum\limits_{n\in F} f_n(t)\| < \dfrac{\varepsilon}{b-a}$ for every finite subset F with $I_\varepsilon \subset F \subset I$ and for all $t \in [a,b]$. From the Proposition 2.2, f is continuous on $[a,b]$, so $f - \sum\limits_{n\in F} f_n$ is continuous. Then, $f - \sum\limits_{n\in F} f_n$ and f are integrable functions on $[a,b]$ for every finite subset F with $I_\varepsilon \subset F \subset I$, hence by Theorem 3.3.5 in ([4],p.96-97), we get $\|\int\limits_a^b f(t)dt - \sum\limits_{n\in F} \int\limits_a^b f_n(t)dt\| = \|\int\limits_a^b (f(t) - \sum\limits_{n\in F} f_n(t))dt\| \le \int\limits_a^b \|f(t) - \sum\limits_{n\in F} f_n(t)\|dt < \varepsilon$. □

Remark 2.1. *Propositions 2.2 and 2.3 are generalizations of two theorems given in ([3],p.240).*

Now let us consider an indexed family $\{x_n\}_{n\in\mathbb{Z}}$ of vectors in H, where $\mathbb{Z}$ is the set of all integer numbers. For an integer $n \ge 0$ and every r with $0 \le r < 1$, let us set $S_n := \sum\limits_{k=-n}^{n} x_k$, $\sigma_n := \dfrac{S_0 + \cdots + S_n}{n+1}$ and $A_r := \sum\limits_{k\in\mathbb{Z}} r^{|k|} x_k$ if $\{r^{|k|} x_k\}_{k\in\mathbb{Z}}$ is summable for $0 \le r < 1$. We call σ_n and A_r to be the n^{th} Cesàro mean, and the r^{th} Abel-Poisson mean of the indexed family $\{x_n\}_{n\in\mathbb{Z}}$, respectively. Following ([12],p.53,54;[13],p.20,153), let us give a definition:

Definition 2.2. *Let* $\{x_n\}_{n\in\mathbb{Z}}$ *be an indexed family in* H.

(i) $\{x_n\}_{n\in\mathbb{Z}}$ *is said to be summable in the sense of Cesaro with the sum* s *if the limit* $\lim\limits_{n\to\infty} \sigma_n$ *exists and say* s.

(ii) $\{x_n\}_{n\in\mathbb{Z}}$ *is said to be summable in the sense of Abel-Poisson if* $\{r^{|k|} x_k\}_{k\in\mathbb{Z}}$ *is summable for every* $0 \le r < 1$ *with sum* A_r *and* $\lim\limits_{r\to 1^-} A_r$ *exists. The limit* $\lim\limits_{r\to 1^-} A_r$ *is called Abel-Poisson sum of* $\{x_n\}_{n\in\mathbb{Z}}$.

We shall use the following notations. Let $GL(H)$ be the group of all invertible bounded linear operators from H to itself, $T := \{e^{it} : -\pi \leq t < \pi\}$ be the topological group of the unit circle. Let us define the function $\varphi : \mathbb{R} \to T$, $\varphi(t) := e^{it}$. This function is a surjective group homomorphism and with kernel $2\pi\mathbb{Z}$. By the first isomorphism theorem we have that $T \cong \mathbb{R}/2\pi\mathbb{Z}$. Further, functions on T naturally identified with 2π-periodic functions on $\mathbb{R}$.

The following definitions are given in [11].

Definition 2.3. *A group homomorphism $\alpha : T \to GL(H)$ is called a linear representation of T in H.*

Definition 2.4. *Let α be a linear representation of T in H. Then,*

(i) α is said to be an isometric linear representation of T in H if $\|\alpha(t)(x)\| = \|x\|$ for all $x \in H$ and $t \in T$.

(ii) α is said to be a bounded linear representation of T in H if there exists an M such that $\|\alpha(t)\| \leq M$ for every $t \in T$.

(iii) α is called a strongly continuous linear representation of T in H if $\lim_{t\to 0} \alpha(t)(x) = x$ for all $x \in H$.

It is easily proved that if α is a strongly continuous linear representation of T in H, then the orbit maps $\alpha_x : T \to H, \alpha_x(t) := \alpha(t)(x)$ for all $x \in H$ are continuous on T. Hence, because of the compactness of T, there exists an $M_x > 0$ for $\forall x \in H$ such that $\|\alpha(t)(x)\| \leq M_x$. This shows that the family $\{\alpha(t)\}_{t\in T}$ of operators is pointwise bounded. By Banach-Steinhaus Theorem it is uniformly bounded. Furthermore, by corollary given in ([11],p.82) there exists an equivalent norm $\| \; \|_\alpha$ to the norm $\| \; \|$ in H relative to α which is an isometric strongly continuous linear representation. Then, in sequel we consider only an isometric strongly continuous linear representation of T in H. We write an isometric strongly continuous representation instead of isometric strongly continuous linear representation.

Let α be an isometric strongly continuous representation of T in H and $x \in H$. Then, since the function $e^{-int}\alpha(t)(x)$ is continuous on T for every $n \in \mathbb{Z}$, the vector valued integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\alpha(t)(x)dt$ exists ([4],p.93).

Definition 2.5. *Let α be an isometric strongly continuous representation of T in $H, n \in \mathbb{Z}$ and $x \in H$. Then, $F_n^\alpha(x)$ defined by $F_n^\alpha(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\alpha(t)(x)dt$ is called the n^{th} Fourier coefficient of x with respect to α. ([11],p.12)*

Note that, since α is an isometric strongly continuous representation, $F_n^\alpha : H \to H$ is a bounded linear operator and $\|F_n^\alpha(x)\| \leq \|x\|$ for every $n \in \mathbb{Z}$, and $x \in H$ ([8], Proposition 2).

In [6–8] it is proved that the family $\{F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ is summable with sum x in sense of Cesàro. In this work, we shall prove directly that the family $\{F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ is summable with sum x in sense of Abel-Poisson.

3. Poisson Kernel

In this section we remind that Poisson Kernel being a vital tool for our main results and give its fundamental properties. By Corollary 29.19 in [1] and Proposition 2.1, it is proved that $\sum_{n\in\mathbb{Z}} r^{|n|}e^{int}$ is uniformly summable on T for every $0 \leq r < 1$ with the sum $P_r(t) = \frac{1 - r^2}{1 - 2rcost + r^2}$,where $P_r(t)$ is called Poisson Kernel and it has the following nice properties.

Theorem 3.1. *([2],p.256,257) The Poisson Kernel satisfies the following:*

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt = 1$ for all $r \in [0, 1)$;

(ii) $P_r(t) > 0$ for all $t, P_r(t) = P_r(-t)$ and $P_r(t)$ is periodic in t with period 2π;

(iii) $P_r(t) < P_r(\delta)$ if $0 < \delta < |t| \leq \pi, 0 \leq r < 1$;

(iv) for each $\delta > 0$, $\lim_{r\to 1^-} P_r(t) = 0$ uniformly in t for $0 < \delta < |t| \leq \pi$.

4. Main Results

Theorem 4.1. *Let α be an isometric strongly continuous representation of T in H and $x \in H$. Then, the indexed family $\{r^{|n|}F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ is summable for every $0 \le r < 1$, and its sum denoted by $A_r^\alpha(x)$, it has the following integral representation* $A_r^\alpha(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\alpha(t)(x)dt$.

Proof. Since the series $\sum_{n=0}^{\infty} r^n$ and $\sum_{n=-\infty}^{-1} r^{-n}$ are convergent for every $0 \le r < 1$ and $\|r^{|n|}F_n^\alpha(x)\| \le r^{|n|}\|x\|$ for each $n \in \mathbb{Z}, 0 \le r < 1$; Corollaries 29.8,29.13,29.18 and 29.19 given in ([1],ch.29) imply that the indexed family $\{r^{|n|}F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ is uniformly and absolutely summable. Let $A_r^\alpha(x) := \sum_{n\in\mathbb{Z}} r^{|n|}F_n^\alpha(x)$ for every $0 \le r < 1$. Since α is an isometric strongly continuous representation, we have $\|e^{-int}r^{|n|}\alpha(t)(x)\| \le r^{|n|}\|x\|$ for all $n \in \mathbb{Z}$, $0 \le r < 1$ and $x \in H$. Hence the same Corollaries above and Proposition 2.1 show that indexed function family $\{e^{-int}r^{|n|}\alpha(t)(x)\}_{n\in\mathbb{Z}}$ is uniformly summable on T, and by Proposition 2.3, we get $\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} r^{|n|}e^{-int}\alpha(t)(x)dt = \sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi} r^{|n|}e^{-int}\alpha(t)(x)dt$. Therefore, $A_r^\alpha(x) = \sum_{n\in\mathbb{Z}} r^{|n|}F_n^\alpha(x) = \sum_{n\in\mathbb{Z}} r^{|n|}(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\alpha(t)(x)dt) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{n\in\mathbb{Z}} r^{|n|}e^{-int}\alpha(t)(x))dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\alpha(t)(x)dt$. From the last equality and (ii) of Theorem 3.1, we get $A_r^\alpha(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\alpha(t)(x)dt$ for every $0 \le r < 1$. □

The operator A_r^α is called the r^{th} Abel-Poisson mean operator of the family $\{F_n^\alpha\}$.

Theorem 4.2. *Let α be an isometric strongly continuous representation of T in H and $x \in H$. Then, the indexed family $\{F_n^\alpha(x)\}_{n\in\mathbb{Z}}$ of Fourier coefficients of x is Abel-Poisson summable to x.*

Proof. Since α is an isometric strongly continuous representation of T in H, we have $\lim_{t\to 0}\alpha(t)(x) = x$. Then, for every $\varepsilon > 0$ there exists a $0 < \rho < \pi$ such that

$$\|\alpha(t)(x) - x\| < \frac{\varepsilon}{2} \tag{1}$$

for all $0 \le |t| < \rho$. Hence (ii) and (iv) of Theorem 3.1, for $\frac{\varepsilon}{4(1+\|x\|)} > 0$ there exists a $\delta > 0$ such that every r, $0 < 1-\delta < r < 1$ and $0 < \rho < |t| \le \pi$, we have

$$0 < P_r(t) < \frac{\varepsilon}{4(1+\|x\|)} \tag{2}$$

Hence considering (1),(2), Theorem 4.1 and Theorem 3.5.5 in [4], we get that

$$\begin{aligned}
\|A_r^\alpha(x) - x\| &= \|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\alpha(t)(x)dt - x\| = \|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)(\alpha(t)(x) - x)dt\| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \|P_r(t)(\alpha(t)(x) - x)\|dt \\
&\le \frac{1}{2\pi}\int_{0<\rho<|t|\le\pi} P_r(t)\|\alpha(t)(x) - x\|dt + \frac{1}{2\pi}\int_{0<|t|\le\rho} P_r(t)\|\alpha(t)(x) - x\|dt \\
&< \frac{1}{2\pi}\int_{0<\rho<|t|\le\pi} \frac{\varepsilon}{4(1+\|x\|)}\|\alpha(t)(x) - x\|dt + \frac{1}{2\pi}\int_{0<|t|\le\rho} P_r(t)\frac{\varepsilon}{2}dt
\end{aligned}$$

$$< \frac{1}{2\pi} \int\limits_{0<\rho<|t|\le\pi} \frac{\varepsilon}{4(1+\|x\|)} (\|\alpha(t)(x)\| + \|x\|)\, dt + \frac{1}{2\pi}\frac{\varepsilon}{2} \int\limits_{0<|t|\le\rho} P_r(t)dt$$

$$< \frac{1}{2\pi}\frac{\varepsilon}{4(1+\|x\|)} 2\|x\| 2\pi + \frac{1}{2\pi}\frac{\varepsilon}{2} 2\pi < \varepsilon$$

□

Remark 4.1. *Theorem 4.2 is stated without proof in ([7], Theorem 9).*

Remark 4.2. *Special cases of this Theorem are given for the Fourier series of functions in a homogeneous Banach spaces on T, $C(T)$ and $L_1(T)$ respectively in ([5],p.16) and ([12],p.56), where $L_1(T)$ is the space of all complex-valued Lebesque integrable functions on T with the norm $\|f\|_{L_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|dt$.*

Proposition 4.3. *Let α be an isometric strongly continuous representation of T in H. Then, the operator A_r^α is a linear and $\left\|A_r^\alpha(x)\right\| \le \|x\|$ for all $x \in H$ and all $r \in [0, 1)$.*

Proof. The operator A_r^α's linearity is clear. On the other hand, since α is an isometric strongly continuous linear representation, we have $\|\alpha(t)(x)\| = \|x\|$ for all $x \in H$. Therefore, by Theorem 4.1 and Theorem 3.5.5 in [4], we get that $\left\|A_r^\alpha(x)\right\| = \|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\alpha(t)(x)dt\| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,\|\alpha(t)(x)\|\, dt = \|x\| \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt = \|x\|$ □

Now,we give some tests for compactness of a set in H in terms of the r^{th} Abel-Poisson mean operator A_r^α. Before the proof and statement of these tests, we give some informations which we will use in the following proof. Let $H_n := \{x : x \in H \text{ and } \alpha(t)(x) = e^{int}x, \ \forall t \in [-\pi, \pi)\}$ for any $n \in \mathbb{Z}$, and $\sum_{k=m}^{n} H_k$ be linear subspace of H spanned by the subset $\bigcup_{k=m}^{n} H_k \subset H$ for $m, n \in \mathbb{Z}$ such that $m \le n$. It is easily seen that H_n is a closed linear subspace of H, and so $\sum_{k=m}^{n} H_k$ is closed. Also $\sum_{k=m}^{n} H_k$ is finite dimensional if each H_k is finite dimensional.

Theorem 4.4. *Let α be a strongly continuous isometric linear representation of T in H and $dim(H_n) < +\infty$ for all $n \in Z$. Then, a subset $S \subset H$ is relatively compact if and only if:*

(*i*) *there exists an $M > 0$ such that $\| x \| \le M$ for $\forall x \in S$;*
(*ii*) *for any $\varepsilon > 0$ there exists $r(\varepsilon)$,$0 < r(\varepsilon) < 1$ such that $\| x - A_r^\alpha(x) \| < \varepsilon$ for $\forall r : r(\varepsilon) < r < 1$ and $\forall x \in S$.*

Proof. Assume that S is a relatively compact subset of H. Then, $\bar{S}$ is a compact subset of H and by Lemma 2.5.2 in [9], $\bar{S}$ is a closed and bounded subset of H. Hence being $S \subset \bar{S}$ shows that S is a bounded subset of H, that is there exists an $M > 0$ such that $\|x\| \le M$ for all $x \in S$ (i). Suppose that condition (*ii*) of the our theorem is false. Then, there exists $\varepsilon_o > 0$ such that for any $0 < \delta < 1$, there exists an r_δ, $0 < \delta < r_\delta < 1$ and an element $x_\delta \in S$, for which the inequality $\varepsilon_o \le \|x_\delta - A_{r_\delta}^\alpha(x_\delta)\|$ holds. So, there exist two sequences $\{r_n\} \subset (0, 1)$ and $\{x_n\} \subset S$ such that $\frac{n}{n+1} < r_n < 1$ and $\varepsilon_o \le \|x_n - A_{r_n}^\alpha(x_n)\|$ for all $n \in \mathbb{N}$. It is clear that $\lim_{n\to+\infty} r_n = 1$. By $x_n \in S$ and relatively compactness of S, there exists a subsequence $\{x_{k_n}\}$ of the sequence $\{x_n\}$ and an element $x_o \in H$ such that $\lim_{n\to+\infty} x_{k_n} = x_o$. Since, $\lim_{r\to 1^-} A_r^\alpha(x_o) = x_o$ and $\lim_{n\to+\infty} r_n = 1$, we have $\lim_{n\to+\infty} A_{r_{k_n}}^\alpha(x_o) = x_o$. According to Proposition 4.3, $\| A_{r_{k_n}}^\alpha(x_o) - A_{r_{k_n}}^\alpha(x_{k_n}) \| \le \| A_{r_{k_n}}^\alpha(x_o - x_{k_n}) \| \le \| x_o - x_{k_n} \|$ for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to+\infty} \| A_{r_{k_n}}^\alpha(x_o) - A_{r_{k_n}}^\alpha(x_{k_n}) \| = 0$. Consequently, the inequality $\varepsilon_o \le \|x_{k_n} - A_{r_{k_n}}^\alpha(x_{k_n})\| \le \|x_{k_n} - x_o\| + \| x_o - A_{r_{k_n}}^\alpha(x_o)\| + \|A_{r_{k_n}}^\alpha(x_o) - A_{r_{k_n}}^\alpha(x_{k_n})\|$ for all $n \in \mathbb{N}$ gives contradiction $0 < \varepsilon_o \le 0$. Thus the set S satisfies condition (*ii*).

Let $\epsilon > 0$ and S be a subset of H satisfying conditions (i) and (ii). Then there exists an $M > 0$ such that $\|x\| \le M$ for $\forall x \in S$ and there exists $r(\varepsilon)$, $0 < r(\varepsilon) < 1$ such that

$$\|x - A_r^\alpha(x)\| < \frac{\varepsilon}{3} \tag{3}$$

for $\forall r : r(\varepsilon) < r < 1$ and $\forall x \in S$. For fixed $r, r(\varepsilon) < r < 1$ there exists an $n \in \mathbb{N}$ such that $2Mr^{n+1}(1-r)^{-1} < \frac{\epsilon}{3}$. Let n be a such natural number and $S_n := \{\sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x) : x \in S\}$. From Proposition 2 in [8], it is known that $\alpha(t)(F_m^\alpha(x)) = e^{imt}F_m^\alpha(x)$. Hence, $F_m^\alpha(x) \in H_m$ for all $x \in H$ and $m \in \mathbb{Z}$, and so $\sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x) \in \sum_{i=-n}^{n} H_i$. On the other hand,

$$\begin{aligned}\sum_{m=-n}^{n} r^{|m|} = \sum_{m=-n}^{-1} r^{|m|} + \sum_{m=0}^{n} r^{|m|} &= (1-r^{n+1})(1-r)^{-1} + (r - r^{n+1})(1-r)^{-1} \\ &= (1 + r - 2r^{n+1})(1-r)^{-1}.\end{aligned} \tag{4}$$

Using the boundedness of S and the inequality $\left\|F_m^\alpha(x)\right\| \le \|x\|$, we obtain

$$\left\|F_m^\alpha(x)\right\| \le M \tag{5}$$

for all $x \in S$ and $m \in \mathbb{Z}$. Hence, using inequalities (4) and (5), we get that

$$\left\|\sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x)\right\| \le \sum_{m=-n}^{n} r^{|m|}\left\|F_m^\alpha(x)\right\| \le \|x\| \sum_{m=-n}^{n} r^{|m|} \le M\left(1 + r - 2r^{n+1}\right)(1-r)^{-1}.$$

Therefore, S_n is a bounded subset of the finite dimensional linear subspace $\sum_{i=-n}^{n} H_i$, so $\bar{S_n}$ is bounded. Hence, $\bar{S_n}$ is compact by Theorem 2.5.3 in [9], and so S_n is totally bounded. Let $\left\{x_1, \dots, x_q\right\}$ be a finite $\frac{\varepsilon}{3}$-net for S_n for any $\varepsilon > 0$. We show that the set $\left\{x_1, \dots, x_q\right\}$ is a finite ε-net for S. Let x be an arbitrary element of S. Since $\sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x) \in S_n$, there exists an $x_i, i \in \{1, \cdots, q\}$ such that

$$\| \sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x) - x_i \| < \frac{\varepsilon}{3}. \tag{6}$$

Using the equality $A_r^\alpha(x) = \sum_{n\in\mathbb{Z}} r^{|n|}F_n^\alpha(x)$, the equality

$$\sum_{|m|>n} r^{|m|} = 2r^{n+1}(1-r)^{-1} \tag{7}$$

and (4), we get that the following inequality

$$\begin{aligned}\| A_r^\alpha(x) - \sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x) \| &= \left\|\sum_{m\in Z} r^{|m|}F_m^\alpha(x) - \sum_{m=-n}^{n} r^{|m|}F_m^\alpha(x)\right\| \\ &= \left\|\sum_{|m|>n} r^{|m|}F_m^\alpha(x)\right\| \le \sum_{|m|>n} r^{|m|}\left\|F_m^\alpha(x)\right\| \\ &\le \|x\| \sum_{|m|>n} r^{|m|} \le 2Mr^{n+1}(1-r)^{-1} < \frac{\varepsilon}{3}\end{aligned} \tag{8}$$

From the inequalities (3), (6) and (8), we get $\| x - x_i \| \leq \| x - A_r^\alpha(x) \| + \| A_r^\alpha(x) - x_i \| \leq \| x - A_r^\alpha(x) \| + \| A_r^\alpha(x) - \sum_{m=-n}^{n} r^{|m|} F_m^\alpha(x) \| + \| \sum_{m=-n}^{n} r^{|m|} F_m^\alpha(x) - x_i \| < \varepsilon$. Thus the set $\left\{x_1, \ldots, x_q\right\}$ is a ε-net for S. Consequently, S is relatively compact by Lemma 8.8.2 in [9]. □

Theorem 4.5. *Let α be a strongly continuous isometric linear representation such that $dim(H_n) < +\infty$ for all $n \in \mathbb{Z}$ and $S \subset H$. Then S is relatively compact if and only if the following conditions are satisfied*

(i) S is bounded subset of H,

(ii) For every $\varepsilon > 0$ there exists a positive number $0 < \delta < \pi$ such that $\|\alpha(t)(x) - x\| < \varepsilon$ for all $0 < |t| < \delta(\varepsilon)$ and $x \in S$.

Proof. Let $\varepsilon > 0$ and S is relatively compact in H. Since S is totally bounded, S is bounded.(i) Let $\{x_1, x_2, ..., x_m\} \subset H$ be an $\frac{\varepsilon}{3}$-net for S. Since $\lim_{t\to 0} \alpha(t)(x_k) = x_k$ for $k \in \{1, 2, ..., m\}$ there exists a $\delta_k \equiv \delta_k(\frac{\varepsilon}{3}) > 0$ such that $\|\alpha(t)(x_k) - x_k\| < \frac{\varepsilon}{3}$ for all $|t| < \delta_k$. Let $\delta := min\{\delta_1, ..., \delta_m\}$. Since $S \subset \bigcup_{k=1}^{m} B(x_k; \frac{\varepsilon}{3})$, if $x \in S$, there exists an $l \equiv l(x), l \in \{1, 2, ..., m\}$ such that $x \in B(x_l; \frac{\varepsilon}{3})$ i.e. $\|x - x_l\| < \frac{\varepsilon}{3}$. Then, $\|\alpha(t)(x) - x\| \leq \|\alpha(t)(x) - \alpha(t)(x_l)\| + \|\alpha(t)(x_l) - x_l\| + \|x_l - x\| \leq \|\alpha(t)(x - x_l)\| + \|\alpha(t)(x_l) - x_l\| + \|x_l - x\| \leq \|\alpha(t)(x_l) - x_l\| + 2\|x_l - x\| < \varepsilon$ for all $|t| < \delta$ and $x \in S$. (ii)

S satifies the conditions (i) and (ii). Let $\varepsilon > 0$ and $\delta \equiv \delta(\varepsilon) > 0$ such that $0 < \delta < \pi$ and $\|\alpha(t)(x) - x\| < \varepsilon$ for all $0 < |t| < \delta(\varepsilon)$ and $x \in S$. Since S is a bounded subset of H, there exists an $M > 0$ such that $\|x\| < M$ for all $x \in S$. By Theorem 3.1-iv. there exists an $r(\varepsilon) > 0$ such that $\int_{\pi \geq |t| \geq \delta} P_r(t) dt < \frac{\varepsilon \pi}{2M}$ for all $r(\varepsilon) < r < 1$. Hence by Theorem 3.1 and Theorem 4.4

$$\begin{aligned}
\|A_r^\alpha(x) - x\| &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \alpha(t)(x) dt - x \right\| = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \left(\alpha(t)(x) - x\right) dt \right\| \\
&\leq \frac{1}{2\pi} \int_{|t| \geq \delta} P_r(t) \|\alpha(t)(x) - x\| dt + \frac{1}{2\pi} \int_{|t| < \delta} P_r(t) \|\alpha(t)(x) - x\| dt \\
&< \frac{1}{2\pi} \int_{|t| \geq \delta} P_r(t) \left(\|\alpha(t)(x)\| + \|x\|\right) dt + \frac{1}{2\pi} \int_{|t| < \delta} P_r(t) \|\alpha(t)(x) - x\| dt \\
&< \frac{1}{2\pi} \int_{|t| \geq \delta} P_r(t) 2\|x\| dt + \frac{1}{2\pi} \frac{\varepsilon}{2} \int_{|t| < \delta} P_r(t) dt < \varepsilon
\end{aligned}$$

for all $0 < r(\varepsilon) < r < 1$ and $x \in S$. Then, S satisfies the condition (ii) of Theorem 4.4. So, S is relatively compact. □

Theorem 4.6. *Let α be a strongly continuous isometric linear representation such that $dim(H_n) < +\infty$ for all $n \in \mathbb{Z}$ and $\emptyset \neq S \subset H$. Then S is relatively compact if and only if for any $\varepsilon > 0$ there exists a positive number $r_o(\varepsilon)$ such that $\|rA_r^\alpha(x) - x\| < \varepsilon$ for all $0 < r_o(\varepsilon) < r < 1$ and $x \in S$.*

Proof. Let S be relatively compact. Suppose that the above condition is not true. Then there exists an $\varepsilon_o > 0$ such that for every $\delta > 0$ there exists an $r_\delta, 0 < \delta < r_\delta$ and a $x_\delta \in S$ such that $\|r_\delta A_{r_\delta}^\alpha(x_\delta) - x_\delta\| \geq \varepsilon_o$. Therefore, there exists a sequence $\{r_n\} \subset \mathbb{R}$ and a sequence $\{x_n\} \subset S$ such that $0 < \frac{n}{n+1} < r_n < 1$ and $\|r_n A_{r_n}^\alpha(x_n) - x_n\| \geq \varepsilon_o$. Since S is relatively compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{k_n}\}$ such that $\lim_{n\to+\infty} x_{k_n} = x_o$ for $x_o \in H$. Considering $\lim_{n\to+\infty} r_n = 1, \frac{n}{n+1} < r_n < 1, \lim_{r\to 1^-} A_r^\alpha(x_o) = x_o$ and the operator

$A_r^\alpha : H \to H$ is a bounded linear operator it follows that $\lim_{n\to+\infty} r_{k_n} A^\alpha_{r_{k_n}}(x_o) = x_o$. Hence,

$$\begin{aligned}
\varepsilon_o &\le \|r_{k_n} A^\alpha_{r_{k_n}}(x_{k_n}) - x_{k_n}\| \\
&\le \|r_{k_n} A^\alpha_{r_{k_n}}(x_{k_n}) - r_{k_n} A^\alpha_{r_{k_n}}(x_o) + r_{k_n} A^\alpha_{r_{k_n}}(x_o) - x_o + x_o - x_{k_n}\| \\
&\le \|r_{k_n}(A^\alpha_{r_{k_n}}(x_{k_n}) - A^\alpha_{r_{k_n}}(x_o))\| + \|r_{k_n} A^\alpha_{r_{k_n}}(x_o) - x_o\| + \|x_o - x_{k_n}\| \\
&\le |r_{k_n}| \|A^\alpha_{r_{k_n}}(x_{k_n} - x_o)\| + \|x_{k_n} - x_o\| + \|r_{k_n} A^\alpha_{r_{k_n}}(x_o) - x_o\| \\
&\le 2\|x_{k_n} - x_o\| + \|r_{k_n} A^\alpha_{r_{k_n}}(x_o) - x_o\|
\end{aligned}$$

for all $n \in \mathbb{N}$ and this gives a contradiction $0 < \varepsilon_o \le 0$. Therefore if S is relatively compact, for every $\varepsilon > 0$ there exists an $0 < r(\varepsilon) < 1$ such that $\|rA_r^\alpha(x) - x\| < \varepsilon$ for all $0 < r_0(\varepsilon) < r < 1$ and $x \in S$.

Let $\varepsilon > 0$. From the condition shows that $\frac{1}{r}(\|x\| + \varepsilon) > \|A_r^\alpha(x)\| > \frac{1}{r}(\|x\| - \varepsilon)$ for all $r_o(\varepsilon) < r < 1$ and $x \in S$. Let us put $\frac{1}{r} = 1 + \delta_r$. Firstly, we show that S is bounded subset of H. If not, there exists a $x_r \in S$ such that $\delta_r(\|x_r\| - \varepsilon) > 2\varepsilon$. For $x_r \in S$, from the inequality $\|A_r^\alpha(x_r)\| > \frac{1}{r}(\|x_r\| - \varepsilon)$, we get $\|A_r^\alpha(x_r)\| > \|x_r\| + \varepsilon > \|x_r\|$. This contradicts to Proposition 4.3. So S is a bounded subset. Finally, we show that S also satisfies the condition (ii) of Theorem 4.4. Since S is a bounded subset of H, there exists an $M > 0$ such that $\|x\| \le M$ for all $x \in S$. Hence, $\lim_{r\to1^{-1}}(1 - r)M = 0$. Therefore, there exists an $0 < r_o^*(\varepsilon) < 1$ such that $(1 - r)M < \varepsilon$ for all $0 < r_o^*(\varepsilon) < r < 1$. Let us take $r(\varepsilon) := max\{r_o(\varepsilon), r_o^*(\varepsilon)\}$. Then by considering Proposition 4.3 for all $0 < r(\varepsilon) < r < 1$ and $x \in S$, we get that

$$\begin{aligned}
\|A_r^\alpha(x) - x\| = \|(1 - r)A_r^\alpha(x) + rA_r^\alpha(x) - x\| &\le (1 - r)\|A_r^\alpha(x)\| + \|rA_r^\alpha(x) - x\| \\
&< (1 - r)M + \|rA_r^\alpha(x) - x\| < 2\varepsilon.
\end{aligned}$$

Since S satisfies all conditions of Theorem 4.4, S is relatively compact. □

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