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On Periodic Solutions To Nonlinear Differential Equations In Banach Spaces

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Abstract. Let A denote the generator of a strongly continuous periodic one-parameter group of bounded linear operators in a complex Banach space H. In this work, an analog of the resolvent operator which is called quasi-resolvent operator and denoted by R_λ is defined for points of the spectrum,some equivalent conditions for compactness of the quasi-resolvent operators R_λ are given.Then using these, some theorems on existence of periodic solutions to the non-linear equations $\Phi(A)x = f(x)$ are given, where $\Phi(A)$ is a polynomial of A with complex cofficients and f is a continuous mapping of H into itself.

1. Introduction

Let $\alpha(t)$ be a sfrongly continuous periodic one-parameter group of bounded linear operators in a Banach space H, A is the generator of α, $D(A)$ is the domain of the definition of the operator A, $\Phi(A)$ is the operator $c_0 I + c_1 A + \cdots + c_{n-1}A^{n-1} + A^n$, where c_j is a complex number, and I is the unit operator in H.

In this paper, we study existence of solutions to the nonlinear equation

$$Ax - \lambda x = f(x), \tag{1}$$

where λ is a complex number, $f : D(A) \to H$ is a continuous mapping, and to the nonlinear equation

$$\Phi(A)x = F(x), \tag{2}$$

where $F : D(A^n) \to H$ is a continuous mapping.

Equations (1) and (2) are abstract forms of periodic nonlinear differential equations in functional spaces. For example, let $H = C[T]$ be the Banach space of all complex continuous functions on $T = \left\{e^{it} : -\pi \le t < \pi\right\}$ with the supremum norm and $\alpha(t)$ is the operator $\alpha(s)x(t) = x(t+s)$. Then $D(A) = C^1(T)$ and $Ax(t) = \frac{d}{dt}x(t)$ is the usual derivative of $x(t)$ [2]. The following periodic functional differential equation $\frac{d}{dt}x(t) - \lambda x = f(t, x(t - \varphi_1(t)), \ldots, x(t - \varphi_m(t)))$, has form (1), and the following so-called periodic high-order functional Duffing equation $\Phi(\frac{d}{dt})x(t) = f(t, x(t - \varphi_1(t)), \ldots, x(t - \varphi_m(t)))$, has form (2), where $f : R \times C^m \to C$ be a

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continuous function, $f(t+2\pi, u_1, u_2, \ldots, u_m) = f(t, u_1, u_2, \ldots, u_m)$ and $\varphi_i : R \to R$ is a 2π-periodic C^1-function, $i = 1, \ldots m$.

The theorem on integral for the operator A, theorems on existence of periodic solutions of a linear differential equation of nth order with constant coefficients and systems of linear differential equations with constant coefficients in Banach spaces are obtained in [5].In the case of an existence of periodic solutions, evident forms of all periodic solutions of a linear differential equation of nth order with constant coefficients and systems of linear differential equations with constant coefficients in Banach spaces are given in terms of resolvent and quasi-resolvent operators of A. Existence of periodic solutions to equations of forms (1) and (2) in classical Banach spaces have been studied in many works. In particular, equations of the forms (1) and (2) for the Banach space of continuous vector-valued functions were considered in papers [3, 4, 7, 8, 14, 15, 19],[21]-[25] and for the Banach space of all summable vector-valued functions on T were considered in [1, 10, 18]. Our approach is based on techniques and results of the theory of periodic one-parameter groups of bounded linear operators in Banach spaces [2, 6, 9, 17].

2. Preliminaries

Denote the group of all invertible bounded linear operators $F : H \to H$ of a complex Banach space H by $GL(H)$.Let T be the one-dimensional torus $\left\{e^{it} : -\pi \le t < \pi\right\}$. Further we shall consider T as the additive group $Q/2\pi Z \simeq \{t : -\pi \le t < \pi\}$ with its euclidean topology, where Q is the field of real numbers.The following definitions are given in [6, 17].

Definition 1.

(i)A homomorphism $\alpha : T \to GL(H)$ will be called a linear representation of T in H.

(ii)Linear representations $\alpha : T \to GL(H)$ and $\beta : T \to GL(V)$ will be called equivalent if there exists a bounded invertible linear operator $B : H \to V$ such that $B\alpha(t) = \beta(t)B$ for all $t \in T$.

(iii)A linear representation α of T in H will be called isometric if $\|\alpha(t)x\|=\|x\|$ for all $t \in T$ and $x \in H$.

(iv)A linear representation α of T in H will be called strongly continuous if $\lim_{t\to 0} \alpha(t)x = x$ for all $x \in H$.

It is known that every strongly continuous linear representation of T on a Banach space is equivalent to a strongly continuous isometric linear representation of T ([17],p.82). In sequel, we consider only strongly continuous isometric linear representations. Let α be a strongly continuous linear representation of T in H, Z be the ring of all integers, $x \in H$ and $n \in Z$. Then by the theorem in ([16], p.314) the Riemann's integral $P_n(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\alpha(t)xdt$ exists. $P_n(x)$ is called the n-th Fourier coefficient of x with respect to α [3, 6, 17].

Proposition 1. *Let α be a strongly continuous isometric linear representation of T in H and $x, y \in H$. Then:*

1. *$\alpha(t)P_n(x) = P_n(\alpha(t)x) = e^{int}P_n(x)$ for all $n \in Z, x \in H$ and $t \in T$;*
2. *$P_nP_m = 0$ for all $m, n \in Z, m \neq n$, and $P_n^2 = P_n$ for all $n \in Z$;*
3. *$\|P_n(x)\| \le \|x\|$ for all $n \in Z$ and $x \in H$.*
4. *If $P_m(x) = P_m(y)$ for each $m \in \mathbb{Z}$, then $x = y$.*

Proof. Proof is given in ([6], p.250; [13], Corollary 1). □

Let $n \in \mathbb{Z}$ and put $H_n := \{x \in H : \alpha(t)x = e^{int}x, \forall t \in T\}$. It is easily proven that H_n is a closed sublinear space of H, $P_n(x) \in H_n$ for all $x \in H$ and $P_n(H) \subset H_n$. Also, for each $x \in H_n$ we have $P_n(x) = \dfrac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-int}\alpha(t)(x)dt = \dfrac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-int}e^{int}xdt = x$. Therefore, $H_n = P_n(H)$. For the $n \in \mathbb{Z}^+$ let us put $s_n(x) := \sum_{k=-n}^{n} P_k(x), \sigma_n(x) := \frac{s_0(x)+s_1(x)+\cdots+s_n(x)}{n+1}, K_n(t) = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)}{2}t}{\sin\frac{t}{2}}\right)^2, Spec(x) := \{in : n \in Z, P_n(x) \neq 0\}, Spec(H) := \bigcup_{x\neq\theta, x\in H} Spec(x), H_f := \{x \in H : Spec(x)$ is finite$\}$. H_f is a subspace of H. In the present paper, we assume that $Spec(H)$ is infinite. The case of the finite $Spec(H)$ is investigated easy and it is omitted. Let α be a strongly continuous isometric linear representation of T in H.

Definition 2. *([6],p.45) A point $x \in H$ will be called a differentiable point of α if there exists $Ax := \lim_{t\to 0} \frac{\alpha(t)x-x}{t}$ in H. Denote the set of all differentiable points of α by $D(A)$.The operator A will be called the generator of the linear representation α([6],p.45). The set $Spec(H)$ is called the spectrum of A. The set $C \setminus Spec(H)$ is called the resolvent set of A.It is easily seen that $H = D(A)$ if and only if $Spec(H)$ is finite.*

Proposition 2. *([13],Proposition 4]) Let α be a strongly continuous isometric linear representation of T in H. Then*

(*i*) *$D(A)$ is a linear subspace of H, $H_f \subset D(A)$ and $\overline{D(A)} = H$;*
(*ii*) *$D(A)$ is $\alpha(T)$-invariant and $\alpha(t)Ax = A\alpha(t)x$ for all $t \in T$, $x \in D(A)$;*
(*iii*) *$AP_n(x) = P_n(Ax) = inP_n(x)$ for all $n \in Z$ and $x \in D(A)$.*

3. Some Inequalities for Norms of the Resolvent and Quasi-Resolvent Operators

Let α be a strongly continuous isometric linear representation of T in H. The linear operator R_λ on H, defined by

$$R_\lambda(x) = (1 - e^{-2\pi\lambda})^{-1} \int_0^{2\pi} e^{-\lambda s}\alpha(s)xds \tag{3}$$

for $\lambda \in C$ such that $\lambda \notin Spec(H)$, is the resolvent of the generator A at the point λ ([6],IV,2.25). Other form of the resolvent operator is given in([11];[5],Theorem 4).

For $\lambda = im \in iZ$, we define the linear operator R_λ as follows

$$R_{im}(x) = (2\pi)^{-1} \int_0^{2\pi} \left(\int_0^t e^{-ims}\alpha(s)xds \right) dt - (1 + \pi)\,(2\pi)^{-1} \int_0^{2\pi} e^{-imt}\alpha(t)xdt \tag{4}$$

The operator $R_{im}(x)$ will be called the quasi-resolvent operator of A for the point im of the spectrum of A. The operator R_λ for $\lambda = 0$ was introduced in [13] and for every $\lambda = im \in iZ$ in [5, 11, 12].

Proposition 3. *Let $\lambda \in C$, $\lambda \notin Spec(H)$. Then $\|R_\lambda\| \le d_\lambda$, where $d_\lambda =\mid \delta(1-e^{-2\pi\lambda}) \mid^{-1}\mid e^{-2\pi\delta}-1 \mid$ for $\lambda = \delta+i\beta, \delta \neq 0$ and $d_\lambda = 2\pi \mid 1 - e^{-2\pi\lambda} \mid^{-1}$ for $\lambda = \delta + i\beta, \delta = 0$.*

Proof. Let $\lambda = \delta + i\beta, \delta \neq 0$. Using formula (3) and isometricity of α, we have $\| R_\lambda(x) \| \le \mid 1 - e^{-2\pi\lambda} \mid^{-1} \int_0^{2\pi} \| e^{-\lambda s}\alpha(s)x \| ds \le \mid \delta(1 - e^{-2\pi\lambda}) \mid^{-1}\mid e^{-2\pi\delta} - 1 \mid \| x \|$. Let $\lambda = \delta + i\beta, \delta = 0$. Similarly $\| R_\lambda(x) \| \le \mid 1 - e^{-2\pi\lambda} \mid^{-1} \int_0^{2\pi} \| e^{-\lambda s}\alpha(s)x \| ds \le 2\pi \mid 1 - e^{-2\pi\lambda} \mid^{-1}\| x \|$. □

Proposition 4. *Let $\lambda = im \in iZ$. Then $\| R_{im} \| \le d_{im} = (1+2\pi)$. In particular, if $P_m(x) = 0$ then $\| R_{im}(x) \| \le \pi \| x \|$.*

Proof. Using formula (4) and isometricity of α, we obtain $\| R_{im}(x) \| \le (2\pi)^{-1} \int_0^{2\pi} \int_0^t dsdt \| x \| +(1 + \pi) \| x \| = (1 + 2\pi) \| x \|$. Assume that $P_m(x) = 0$. Similarly, we have $\| R_{im}(x) \| \le (2\pi)^{-1} \int_0^{2\pi} (\int_0^t \| e^{-ims}\alpha(s)x \| ds)dt \le (2\pi)^{-1} \int_0^{2\pi} \int_0^t dsdt \| x \| = \pi \| x \|$. □

Corollary 1. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a set of complex numbers and d_{λ_k} is the real number defined in Proposition 3 and 4. Then the inequality $\| R_{\lambda_1}R_{\lambda_2} \cdots R_{\lambda_n}(x) \| \le d_\Lambda \|x\|$ holds, where $d_\Lambda = d_{\lambda_1}d_{\lambda_2}...d_{\lambda_n}$.*

Proof. It follows easily from Proposition 3 and 4 by induction. □

Proposition 5. *Let α be a strongly continuous isometric linear representation of T in H. Then :*

(*i*) *$R_\lambda - R_{im} = (\lambda - im)R_\lambda \circ R_{im} - (\lambda - im)^{-1}P_m$ for all $m \in Z$ and $\lambda \in C \setminus Spec(H)$;*
(*ii*) *$R_\lambda \circ R_{im} = R_{im} \circ R_\lambda$ for all $\lambda \in C$ and $m \in Z$.*

Proof. (*i*). Let $x \in H$, $\lambda \in C \setminus Spec(H)$ and $m \in Z$. For $n \neq m$, by Theorems 3,4 in [5] and $P_nP_m = 0$ in Proposition 1,we obtain $P_n(R_\lambda - R_{im}) = (in - \lambda)^{-1}P_n - (in - im)^{-1}P_n = P_n((\lambda - im)R_\lambda R_{im} - (\lambda - im)^{-1}P_m)$. Similarly, $P_m(R_\lambda - R_{im}) = (im - \lambda)^{-1}P_m - P_mR_{im} = (im - \lambda)^{-1}P_m - P_m = P_m((\lambda - im)R_\lambda R_{im} - (\lambda - im)^{-1}P_m)$. Hence $P_n(R_\lambda(x) - R_{im}(x)) = P_n(im - \lambda)R_\lambda \circ R_{im}(x) - (im - \lambda)^{-1}P_m(x)$) for every $n \in Z$. By Proposition 1, we obtain equality (*i*). A proof of (*ii*) is similar. □

Remark 1. *Equation (i) in Proposition 5 is a generalization for operators R_{im} of the resolvent equation ([6],p.239,IV,1.2).*

4. Conditions for Compactness of Operators R_λ

First we give a test for compactness of a set in the Banach space of a linear representation.

Theorem 1. *Let α be a strongly continuous isometric linear representation of T in H, satisfying the condition $dim(H_n) < +\infty$ for all $n \in Z$.Then a subset $S \subset H$ is relatively compact if and only if:*

(*i*) *there exists an $M > 0$ such that $\| x \| \leq M$ for all $x \in S$;*
(*ii*) *for any $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $\| x - \sigma_n(x) \| < \varepsilon$ for all $n \in \mathbb{N}, n \geq n(\varepsilon)$,and $x \in S$.*

Proof. Let S be a relatively compact subset of H. Then S is bounded that is there exists an $M > 0$ such that $\| x \| \leq M$ for all $x \in S$. Suppose that (*ii*) of the our theorem is false. Then there exists an $\varepsilon_0 > 0$ such that for any integer $m > 0$ there exist an $n_m > m$ and an element $x_{n_m} \in S$, satisfying the condition $\| x_{n_m} - \sigma_{n_m}(x_{n_m}) \| \geq \varepsilon_0$. By $x_{n_m} \in S$ and relatively compactness of S, there exist a subsequence $\left\{x'_k\right\}$ of the sequence $\{x_{n_m}\}$ and an element $x_o \in H$ such that $\lim_{k\to\infty} \| x'_k - x_o \| = 0$. We have $\varepsilon_0 \leq \| x'_k - \sigma_k(x'_k) \| \leq \| x'_k - x_o \| + \| x_o - \sigma_k(x_o) \| + \| \sigma_k(x_o) - \sigma_k(x'_k) \|$. According to Proposition 3 in [13], $\| \sigma_k(x_o) - \sigma_k(x'_k) \| = \| \sigma_k(x_o - x'_k) \| \leq \| x_o - x'_k \|$. Using Theorem 1 in [13], we obtain $\varepsilon_0 \leq 2 \| x_o - x'_k \| + \| x_o - \sigma_k(x_o) \| \to 0$. This is a contradiction shows that the set S satisfies condition (*ii*).

Conversely, let S be a subset of H satisfying conditions (*i*) and (*ii*). For fixed n, we consider the set $S_n = \{\sigma_n(x) : x \in S\}$. We note that $\sigma_n(x) \in \sum_{i=-n}^{n} H_i$ for any f. Therefore S_n is a subset of the finite dimensional closed subspace $\sum_{i=-n}^{n} H_i$ of H, by Proposition 3 in [13] S_n is bounded, hence $\overline{S_n} \subset \sum_{i=-n}^{n} H_i$ is bounded. Therefore $\overline{S_n}$ is compact and so S_n is totally bounded. Let $\varepsilon > 0$ and $\left\{x_1, \ldots, x_q\right\}$ a finite ε-net for S_n. We show that it is a finite 2ε-net for S. Let x be an arbitrary element of S. There exists an x_i such that $\| \sigma_n(x) - x_i \| < \varepsilon$. Then $\| x - x_i \| \leq \| x - \sigma_n(x) \| + \| \sigma_n(x) - x_i \| < 2\varepsilon$. Thus the set $\left\{x_1, \ldots, x_q\right\}$ is a 2ε-net for S. Therefore S is relatively compact. □

Remark 2. *Theorem 1 was given without of proof in the paper ([12], Theorem 11).*

Theorem 2. *Let α be a strongly continuous isometric linear representation of T in H. Then following three conditions are equivalent:*

(*i*) *$dim(H_n) < +\infty$ for all $n \in Z$;*
(*ii*) *the operator R_λ is compact for any $\lambda \in C$;*
(*iii*) *the operator R_λ is compact for some $\lambda \in C$.*

Proof. (*i*) $\to$ (*ii*). First we prove that the operator R_0 is compact. Consider the set $B = \{x \in H : \| x \| \leq 1\}$. We prove that the set $R_0(B)$ satisfies conditions of Theorem 1. By Proposition 4, the operator R_0 is bounded. Hence the set $R_0(B)$ is bounded. Prove that set $R_0(B)$ satisfies condition (*ii*) of Theorem 1. We need the following lemma.

Lemma 1. *Let $x \in H$ such that $P_0(x) = 0$. Then $\alpha(t)R_0(x) = \int_0^t \alpha(s)xds + R_0(x)$.*

Proof. It is given in ([13],Lemma 2;[5],Lemma 2). □

According to Lemma 1, we get $\alpha(t)R_0(x) - R_0(x) = \int_0^t \alpha(s)xds$. From this equality, using the inequality $||x|| \leq 1$, we obtain

$$||\alpha(t)R_0(x) - R_0(x)|| \leq \left\| \int_0^t \alpha(s)xds \right\| \leq \int_0^t ||x||\, ds = |t|\, ||x|| \leq |t|\,. \tag{5}$$

for all $x \in B$. From this inequality, we obtain that for any $\varepsilon > 0$ there exists a $0 < \delta < \frac{\varepsilon}{2}$ such that $||\alpha(t)R_0(x) - R_0(x)|| < \frac{\varepsilon}{2}$ for all $|t| < \delta < \pi$ and all $x \in B$. Since $K_n(t) = K_n(-t)$ for all $t \in [-\pi, \pi]$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$ and $\sin\frac{\delta}{2} \leq \sin\frac{t}{2}$ for all $t \in [\delta, \pi)$, using Proposition 3 in [13] and Proposition 4, inequalities (5) and $K_n(t) = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)}{2}t}{\sin\frac{t}{2}}\right)^2 \le \frac{1}{(n+1)\sin^2\frac{\delta}{2}}$ for all $t \in [\delta, \pi]$, we obtain $\|\sigma_n(R_0x) - R_0x\| = \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\alpha(t)R_0xdt - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)R_0xdt\right\| \le \frac{1}{2\pi}\int_{-\pi}^{-\delta} \|K_n(t)(\alpha(t)R_0x - R_0x)\|\, dt + \frac{1}{2\pi}\int_{-\delta}^{\delta} \|K_n(t)(\alpha(t)R_0x - R_0x)\|\, dt + \frac{1}{2\pi}\int_{\delta}^{\pi} \|K_n(t)(\alpha(t)R_0x - R_0x)\|\, dt \le \frac{1}{\pi}\int_{\delta}^{\pi} \|K_n(t)\|\, \|\alpha(t)R_0x - R_0x\|\, dt + \frac{1}{2\pi}\int_{-\delta}^{\delta} \|K_n(t)\|\, \|\alpha(t)R_0x - R_0x\|\, dt \le \frac{1}{\pi(n+1)\sin^2\frac{\delta}{2}}\int_{\delta}^{\pi} \|\alpha(t)R_0x - R_0x\|\, dt + \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(t)\, |t|\, dt \le \frac{2\|R_0(x)\|}{(n+1)\sin^2\frac{\delta}{2}} + \delta \le \frac{2(2\pi+1)}{(n+1)\sin^2\frac{\delta}{2}} + \delta = \frac{2(2\pi+1)}{(n+1)\sin^2\frac{\varepsilon}{4}} + \frac{\varepsilon}{2}$ for all $x \in B$. On the other hand, there exists an $n \in \mathbb{N}, n > n_0$ such that $\frac{2(2\pi+1)}{(n+1)\sin^2\frac{\varepsilon}{4}} < \frac{\varepsilon}{2}$ for all $n > n_0$.Hence, we obtain that $\|\sigma_n(R_0x) - R_0x\| < \varepsilon$ for all $n > n_0$ and all $x \in B$.This shows that the set $R_0(B)$ satisfies the condition (ii) of Theorem 1,so it is relatively compact and the operator R_0 is compact.

Let $\lambda \notin iZ$. Using Proposition 5, we obtain $R_\lambda = R_0 - \lambda R_\lambda \circ R_0 + \frac{1}{\lambda}P_0$. Since R_0 is compact, the operator $(R_\lambda \circ R_0)$ is also compact. The operator P_0 is compact as a projection operator onto the finite-dimensional subspace H_0. Therefore the operator R_λ is compact. Similarly, using Proposition 5, we obtain that the operator R_{im} is also compact. The implication $(ii) \to (iii)$ is obvious. $(iii) \to (i)$. Let R_λ be a compact operator for some $\lambda \in C$. Assume that $\lambda \notin iZ$. According to Theorem 3 in [5], since $R_\lambda(x) = (im - \lambda)^{-1}x$ for all $x \in H_m$ and all $m \in Z$, we have $R_\lambda(H_m) = H_m$. Using compactness of R_λ and Theorem 8.1.13 in ([20], p.288) we obtain that $dim(H_m) < +\infty$ for all $m \in Z$. In the case $\lambda \in iZ$, using Theorem 3 in [5], we obtain that $dim(H_m) < +\infty$ for all $m \in Z$. The proof is completed. □

5. Theorems on Existence of a Solution to a Periodic Nonlinear Differential Equation in a Banach Space

Theorem 3. *Let α be a strongly continuous isometric linear representation of T in H, $a \in H$ and $\lambda \in C$.*

(*i*) *Assume that $\lambda \notin Spec(H)$. Then the equation $Ax - \lambda x = a$ has the unique solution $x = -R_\lambda(a)$ for every $a \in H$, where R_λ has the form (4);*

(*ii*) *Assume that $\lambda \in Spec(H)$, $\lambda = im$ for some $m \in Z$. Then the equation $Ax - imx = a$ has a solution if and only if $P_m(a) = 0$. In the case $P_m(a) = 0$, a general solution of the equation $Ax - imx = a$ has the form $x = -R_{im}(a) + c$, where R_{im} has the form (5) and c is an arbitrary element of H_m.*

Proof. It is given in ([5],Theorem 7). □

Now we consider a solution to the nonlinear equation $Ax - \lambda x = f(x)$ in H, where λ is a complex number and $f : H \to H$ is a continuous mapping.

Theorem 4. *Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H, $\lambda \in C, \lambda \notin Spec(H)$ and $dim(H_n) < +\infty$ for all $n \in Z$. Assume that $f_1 : H \to H$ and $f_2 : H \to H$ are continuous functions satisfying the following conditions:*

(i) there exists a real number $M_1 > 0$ such that $M_1 < d_\lambda^{-1}$ and $\|f_1(x)\| \le M_1 \|x\|$ for all $x \in H$, where d_λ is as in Proposition 3;

(ii) there exists a real number $M_2 > 0$ such that $\|f_2(x)\| \le M_2$ for all $x \in H$.
Then the equation $Ax - \lambda x = f_1(x) + f_2(x)$ has a solution.

Proof. According to Theorems 3,4 in [5], a solution of the equation $Ax - \lambda x = f_1(x) + f_2(x)$ is reduced to that for equation $x = -R_\lambda(f_1(x) + f_2(x))$. We prove that the operator $-R_\lambda(f_1 + f_2)$ satisfies the condition of the Schauder's fixed point theorem. For $r > 0$ put $B_r = \{x \in H : \|x\| \le r\}$. We choose $\frac{d_\lambda M_2}{1-d_\lambda M_1} \le r$. Then, using inequalities $M_1 < d_\lambda^{-1}$, $\|f_1(x)\| \le M_1 \|x\|$ and $\|f_2(x)\| \le M_2$ in our theorem, we obtain that $\|-R_\lambda(f_1(x) + f_2(x))\| \le d_\lambda(M_1 r + M_2) \le r$ for all $x \in B_r$ that is the operator $-R_\lambda(f_1 + f_2)$ is a mapping of B_r into B_r. Since the operator R_λ is compact, the operator $-R_\lambda(f_1 + f_2)$ is continuous and the set $-R_\lambda(f_1 + f_2)(B_r)$ is conditionally compact, according to Shauder's theorem, the equation $-R_\lambda(f_1 + f_2)(x) = x$ has a solution and the theorem is proved. □

Corollary 2. *Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H, $\lambda \in C, \lambda \notin Spec(H)$ and $dim(H_n) < +\infty$ for all $n \in Z$. Assume that $B : H \to H$ is a bounded linear operator such that $\|B\| < d_\lambda^{-1}$. Then the equation $Ax - \lambda x - Bx = a$ has a solution for every $a \in H$.*

Proof. It is a particular case of Theorem 4. □

Theorem 5. *Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H, $\lambda = im \in Spec(H), m \in Z$ and $dim(H_n) < +\infty$ for all $n \in Z$. Assume that $f_1 : H \to H$ and $f_2 : H \to H$ are continuous functions satisfying the following conditions:*

(*i*) *there exists a real number $M_1 > 0$ satisfying $M_1 < d_\lambda^{-1}$ and $\left\|f_1(x)\right\| \leq M_1 \|x\|$ for all $x \in H$, where d_λ is as in Proposition 3;*
(*ii*) *there exists a real number $M_2 > 0$ satisfying $\left\|f_2(x)\right\| \leq M_2$ for all $x \in H$.*
(*iii*) *$P_m(f_1(x) + f_2(x)) = 0$ for all $x \in H$.*

Then the equation $Ax - \lambda x = f_1(x) + f_2(x)$ has a solution.

Proof. It is similar to proofs of Theorems 3(ii) and 4. □

Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H, A is the generator of α and $\Phi(A)$ is the operator

$$\Phi(A) = c_0 I + c_1 A + \cdots + c_{n-1}A^{n-1} + A^n, \tag{6}$$

where $n > 0$, c_i is a complex number ($i = 0, \ldots, n-1$) and I is the unit operator in H. Denote the set of all $x \in D(A)$ such that $Ax \in D(A)$ by $D(A^2)$. Similarly, denote the set of all $x \in D(A^{n-1})$ such that $Ax \in D(A)$ by $D(A^n)$.

Proposition 6. *Let α be a strongly continuous isometric linear representation of T in H and $n \in Z, n \geq 1$. Then $D(A^n) = R_{\lambda_1} R_{\lambda_2} \cdots R_{\lambda_n}(H)$ for all $\lambda_i \in C, i = 1, 2, \ldots n$.*

Proof. It follows from Theorem 4 in [5] by induction. □

Now we consider a solution to equation (2). This equation may be written in the form

$$(\lambda_1 I - A) \ldots (\lambda_n I - A)x = (-1)^n f(x). \tag{7}$$

Proposition 7. *Let $\Phi(A)$ be the linear operator (6), $\lambda_1, \ldots, \lambda_n$ are complex roots of the polynomial $\Phi(\lambda)$.*

(*i*) *Assume that $\lambda_1, \ldots, \lambda_n \notin iZ$. Then an element $x_0 \in D(A^n)$ is a solution to equation* (7) *if and only if it is a solution to the equation*

$$x = (-1)^n R_{\lambda_1} \ldots R_{\lambda_n}(f(x)); \tag{8}$$

(*ii*) *Assume that $\lambda_1 = im_1, \ldots, \lambda_r = im_r \in iZ$, where $r > 0$, $\lambda_{r+1}, \ldots, \lambda_n \notin iZ$ and $P_{m_1} f(x) = \cdots = P_{m_r} f(x) = 0$ for all $x \in H$. Then an element $x_0 \in D(A^n)$ is a solution to equation* (7) *if and only if it is a solution to equation* (8).

Proof. (i) Assume that $x_0 \in D(A^n)$ is a solution to equation (7). By Theorem 4-(*iii*) in [5], applying operators $R_{\lambda_1}, \ldots, R_{\lambda_n}$ to equality (7), we obtain $x_0 = (-1)^n R_{\lambda_1} \ldots R_{\lambda_n}(f(x_0))$. Conversely, assume that $x_0 \in D(A^n)$ such that $x_0 = (-1)^n R_{\lambda_1} \ldots R_{\lambda_n}(f(x_0))$. According to Proposition 6, $R_{\lambda_1} \ldots R_{\lambda_n}(f(x_0))$ is an element of $D(A^n)$. By Theorem 4-(*iii*) in [5], applying operators $(\lambda_1 I - A), \ldots, (\lambda_n I - A)$ to $x_0 = (-1)^n R_{\lambda_1} \ldots R_{\lambda_n}(f(x_0))$, we obtain $(\lambda_1 I - A) \ldots (\lambda_n I - A)x_0 = (-1)^n f(x_0)$.

(ii) Let $\lambda_1 = im_1, \ldots, \lambda_r = im_r \in iZ$, where $r > 0$, $\lambda_{r+1}, \ldots, \lambda_n \notin iZ$ and $P_{m_1} f(x) = \cdots = P_{m_r} f(x) = 0$ for all $x \in H$. Assume that $x_0 \in D(A^n)$ is a solution to equation (7). By Theorem 4-(*ii*) in [5], applying $R_{\lambda_{r+1}}, \ldots, R_{\lambda_n}$ to equality (7), we obtain $(\lambda_1 I - A) \ldots (\lambda_r I - A)x_0 = (-1)^n R_{\lambda_{r+1}} \ldots R_{\lambda_n} f(x_0)$. From Theorem 4 in [5] and $P_{m_1} f(x_0) = 0$, we get $P_{m_1}(R^{m_{r+1}}_{\lambda_{r+1}} \ldots R_{\lambda_n} f(x_0)) = R_{\lambda_{r+1}} \ldots R_{\lambda_n} P_{m_1} f(x_0) = 0$. According to Theorem 1, we have $(\lambda_2 I - A) \ldots (\lambda_r I - A)x_0 = (-1)^n R_{\lambda_1} R_{\lambda_{r+1}} \ldots R_{\lambda_n} f(x_0)$. Similarly, by induction, $x_0 = (-1)^n R_{\lambda_1} \ldots R_{\lambda_n}(f(x_0))$. □

Proposition 8. *Assume that $\Phi(A)$ is the linear operator (6), $a \in H$, $\lambda_1, \ldots, \lambda_k$ are different complex roots of the polynomial $\Phi(\lambda)$ and s_i is the multiplicity of λ_i, $s_1 + \cdots + s_k = n$.*

(i) *If $\lambda_1, \ldots, \lambda_k \notin iZ$, then for any $a \in H$ there exists the unique solution to equation*

$$\Phi(A)x = a \tag{9}$$

in H and it is $x = (-1)^n R_{\lambda_1}^{s_1} \ldots R_{\lambda_k}^{s_k}(a)$.

(ii) *If $\lambda_1 = im_1, \ldots, \lambda_r = im_r \in iZ$, where $r > 0$, and $\lambda_{r+1}, \ldots, \lambda_k \notin iZ$, then a solution to equation* (9) *exists if and only if*

$$P_{m_1}(a) = \cdots = P_{m_r}(a) = 0. \tag{10}$$

For $a \in H$, satisfying condition (10)*, a general solution to equation (9) is*

$$x = (-1)^n R_{\lambda_1}^{s_1} \ldots R_{\lambda_r}^{s_r} R_{\lambda_{r+1}}^{s_{r+1}} \ldots R_{\lambda_k}^{s_k}(a) + b_1 + \cdots + b_r,$$

where b_k is an arbitrary element of $H_{m_k} = \left\{x \in H : \alpha(t)x = e^{im_k t}x, \quad \forall \quad t \in T\right\}$.

Proof. It follows from Proposition 7, Theorem 3 and Theorems 3,4 in [5]. □

We consider a solution to the nonlinear equation $\Phi(A)x = f(x)$ in H, where $\Phi(A)$ is the linear operator (6) and $f : H \to H$ is a continuous mapping.

Theorem 6. *Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H such that $dim(H_k) < +\infty$ for all $k \in Z$ and $\Phi(A)$ is the linear operator (6) such that $\lambda_i \notin iZ$ for all $i = 1, 2, \ldots, n$. Assume that $f_1 : H \to H$ and $f_2 : H \to H$ are continuous functions satisfying the following conditions:*

(i) *there exists a real number $M_1 > 0$ such that $M_1 < (d_{\lambda_1} d_{\lambda_2} \cdots d_{\lambda_n})^{-1}$ and $\left\|f_1(x)\right\| \leq M_1 \|x\|$ for all $x \in H$, where d_λ is as in Proposition 3;*

(ii) *there exists a real number $M_2 > 0$ such that $\left\|f_2(x)\right\| \leq M_2$ for all $x \in H$.*

Then the equation $\Phi(A)x = f_1(x) + f_2(x)$ has a solution.

Proof. It follows from Theorem 4 by induction. □

Theorem 7. *Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H such that $dim(H_k) < +\infty$ for all $k \in Z$ and $\Phi(A)$ is the linear operator (6) such that $\lambda_1 = im_1, \ldots, \lambda_r = im_r \in iZ$, where $r > 0$, and $\lambda_{r+1}, \ldots, \lambda_n \notin iZ$. Assume that $f_1 : H \to H$ and $f_2 : H \to H$ are continuous functions satisfying the following conditions:*

(i) *there exists a real number $M_1 > 0$ such that $M_1 < (d_{\lambda_1} d_{\lambda_2} \cdots d_{\lambda_n})^{-1}$ and $\left\|f_1(x)\right\| \leq M_1 \|x\|$ for all $x \in H$;*

(ii) *there exists a real number $M_2 > 0$ such that $\left\|f_2(x)\right\| \leq M_2$ for all $x \in H$.*

(iii) *$P_{m_i}(f_1(x) + f_2(x)) = 0$ for all $x \in H$ and $i = 1, \ldots, r$.*

Then the equation $\Phi(A)x = f_1(x) + f_2(x)$ has a solution.

Proof. It follows from Theorems 4 and 5 by induction. □

Now we consider the existence and uniqueness of a solution of equation (7).

Theorem 8. . *Let $\alpha(t)$ be a strongly continuous isometric linear representation of T in H and $\lambda \in C, \lambda \notin iZ$. Assume that $f : D(A) \to H$ is a function such that there exists a real number $M > 0$ satisfying $M < d_\lambda^{-1}$ and $\left\|f(x) - f(y)\right\| \leq M\left\|x - y\right\|$ for all $x, y \in D(A)$, where d_λ is as in Proposition 3. Then the equation $Ax - \lambda x = f(x)$ has the unique solution.*

Proof. According to Proposition 7, equation $Ax-\lambda x = f(x)$ has a unique solution if and only if so for equation $x = -R_\lambda f(x)$. We prove that the operator $-R_\lambda f$ is a contracting mapping. We have $\| -R_\lambda f(x) - (-R_\lambda f(y))\| = \|R_\lambda(f(x) - f(y))\| < d_\lambda M\|x - y\|$. Since $d_\lambda M < 1$, the operator $-R_\lambda f$ is a contracting mapping. Hence the equation $x = -R_\lambda f(x)$ has a unique solution. □

Remark 3. *Results, which are similar to Theorem 8, are true for the case $\lambda \in iZ$ and for the equation $\Phi(A)x = f(x)$.*

We note that the above results on periodic solutions are applicable also to functional differential equations of first-order. Let $H = C[T]$ be the Banach space of all complex continuous 2π-periodic functions with the norm $\| x \| = \max_{t\in T} | x(t) |$, and $\alpha(t)$ is the operator $\alpha(s)x(t) = x(t + s)$, where $T = R/Z \approx [0, 2\pi)$. Then $\alpha(t)$ ($t \in T$) is a strongly continuous periodic one-parameter group of bounded linear operators on $C[T]$.Let A be the generator of the group $\alpha(t), t \in T$.It is known that $A = \frac{d}{dt}$ and $x(t) \in D(A)$ if and only if $\frac{d}{dt}x(t) \in C[T]$. We consider the following differential equation in C[T] with multiple deviating arguments

$$\frac{d}{dt}x(t) - \lambda x(t) = f(t, x(t - \varphi_1(t)), \ldots, x(t - \varphi_m(t))) \tag{11}$$

where $\lambda \in C$, $f : R \times C^m \to C$ be a continuous function, $f(t + 2\pi, u_1, u_2, \ldots, u_m) = f(t, u_1, u_2, \ldots, u_m)$ and $\varphi_i : R \to R$ is a 2π-periodic C^1-function, $i = 1, \ldots m$.Since the mapping $F : C[T] \to C[T]$, defined by $x(t) \to f(t, x(t-\varphi_1(t)), \ldots, x(t-\varphi_m(t)))$, is continuous, the Eq(11) has the form $(Ax-\lambda x = f(x)) : \frac{d}{dt}x(t)-\lambda x(t) = F(x)$.Hence the above results on periodic solutions are applicable to the Eq(11).

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