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New Matrix Domain Derived by the Matrix Product

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Abstract. In this work, we define new sequence spaces by using the matrix obtained by product of factorable matrix and generalized difference matrix of order m. Afterward, we investigate topological structure which are completeness, AK-property, AD-property. Also, we compute the $\alpha-$, $\beta-$ and $\gamma-$ duals, and obtain bases for these sequence spaces. Finally we give necessary and sufficient conditions on matrix transformation between these new sequence spaces and c, ℓ_∞.

1. Introduction

The matrix domain plays an important role to construct a new sequence space. In studies on the sequence space, generally there are some approaches. Most important of them are determination of topologies, matrix mappings and inclusion relations. These methods are applied to study the matrix domain λ_A of an infinite matrix A in a sequence space defined by $\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}$. Recently, in literature, there are many studies done by using the matrix domain. Some of them can be found in the following list [1-10, 16, 17, 23, 26, 27].

Quite recently, some new sequence spaces are defined by using the generalized weighted mean and the generalized difference operator of order m or by combining both of them.

Now we will give short literature information in consist of recent works about the concepts mentioned above as follows: In [19], the difference sequence spaces first defined by Kızmaz. Further, the authors including Ahmad and Mursaleen [1], Çolak and Et [13], Altay and Başar [6], Karakaya and Polat [15] and the others have defined and studied new sequence spaces by considering matrices that represent difference operators and its generalizations. The article concerning this work can be found in the list of references [7, 8, 12, 18].

On the other hand, by using generalized weighted mean, several authors defined some new sequence spaces and studied some properties. Some of them are as follows: Malkowsky and Savaş [21] have defined the sequence spaces $z(u, v, \lambda)$ which consist of all sequences such that $G(u, v)$-transform of them are in $\lambda \in \left\{\ell_\infty, c, c_0, \ell_p\right\}$. Başar and Altay [3, 5] have defined the sequence spaces of nonabsolute type derived by generalized weighted mean.

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In this work, our purpose is to introduce new sequence spaces by combining the generalized weighted mean and difference matrix B^m and to investigate topological structure which are completeness, AK-and AD-properties, also to compute the $\alpha-, \beta-, \gamma-$ duals and basis of sequence spaces. In addition, we characterize some matrix mappings on these spaces.

The results related to the matrix domain of the matrix B^m are more general and more comprehensive than the corresponding consequences of matrix domain of operator Δ^m and some others. Therefore, in many ways, our work is more general than earlier studies.

2. Some Basic Definitions and Notations

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs and ℓ_1 we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to C$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a $BK-space$. The sequence spaces ℓ_∞, c and c_0 are $BK-space$ with the sup-norm defined by $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$. An FK-space λ is said to have $AK\text{-}property$, if $\phi \subset \lambda$ and $\{e^{(k)}\}$ is a basis for λ, where $e^{(k)}$ is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in N$ and $\phi = span\{e^{(k)}\}$, the set of all finitely non-zero sequences. If ϕ is dense in λ, then λ is called an *AD-space*, thus AK implies AD. For example, the spaces c_0 and cs are AK-spaces.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in N$. Then, we write $Ax = \{(Ax)_n\}$, the $A-$transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $x \in \lambda$ implies that $Ax \in \mu$, then we say that A defines a matrix mapping from λ into μ and denote it by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we mean the class of all infinite matrices A such that $A : \lambda \to \mu$. Also we denote all finite subsets of $\mathbb{N}$ by F. We write $e = (1, 1, 1, ...)$ and U for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in U$, let $1/u = (1/u_n)$. Let $u, v \in U$ and define the matrix $G(u, v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k; & \text{if } 0 \leq k \leq n \\ 0; & \text{if } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$, where u_n depends only on n and v_k only on k. The matrix $G(u, v)$, defined above, is called as *generalized weighted mean* or *factorable matrix*.

The continuous dual X' of a normed space X is defined as the space of all bounded linear functionals on X. If A is triangle, that is $a_{nk} = 0$ if $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$, and λ is a sequence space, then $f \in \lambda'_A$ if and only if $f = g \circ A,\ g \in \lambda'$.

Let X be a seminormed space. A set $Y \subset X$ is called *fundamental* if the span of Y is dense in X. One of the useful results on fundamental set which is an application of Hahn-Banach Theorem as follows: If Y is the subset of a seminormed space X and $f \in X'$, $f(Y) = 0$ implies $f = 0$, then Y is fundamental ([29], p.39).

3. The Sequence Spaces $(\ell_{\infty B})(u, v, m)$, $c_B(u, v, m)$ and $(c_{0B})(u, v, m)$

In this section, we define the new sequence spaces $(\ell_{\infty B})(u, v, m)$, $c_B(u, v, m)$ and $(c_{0B})(u, v, m)$ derived by the composition of the generalized weighted mean $G(u, v)$ and the generalized difference matrix B^m of order m, the generalization of the matrix Δ^m of the difference matrix of order m, where $m \in \mathbb{N}$. Throughout the text, ν denotes any of the spaces ℓ_∞, c and c_0. Furthermore we prove that these new sequence spaces are complete normed linear space, and we compute their $\alpha-, \beta-$ and $\gamma-$ duals. In this section additionally we give the bases for the spaces $c_B(u, v, m)$,$(c_{0B})(u, v, m)$ and finally we show that these spaces have $AK-$ and AD properties.

The work of Euler spaces of difference sequences of order m was studied by Polat Başar in [24]. Later, in [11], Başarır and Kayıkçı defined the matrix $B^m = (b^m_{nk})$ by

$$b^m_{nk} = \begin{cases} \binom{m}{n-k} r^{m-n+k} s^{n-k}, & (\max\{0, n-m\} \leq k \leq n) \\ 0, & (0 \leq k < \max\{0, n-m\}) \text{ or } (k > n) \end{cases}$$

It is easily check that if $r = 1$ and $s = -1$, we obtain that $B^m = \Delta^m$ (*see*,[2]).

We define new sequence spaces $\nu(u, v, m)$ by

$$\nu(u, v, m) = \left\{ x = (x_n) \in w : B(u, v, m)x = \left\{ \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{m}{j-k} r^{m-j+k} s^{j-k} u_n v_j x_k \right\} \in \nu \right\}.$$

We can redefine the spaces $\nu(u, v, m)$ as the matrix domain of the triangle $B(u, v, m)$ in the spaces ν, that is

$$\nu(u, v, m) = \nu_{B(u,v,m)}.$$

This definition includes the following special cases: i) If $r = 1$ and $s = -1$, $\nu(u, v, m) = \nu(u, v, \Delta^m)$. ii) If $r = 1$, $s = -1$ and $m = 1$, then $\nu(u, v, m) = \nu(u, v, \Delta)$ (*see*,[25]). iii) If $u_k = \frac{1}{\lambda_k}$, $v_i = \lambda_i - \lambda_{i-1}$, $m = 1, r = 1, s = -1$ and $\nu \in \{c, c_0\}$ then $\nu(u, v, m) = c_0^{\lambda}(\Delta)$ and $c^{\lambda}(\Delta)$(*see*, [22]).

Define the sequence $y = (y_k)$; which will be frequently used as the $B(u, v, m)$-transform of a sequence $x = (x_k)$ i.e. for $(m, n \in \mathbb{N})$,

$$y_n = \{B(u, v, m)x\}_n = \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{m}{j-k} r^{m-j+k} s^{j-k} u_n v_j x_k. \tag{1}$$

Since the proof for any one of these new sequence spaces may also be obtained in the similar way for the other spaces, to avoid the repetition of the similar statements, we give the proof for only one of those spaces.

Theorem 3.1. *The sequence space $\nu(u, v, m)$ is a complete normed linear space with respect to the norm defined by*

$$\|x\|_{\nu(u,v,m)} = \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{m}{j-k} r^{m-j+k} s^{j-k} u_n v_j x_k \right| = \|y\|_{\infty} \tag{2}$$

Proof. The linearity of $\nu(u, v, m)$ with respect to the coordinate-wise addition and scalar multiplication follows from the following inequality satisfying for $x = (x_k)$; $t = (t_k) \in \nu(u, v, m)$ and $\alpha, \beta \in \mathbb{R}$

$$\sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{m}{j-k} r^{m-j+k} s^{j-k} u_n v_j (\alpha x_k + \beta t_k) \right| \leq |\alpha| \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{m}{j-k} r^{m-j+k} s^{j-k} u_n v_j x_k \right| \tag{3}$$

$$+ |\beta| \sup_{n \in \mathbb{N}} \left| \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{m}{j-k} r^{m-j+k} s^{j-k} u_n v_j t_k \right|.$$

After this step, we must show that the spaces $\nu(u, v, m)$ holds the norm conditions and the completeness with respect to given norm. It is easy to show that (2) holds the norm condition for the spaces $\nu(u, v, m)$. We now consider the space $(\ell_{\infty B})(u, v, m)$. To prove the completeness of the space $(\ell_{\infty B})(u, v, m)$, let us take any Cauchy sequence (x^n) in the space $(\ell_{\infty B})(u, v, m)$. Then for a given $\varepsilon > 0$, there exists a positive integer $N_0(\varepsilon)$ such that $\|x^n - x^r\|_{\nu(u,v,m)} < \varepsilon$ for all $n, r > N_0(\varepsilon)$. Hence fixed $i \in \mathbb{N}$,

$$\left| B(u, v, m) \left(x_i^n - x_i^r \right) \right| < \varepsilon$$

for all $n, r \geq N_0(\varepsilon)$. Therefore the sequence $(B(u,v,m)x^n)_{i\in\mathbb{N}}$ is a Cauchy sequence of real numbers for every $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, that is;

$$(B(u,v,m)x^r)_{i\in\mathbb{N}} \to (B(u,v,m)x)_{i\in\mathbb{N}}$$

as $r \to \infty$. So we have

$$\left|B(u,v,m)\left(x_i^n - x_i\right)\right| < \varepsilon$$

for every $n \geq N_0(\varepsilon)$ and as $r \to \infty$. This implies that $\|x^n - x\|_{\nu(u,v,m)} < \varepsilon$ for every $n \geq N_0(\varepsilon)$. Now we must show that $x \in (\ell_{\infty B})(u,v,m)$. We have

$$\sup_n \left|(B(u,v,m)x)_n\right| \leq \|x^n\|_{\nu(u,v,m)} + \|x^n - x\|_{\nu(u,v,m)} = O(1).$$

This implies that $x = (x_i) \in (\ell_{\infty B})(u,v,m)$. Therefore $(\ell_{\infty B})(u,v,m)$ is a Banach space. It can be shown that $c_B(u,v,m)$ and $(c_{0B})(u,v,m)$ are closed subspaces of $(\ell_{\infty B})(u,v,m)$ which leads us to the consequence that the spaces $c_B(u,v,m)$ and $(c_{0B})(u,v,m)$ are also the Banach spaces with the norm (2).

Furthermore, since $(\ell_{\infty B})(u,v,m)$ is a Banach space with the continuous coordinates, i.e.;

$\|B(u,v,m)(x^n - x)\|_{\nu(u,v,m)} \to 0$ implies $\left|B(u,v,m)\left(x_i^n - x_i\right)\right| \to 0$ for all $i \in \mathbb{N}$. Therefore, it is a *BK*-space. □

Theorem 3.2. *The sequence spaces $(\ell_{\infty B})(u,v,m)$, $c_B(u,v,m)$ and $(c_{0B})(u,v,m)$ are linearly isomorphic to the spaces ℓ_∞, c, and c_0, respectively, i.e., $(\ell_{\infty B})(u,v,m) \cong \ell_\infty$, $c_B(u,v,m) \cong c$ and $(c_{0B})(u,v,m) \cong c_0$.*

Proof. To prove the fact $(c_{0B})(u,v,m) \cong c_0$, we should show the existence of a linear bijection between the spaces $(c_{0B})(u,v,m)$ and c_0. Consider the transformation T defined with the notation (1), from $(c_{0B})(u,v,m)$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Let $y \in c_0$ and define the sequence $x = (x_k)$ by

$$x_n = \sum_{k=1}^{n} \left[\frac{(-1)^{n-k}}{u_k} \sum_{i=k}^{k+1} \binom{m+n-i-1}{n-i} \frac{s^{n-i}}{r^{m+n-i}\, v_i} \right] y_k. \tag{4}$$

Then

$$\lim_{n\to\infty} (B(u,v,m)x)_n = \lim_{n\to\infty} y_n = 0.$$

Thus we have that $x \in (c_{0B})(u,v,m)$. Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says us that the spaces $(c_{0B})(u,v,m)$ and c_0 are linearly isomorphic. In the same way, it can be shown that $c_B(u,v,m)$ and $(\ell_{\infty B})(u,v,m)$ are linearly isomorphic to c and ℓ_∞, respectively, and so we omit the details. □

Now we give the definition of a Schauder basis of a normed space. If a normed sequence space ν contains a sequence (b_n) such that, for every $x \in \nu$, there is unique sequence of scalars (α_n) for which

$$\left\| x - \sum_{k=0}^{n} \alpha_n b_k \right\| \to 0 \text{ as } n \to \infty.$$

Then (b_n) is called a *Schauder basis* for ν. The series $\sum \alpha_k b_k$ has the sum x that is called the *expansion* of x in (b_n), and we write $x = \sum \alpha_k b_k$, (Maddox [20]; p.98).

Because of the isomorphism T defined in Theorem 3.2 the inverse image of the bases of spaces c_0 and c is onto, and so they are the bases of the new spaces $(c_{0B})(u,v,m)$ and $c_B(u,v,m)$, respectively. Therefore, we give the following theorem without proof.

Theorem 3.3. *Let $\alpha_k = (B(u,v,m)x)_k$ for all $k \in \mathbb{N}$. Define the sequence $d^{(k)} = \left\{d_n^{(k)}\right\}_{n\in\mathbb{N}}$ of the elements of the space $(c_{0B})(u,v,m)$ by*

$$d_n^{(k)}(B^m) = \begin{cases} 0; & (n<k) \\ \frac{(-1)^{n-k}}{u_k}\sum_{i=k}^{k+1}\begin{pmatrix} m+n-i-1 \\ n-i \end{pmatrix}\frac{s^{n-i}}{r^{m+n-i}v_i}; & (n \geq k) \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then the following assertions are true:

i) The sequence $\left\{d^{(k)}\right\}_{k\in\mathbb{N}}$ is basis for the space $(c_{0B})(u,v,m)$, and any $x \in (c_{0B})(u,v,m)$ has a unique representation in the form

$$x = \sum_k \alpha_k d^{(k)}$$

ii) The set $\left\{e, b^{(k)}\right\}$ is a basis for the space $c_B(u,v,m)$, and any $x \in c_B(u,v,m)$ has a unique representation in the form

$$x = le + \sum_k (\alpha_k - l)\, d^{(k)},$$

where $l = \lim_{k\to\infty}(B(u,v,m)x)_k$.

Theorem 3.4. *The sequence space $(c_{0B})(u,v,m)$ has AD-property whenever $u \in c_0$.*

Proof. For this, we prove that the set ϕ, the space of all finitely non-zero sequences, is dense in $(c_{0B})(u,v,m)$. Suppose that $f \in [(c_{0B})(u,v,m)]'$. Then there exists a functional g over the space c_0 such that $f(x) = g(B(u,v,m)x)$ for some $g \in c_0' = \ell_1$. Since c_0 has $AK-property$ and $c_0' \cong \ell_1$, we have

$$f(x) = \sum_{j=0}^{\infty} a_j (B(u,v,m)x)_j$$

for some $a = (a_j) \in \ell_1$. Since the matrix domain generated by $B(u,v,m)$ over c_0 is a expansion, the inclusion $c_0 \subset (c_{0B})(u,v,m)$ holds. Hence the inclusion $\phi \subset (c_{0B})(u,v,m)$ holds. For any $f \in [(c_{0B})(u,v,m)]'$ and $e^{(k)} \in \phi \subset (c_{0B})(u,v,m)$, we have

$$f\left(e^{(k)}\right) = \sum_{j=1}^{\infty} a_j \left(B(u,v,m)e^{(k)}\right)_j = \left\{B'(u,v,m)a\right\}_k$$

where $B'(u,v,m)$ is the transpose of the matrix $B(u,v,m)$. Hence, from Hahn-Banach Theorem, ϕ is dense in $(c_{0B})(u,v,m)$ if and only if $B'(u,v,m)a = \theta$ for $a \in \ell_1$ implies $a = \theta$. Since the null space of the operator $B'(u,v,m)$ on w is $\{\theta\}$, $(c_{0B})(u,v,m)$ has AD property. □

We now give the details about duals of the sequence spaces $(\ell_{\infty B})(u,v,m)$, $c_B(u,v,m)$ and $(c_{0B})(u,v,m)$. For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \tag{5}$$

With notation of (5), the $\alpha-$, $\beta-$ and $\gamma-$ duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined in [14] by

$$\lambda^\alpha = S(\lambda,\ell_1),\ \lambda^\beta = S(\lambda,cs) \text{ and } \lambda^\gamma = S(\lambda,bs).$$

Now, we need the following Lemmas due to Stieglitz and Tietz [28] for the next theorems.

Lemma 3.5. *$A \in (c_0 : \ell_1)$ if and only if*

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Lemma 3.6. *$A \in (c_0 : c)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty,$$

$$\lim_{n \to \infty} a_{nk} - \alpha_k = 0.$$

Lemma 3.7. *$A \in (c_0 : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty.$$

Theorem 3.8. *Let $u, v \in U$, $a = (a_k) \in w$ and the matrix $V^m = \left(v^m_{nk}\right)$ by*

$$v^m_{nk} = \begin{cases} \frac{(-1)^{n-k}}{u_k} \sum_{i=k}^{k+1} \begin{pmatrix} m+n-i-1 \\ n-i \end{pmatrix} \frac{s^{n-i}}{r^{m+n-i} v_i} a_n; & (0 \le n \le k) \\ 0; & (k > n) \end{cases} \tag{6}$$

for all $k, n \in \mathbb{N}$. Then the $\alpha-$dual of the space $\nu(u, v; m)$ is the set

$$b^m_B = \left\{ a = (a_n) \in w : \sup_{K \in F} \sum_n \left| \sum_{k \in K} v^m_{nk} \right| < \infty \right\}.$$

Proof. Let $a = (a_n) \in w$ and consider the matrix $B^{-1}(u, v, m)$ which is inverse of the matrix $B(u, v, m)$ and sequence $a = (a_n)$. Bearing in mind the relation (1), we immediately derive that

$$a_n x_n = \sum_{k=1}^{n} \frac{(-1)^{n-k} a_n}{u_k} \sum_{i=k}^{k+1} \begin{pmatrix} m+n-i-1 \\ n-i \end{pmatrix} \frac{s^{n-i}}{r^{m+n-i} v_i} y_k = (V^m y)_n \tag{7}$$

for all $n, k \in \mathbb{N}$. We therefore observe by (7) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \nu(u, v; m)$ if and only if $V^m y \in \ell_1$ whenever $y \in \nu$. Then, we derive by Lemma (3.5) that

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} v^m_{nk} \right| < \infty$$

which yields the consequence that $[(c_{0B})(u, v, m)]^\alpha = [c_B(u, v, m)]^\alpha = [(\ell_{\infty B})(u, v, m)]^\alpha = b^m_B$. □

Theorem 3.9. *Let $u, v \in U$, $a = (a_n) \in w$ and the matrix $C = (c_{nk})$ defined by*

$$c^m_{nk} = \begin{cases} \frac{1}{u_k} \sum_{i=k}^{n} (-1)^{n-k} a_i \sum_{p=i}^{i+1} \begin{pmatrix} m+i-p-1 \\ i-p \end{pmatrix} \frac{s^{i-p}}{r^{m+i-p} v_p}; & (0 < k \le n) \\ 0; & (k > n) \end{cases} \tag{8}$$

and also define the sets c_{1B}, c_{2B}, c_{3B}, c_{4B} by

$$\begin{aligned} c_{1B} &= \left\{ a = (a_n) \in w : \sup_n \sum_n |c_{nk}| < \infty \right\}; \\ c_{2B} &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} c_{nk} \text{ exists for each } k \in N \right\}; \\ c_{3B} &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n \to \infty} c_{nk} \right| \right\}; \\ c_{4B} &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} \sum_k c_{nk} \text{ exists} \right\}. \end{aligned}$$

Then, the sets $[c_{0B}(u,v,m)]^{\beta}$, $[c_B(u,v,m)]^{\beta}$ and $[\ell_{\infty B}(u,v,m)]^{\beta}$ are the sets $c_{1B}\cap c_{2B}$, $c_{1B}\cap c_{2B}\cap c_{4B}$ and $c_{2B}\cap c_{3B}$ respectively.

Proof. Since the proof may be obtained by the similar way for the spaces $c_B(u,v,m)$ and $(\ell_{\infty B})(u,v,m)$, we only give the proof for the space$(c_{0B})(u,v,m)$. Consider the equation with (8)

$$\begin{aligned}\sum_{k=1}^{n} a_k x_k &= \sum_{k=1}^{n}\left[\sum_{i=1}^{k}\frac{(-1)^{n-k}}{u_k}\sum_{i=k}^{k+1}\begin{pmatrix} m+n-i-1 \\ n-i\end{pmatrix}\frac{s^{n-i}}{r^{m+n-i}v_i}y_i\right]a_k \\ &= (C^m y)_n .\end{aligned} \tag{9}$$

Thus, we deduce from Lemma 3.6 and (9) that $ax=(a_nx_n)\in cs$ whenever $x\in c_{0B}(u,v,m)$ if and only if $C^m y\in cs$ whenever $y\in c_0$. Therefore we derive by Lemma 3.6 which shows that $\{(c_{0B})(u,v,m)\}^{\beta}=c_{1B}\cap c_{2B}$. □

Theorem 3.10. *The $\gamma-$dual of $v(u,v;m)$ is the set c_{1B}.*

Proof. This may be obtained in the similar way used in the proof of Theorem 3.9 with Lemma 3.7 instead of Lemma 3.6. So, we omit the details. □

4. Matrix Transformations on Space $c_B(u,v,m)$

In this section, we directly prove the theorems which characterize the classes $(c_B(u,v,m):\ell_\infty)$ and $A\in(c_B(u,v,m):c)$. We shall write throughout for brevity that

$$\tilde{a}_{nk}=\frac{a_{nk}}{u_k}\sum_{p=k}^{\infty}\left[(-1)^{p-k}\sum_{j=p}^{p+1}\begin{pmatrix} m+p-j-1 \\ p-j\end{pmatrix}\frac{s^{p-j}}{r^{m+p-j}v_j}\right]$$

for all $n,k\in\mathbb{N}$.

Theorem 4.1. *$A\in(c_B(u,v,m):\ell_\infty)$ if and only if*

$$\sup_n\sum_{k=0}^{\infty}\left|\frac{a_{nk}}{u_k}\sum_{p=k}^{\infty}\left[(-1)^{p-k}\sum_{j=p}^{p+1}\begin{pmatrix} m+p-j-1 \\ p-j\end{pmatrix}\frac{s^{p-j}}{r^{m+p-j}v_j}\right]\right|<\infty, \tag{10}$$

exists for all $k,n\in\mathbb{N}$.

Proof. Suppose that $A\in(c_B(u,v,m):\ell_\infty)$. Then Ax exists and is in ℓ_∞ for all $x\in c_B(u,v,m)$. So, we can consider the following equality

$$\sum_{k=0}^{n}a_{nk}x_k=\sum_{k=0}^{n}\frac{a_{nk}}{u_k}\sum_{p=k}^{n}\left[(-1)^{p-k}\sum_{j=p}^{p+1}\begin{pmatrix} m+p-j-1 \\ p-j\end{pmatrix}\frac{s^{p-j}}{r^{m+p-j}v_j}\right]y_k\,;(n\in N).$$

which yields us under our assumptions as $n\to\infty$ that

$$\sum_{k=0}^{\infty}a_{nk}x_k=\sum_{k=0}^{\infty}\frac{a_{nk}}{u_k}\sum_{p=k}^{\infty}\left[(-1)^{p-k}\sum_{j=p}^{p+1}\begin{pmatrix} m+p-j-1 \\ p-j\end{pmatrix}\frac{s^{p-j}}{r^{m+p-j}v_j}\right]y_k\,;\ (n\in N). \tag{11}$$

Using (11) under our assumptions. we get that

$$\sup_n\sum_{k=0}^{\infty}|\tilde{a}_{nk}|<\infty$$

Conversely we assume that (10) holds. Then we have to show that $A \in (c_B(u,v,m) : \ell_\infty)$. To do this, let $\tilde{A} = (\tilde{a}_{nk})$ be a matrix connected with $A = (a_{nk})$. For any $y \in \ell_\infty$ and from (10), we have $\left|\tilde{A}y\right| < \sup_n \sum_{k=0}^{\infty} |\tilde{a}_{nk}| < \infty$. So $\tilde{A}y \in \ell_\infty$. By considering (1), we get

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk}y_k \tag{12}$$

This means that $A \in (c_B(u,v,m) : \ell_\infty)$. Hence this completes the proof. □

Theorem 4.2. *$A \in (c_B(u,v,m) : c)$ if and only if (10) holds, and*

$$\lim_{n\to\infty} \tilde{a}_{nk} = \alpha_k \text{ for each } k \in N \tag{13}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \tilde{a}_{nk} = \alpha \tag{14}$$

Proof. Suppose that A satisfies the conditions (10), (13) and (14). Let us take any $x \in c_B(u,v,m)$. Then Ax exists and it is trivial that the sequence $y = (y_k)$ connected with the sequence $x = (x_k)$ by the relation (1) is in c such that $y_k \to l$ as $k \to \infty$. From (10) and (13), we have

$$\sum_{j=0}^{k} \left|\alpha_j\right| \leq \sup_n \sum_{j=0}^{k} \left|\tilde{a}_{nj}\right| < \infty$$

holds for every $k \in N$. Hence we get $(\alpha_k) \in \ell_1$. Now considering (12), let us write

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk}(y_k - l) + l\sum_{k=0}^{\infty} \tilde{a}_{nk} \tag{15}$$

After by letting $n \to \infty$ in (15), we get

$$(Ax)_n \to \sum_{k=0}^{\infty} \alpha_k (y_k - l) + l\alpha$$

which shows that $A \in (c_B(u,v,m) : c)$.

Conversely suppose that $A \in (c_B(u,v,m) : c)$. Then since the inclusion $c \subset \ell_\infty$ holds, the necessity of (10) is immediately obtained from Theorem 4.1. To prove the necessity of (13), consider the sequence $x = x^{(k)} = \left\{x_n^{(k)}\right\} \in c_B(u,v,m)$ defined by

$$x_n^{(k)} = \begin{cases} \frac{(-1)^{n-k}}{u_k} \sum_{i=k}^{k+1} \begin{pmatrix} m+n-i-1 \\ n-i \end{pmatrix} \frac{s^{n-i}}{r^{m+n-i}v_i}; & (0 \leq n \leq k) \\ 0; & (k > n) \end{cases}$$

for each $k \in N$. Since Ax exists and in c for every $x \in c_B(u,v,m)$, one can easily see that

$Ax^{(k)} = \{\tilde{a}_{nk}\}_{n\in N} \in c$ for each $k \in N$ which shows the necessity of (13). Similarly by putting $x = e$ in (12), we also obtain that $Ax = \left\{\sum_{k=0}^{\infty} \tilde{a}_{nk}\right\}_{n\in N}$ belongs to the space c and this shows the necessity of (14). Hence this completes the proof. □

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