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# Inverse Problems for Sturm–Liouville Difference Equations

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**Abstract.** We consider a discrete Sturm–Liouville problem with Dirichlet boundary conditions. We show that the specification of the eigenvalues and weight numbers uniquely determines the potential. Moreover, we also show that if the potential is symmetric, then it is uniquely determined by the specification of the eigenvalues. These are discrete versions of well-known results for corresponding differential equations.

## 1. Introduction

Consider the eigenvalue problem consisting of the Sturm–Liouville differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in \mathbb{R} \tag{1}$$

and Dirichlet boundary conditions

$$y(0) = y(1) = 0, \tag{2}$$

where the square integrable function $q : \mathbb{R} \to \mathbb{R}$ is referred to as the potential. In the theory of inverse problems, it is assumed that certain spectral data are known, and the problem is to find the potential. This theory is well developed in the continuous case, and we refer to [3, 4, 6, 8, 10–12] for further reading. It is well known [9, Chapter 0] that the problem (1)–(2) has infinitely many simple real eigenvalues $\lambda_1 < \lambda_2 < \dots$ with corresponding orthogonal eigenfunctions. Defining the weight numbers by the integral over $[0, 1]$ of the square of normalized eigenfunctions, we have the following well-known result.

**Theorem 1.1 (See [4, Theorem 1.4.2]).** *If two eigenvalue problems of the form* (1)–(2) *have the same eigenvalues and the same weight numbers, then their potentials are the same almost everywhere on* $[0, 1]$.

Moreover, if the spectrum is symmetric, i.e., $q(x) = q(1 - x)$ for all $x \in [0, 1]$, then we have the following improvement of Theorem 1.1.

**Theorem 1.2 (See [4, Theorem 1.4.3]).** *If two eigenvalue problems of the form* (1)–(2) *have symmetric potentials and the same eigenvalues, then their potentials are the same almost everywhere on* $[0, 1]$.

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In this paper, we let $N \in \mathbb{N}$ and consider the discrete eigenvalue problem consisting of the Sturm–Liouville difference equation

$$-\Delta^2 y_k + q_k y_{k+1} = \lambda y_{k+1}, \quad k \in \mathbb{Z} \tag{3}$$

and Dirichlet boundary conditions

$$y_0 = y_{N+1} = 0, \tag{4}$$

where the sequence $q = \{q_k\}_{k \in \mathbb{Z}}$ is referred to as the *potential*. As usual, $\Delta$ is the forward difference operator (see, e.g., [1, 7]) defined by

$$\Delta y_k = y_{k+1} - y_k \quad \text{so that} \quad \Delta^2 y_k = y_{k+2} - 2y_{k+1} + y_k.$$

It is well known [2, Section 4.5] that the problem (3)–(4) has $N$ simple real eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ with corresponding orthogonal eigenfunctions. Defining the weight numbers as the sum over $[0, N] \cap \mathbb{Z}$ of the squares of normalized eigenfunctions, we will prove the following analogue of Theorem 1.1.

**Theorem 1.3.** *If two eigenvalue problems of the form* (3)–(4) *have the same eigenvalues and the same weight numbers, then their potentials are exactly the same on* $[0, N-1] \cap \mathbb{Z}$.

Moreover, if the spectrum is symmetric, i.e., $q_k = q_{N-1-k}$ for all $k \in [0, N-1] \cap \mathbb{Z}$, then we have the following improvement of Theorem 1.3.

**Theorem 1.4.** *If two eigenvalue problems of the form* (3)–(4) *have symmetric potentials and the same eigenvalues, then their potentials are exactly the same on* $[0, N-1] \cap \mathbb{Z}$.

The set up of this paper is summarized as follows. Section 2 contains some preliminary results about the discrete eigenvalue problem (3)–(4) and the proof of the uniqueness result, Theorem 1.3. We also refer to [13] for the more general case of Jacobi operators. We note that our proof, unlike the proof of the corresponding continuous result, Theorem 1.1, follows neither the methods of Marčenkov [12] (who uses Parseval's equality) nor Levinson [10] (who uses the contour integral method) but is based on a matrix method that is taylored specifically to the discrete case. In Section 3, we discuss the case of symmetric potentials and prove Theorem 1.4. We provide a simple proof based on Theorem 1.3. Theorem 1.4 is not new, it has been proved directly with different methods in [5, Theorem 1]. In Section 4, we present an example. Finally, we offer some remarks and directions for future research in Section 5.

## 2. The Uniqueness Result

We introduce the notation used in this article. Let $\varphi(\lambda)$ and $\psi(\lambda)$ be the (clearly unique) solutions of (3) satisfying

$$\varphi_0(\lambda) = 0, \quad \Delta\varphi_0(\lambda) = 1 \quad \text{and} \quad \psi_{N+1}(\lambda) = 0, \quad \Delta\psi_{N+1}(\lambda) = 1$$

and define the *characteristic function* $\Omega : \mathbb{R} \to \mathbb{R}$ by

$$\Omega(\lambda) := \psi_k(\lambda)\Delta\varphi_k(\lambda) - \varphi_k(\lambda)\Delta\psi_k(\lambda) = -\varphi_{N+1}(\lambda) = \psi_0(\lambda)$$

(and we note that this expression does not depend on $k \in \mathbb{Z}$ as its forward difference can easily be seen, using the discrete product rule, to be equal to zero). It is clear that the zeros of the characteristic functions are the eigenvalues of the problem (3)–(4), and it can easily be seen from (3) that $\Omega$ is a polynomial of degree $N$ with leading coefficient $(-1)^{N-1}$ so that $\Omega$ can be written as

$$\Omega(\lambda) = (-1)^{N-1} \prod_{j=1}^{N} (\lambda - \lambda_j), \tag{5}$$

where $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ are the (real and simple) zeros of $\Omega$ and hence the eigenvalues of (3)–(4).

We now consider the equation (3) with $q_k \equiv 0$, i.e.,

$$-\Delta^2 y_k = \lambda y_{k+1}, \quad k \in \mathbb{Z}, \tag{6}$$

and denote by $S(\lambda)$ the (again unique) solution of (6) satisfying the initial conditions $S_0(\lambda) = 0$ and $\Delta S_0(\lambda) = 1$. We show the following crucial auxiliary result.

**Lemma 2.1.** *Define a function* $K : \mathbb{N}_0^2 \to \mathbb{R}$ *by*

$$\begin{aligned} &K(i,j) := 0 \quad \text{for all} \quad 0 \le j < i, \\ &K(i,i) := 1 \quad \text{for all} \quad i \in \mathbb{N}_0, \\ &K(i,0) := 0 \quad \text{for all} \quad i \in \mathbb{N}, \end{aligned}$$

*and for* $i \in \mathbb{N}$, $1 \le j \le i$, *recursively by*

$$K(i+1,j) := K(i,j-1) + K(i,j+1) + q_{i-1}K(i,j) - K(i-1,j). \tag{7}$$

*Then*

$$\varphi_i(\lambda) = \sum_{k=0}^{i} K(i,k)S_k(\lambda) \tag{8}$$

*for all* $\lambda \in \mathbb{R}$ *and all* $i \in \mathbb{N}_0$.

*Proof.* We prove (8) by induction. First, note that

$$\varphi_0(\lambda) = 0 = K(0,0)S_0(\lambda)$$

and

$$\varphi_1(\lambda) = 1 = K(1,0)S_0(\lambda) + K(1,1)S_1(\lambda)$$

show that (8) holds for $i = 0$ and $i = 1$. Now we assume that (8) holds for all $i \in [0,m] \cap \mathbb{Z}$ with some $m \in \mathbb{N}$. Then

$$\begin{aligned}
\varphi_{m+1}(\lambda) &= (2-\lambda+q_{m-1})\varphi_m(\lambda) - \varphi_{m-1}(\lambda) \\
&= (2-\lambda+q_{m-1})\sum_{k=0}^{m} K(m,k)S_k(\lambda) - \sum_{k=0}^{m-1} K(m-1,k)S_k(\lambda) \\
&= \sum_{k=1}^{m} K(m,k)(2-\lambda)S_k(\lambda) + \sum_{k=1}^{m} q_{m-1}K(m,k)S_k(\lambda) - \sum_{k=1}^{m-1} K(m-1,k)S_k(\lambda) \\
&= \sum_{k=1}^{m} K(m,k)(S_{k+1}(\lambda)+S_{k-1}(\lambda)) + \sum_{k=1}^{m} q_{m-1}K(m,k)S_k(\lambda) - \sum_{k=1}^{m-1} K(m-1,k)S_k(\lambda) \\
&= \sum_{k=2}^{m+1} K(m,k-1)S_k(\lambda) + \sum_{k=0}^{m-1} K(m,k+1)S_k(\lambda) + \sum_{k=1}^{m} q_{m-1}K(m,k)S_k(\lambda) - \sum_{k=1}^{m-1} K(m-1,k)S_k(\lambda) \\
&= \sum_{k=1}^{m+1} K(m,k-1)S_k(\lambda) + \sum_{k=1}^{m} K(m,k+1)S_k(\lambda) + \sum_{k=1}^{m} q_{m-1}K(m,k)S_k(\lambda) - \sum_{k=1}^{m} K(m-1,k)S_k(\lambda) \\
&= \sum_{k=0}^{m+1} K(m+1,k)S_k(\lambda).
\end{aligned}$$

Thus (8) holds for $i = m+1$. This completes the proof. □

**Corollary 2.2.** *Define the $N \times N$-matrices $\Phi$ and $\mathcal{S}$ by defining its entry in the ith row and jth column, $1 \le i, j \le N$, by*

$$\Phi_{ij} = \varphi_i(\lambda_j) \quad \text{and} \quad \mathcal{S}_{ij} = S_i(\lambda_j).$$

*Then there exists a lower triangular matrix $\mathcal{K}$ with entries 1 on the diagonal and independent of $\lambda$ such that*

$$\Phi = \mathcal{K}\mathcal{S}.$$

*Proof.* By Lemma 2.1, we have

$$\varphi_i(\lambda_j) = \sum_{k=1}^{i} K(i,k) S_k(\lambda_j) \quad \text{and} \quad K(i,i) = 1$$

for all $i, j \in \{1, \dots, N\}$. Hence, by defining $\mathcal{K}$ by defining its entry in the $i$th row and $j$th column, $1 \le i, j \le N$, by

$$\mathcal{K}_{ij} = K(i,j),$$

we arrive at $\Phi = \mathcal{K}\mathcal{S}$. □

**Remark 2.3.** *Let us define the* weight numbers

$$\alpha_j := \sum_{i=1}^{N} (\varphi_i(\lambda_j))^2 \quad \text{for} \quad j \in \{1, \dots, N\}. \tag{9}$$

*Note that $\Phi^T \Phi$ is a diagonal matrix with diagonal entries $\alpha_j$.*

Now we are ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* Together with (3)–(4), we consider the same problem with $q$ replaced by $\tilde{q}$. The eigenvalues, weight numbers etc. of that eigenvalue problem are denoted by $\tilde{\lambda}$, $\tilde{\alpha}_j$ and so on. By Corollary 2.2, we have

$$\Phi = \mathcal{K}\mathcal{S} \quad \text{and} \quad \tilde{\Phi} = \tilde{\mathcal{K}}\mathcal{S}. \tag{10}$$

From the assumption, we have

$$\mathcal{S}^T \mathcal{K}^T \mathcal{K} \mathcal{S} = \Phi^T \Phi = \tilde{\Phi}^T \tilde{\Phi} = \mathcal{S}^T \tilde{\mathcal{K}}^T \tilde{\mathcal{K}} \mathcal{S}. \tag{11}$$

Due to Remark 2.3, each occurring matrix in (11) is invertible. Thus we may multiply (11) from the left with $(\mathcal{S}^T)^{-1}$ and from the right with $\mathcal{S}^{-1}$ to arrive at

$$\mathcal{K}^T \mathcal{K} = \tilde{\mathcal{K}}^T \tilde{\mathcal{K}}. \tag{12}$$

By Corollary 2.2, both $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are lower triangular matrices with 1 on the diagonal, and hence it is easy to show that (12) implies

$$\mathcal{K} = \tilde{\mathcal{K}}. \tag{13}$$

Using (13) in (10), we find $\Phi = \mathcal{K}\mathcal{S} = \tilde{\mathcal{K}}\mathcal{S} = \tilde{\Phi}$ i.e.,

$$\varphi_i(\lambda_j) = \tilde{\varphi}_i(\lambda_j) \quad \text{for all} \quad i, j \in \{1, \dots, N\}.$$

Inserting $\varphi_i(\lambda_1) = \tilde{\varphi}_i(\lambda_1)$ for all $i \in \{0, \dots N+1\}$ in (3) and using that $\varphi_i(\lambda_1) \neq 0$ for all $i \in \{1, \dots, N\}$ (it is known [1, 2, 7] that the $j$th eigenfunction has exactly $j-1$ generalized zeros in the open interval $(0, N+1)$; in particular, the first eigenfunction has no generalized zero in the open interval $(0, N+1)$ and hence no zero in there), the claim follows. □

## 3. Symmetric Potentials

Since both $\varphi(\lambda_j)$ and $\psi(\lambda_j)$ are eigenfunctions corresponding to the eigenvalue $\lambda_j$, $j \in \{1, \dots, N\}$, there exist numbers $\beta_j \in \mathbb{R}$ such that

$$\psi_i(\lambda_j) = \beta_j \varphi_i(\lambda_j) \tag{14}$$

for all $i \in \mathbb{N}_0$ and $j \in \{1, \dots, N\}$. We now show the following crucial auxiliary result.

**Lemma 3.1.** *The numbers $\beta_j$ from* (14) *and the weight numbers $\alpha_j$ from* (9) *satisfy the relation*

$$\beta_j \alpha_j = -\dot{\Omega}(\lambda_j), \quad j \in \{1, \dots, N\}. \tag{15}$$

*Proof.* Using the discrete product rule and (3), we find for any $i \in \mathbb{N}_0$ and $\lambda, \mu \in \mathbb{R}$ that

$$\begin{aligned} &\Delta\left[\psi_k(\mu)\Delta\varphi_k(\lambda) - \varphi_k(\lambda)\Delta\psi_k(\mu)\right] \\ &\quad = \psi_{k+1}(\mu)\Delta^2\varphi_k(\lambda) + (\Delta\psi_k(\mu))(\Delta\varphi_k(\lambda)) - \varphi_{k+1}(\lambda)\Delta^2\psi_k(\mu) - (\Delta\varphi_k(\lambda))(\Delta\psi_k(\mu)) \\ &\quad = \psi_{k+1}(\mu)(q_k - \lambda)\varphi_{k+1}(\lambda) - \varphi_{k+1}(\lambda)(q_k - \mu)\psi_{k+1}(\mu) \\ &\quad = (\mu - \lambda)\varphi_{k+1}(\lambda)\psi_{k+1}(\mu). \end{aligned}$$

Summing this equation from $k = 0$ until $k = N$, we obtain

$$\sum_{k=0}^{N} \varphi_{k+1}(\lambda)\psi_{k+1}(\mu) = \frac{\Omega(\lambda) - \Omega(\mu)}{\mu - \lambda}$$

from which we deduce

$$\sum_{k=0}^{N} \varphi_{k+1}(\lambda)\psi_{k+1}(\lambda) = -\dot{\Omega}(\lambda)$$

since $\psi$, being a polynomial, is continuous in $\lambda$. Using this for $\lambda = \lambda_j$ and in view of (9) and (14), the proof is complete. □

Now we are ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* Using the same notation as in the proof of Theorem 1.3, we show that $\alpha_j = \tilde{\alpha}_j$ for all $j \in \{1, \dots, N\}$, and then the statement follows from Theorem 1.3. Let $j \in \{1, \dots, N\}$ and note that

$$\varphi_0(\lambda_j) = \varphi_{N+1}(\lambda_j) = 0$$

and

$$\Delta^2\varphi_k(\lambda_j) = (q_k - \lambda_j)\varphi_{k+1}(\lambda_j)$$

holds for all $k \in \mathbb{Z}$. Let us introduce a new sequence $\omega$ by

$$\omega_k := \varphi_{N+1-k}(\lambda_j), \quad k \in \mathbb{Z}.$$

Then we have

$$\omega_0 = \omega_{N+1} = 0$$

and for $k \in \mathbb{Z}$,

$$\begin{aligned} (q_k - \lambda_j)\omega_{k+1} &= (q_k - \lambda_j)\varphi_{N+1-(k+1)}(\lambda_j) \\ &= (q_{N-k-1} - \lambda_j)\varphi_{(N-k-1)+1}(\lambda_j) = \Delta^2\varphi_{N-k-1}(\lambda_j) \\ &= \varphi_{(N-k-1)+2}(\lambda_j) - 2\varphi_{(N-k-1)+1}(\lambda_j) + \varphi_{N-k-1}(\lambda_j) \\ &= \varphi_{N+1-(k+2)}(\lambda_j) - 2\varphi_{N+1-(k+1)}(\lambda_j) + \varphi_{N+1-k}(\lambda_j) \\ &= \omega_{k+2} - 2\omega_{k+1} + \omega_k = \Delta^2\omega_k. \end{aligned}$$

Hence $\omega$ is also an eigenfunction corresponding to the eigenvalue $\lambda_j$. In fact, since

$$\omega_N = \varphi_1(\lambda_j) = 1 = -(-1) = -\psi_N(\lambda_j),$$

we have $\omega = -\psi(\lambda_j)$ and thus

$$\varphi_{N+1-k}(\lambda_j) = \omega_k = -\psi_k(\lambda_j) = -\beta_j\varphi_k(\lambda_j)$$

(observe (14)) and therefore

$$\varphi_k(\lambda_j) = -\beta_j\varphi_{N+1-k}(\lambda_j) = \beta_j^2\varphi_k(\lambda_j)$$

so that $\beta_j \in \{-1, 1\}$. Using this in Lemmma 3.1, we find

$$0 < \alpha_j = \beta_j^2\alpha_j = -\beta_j\dot{\Omega}(\lambda_j),$$

and since (5) implies $\Omega(\lambda) < 0$ for all $\lambda < \lambda_1$ and hence $\dot{\Omega}(\lambda_1) > 0$ and thus $(-1)^{j-1}\dot{\Omega}(\lambda_j) > 0$, we conclude that

$$\beta_j = (-1)^j \quad \text{and} \quad \alpha_j = (-1)^{j-1}\dot{\Omega}(\lambda_j).$$

The same holds for $\tilde{\alpha}_j$, so indeed we have $\alpha_j = \tilde{\alpha}_j$. □

## 4. An Example

Now we look at a simple example with $N = 2$. We have

$$\varphi_2(\lambda) = 2 - \lambda + q_0, \quad \varphi_3(\lambda) = (2 - \lambda + q_1)(2 - \lambda + q_0) - 1$$

and $\psi_2(\lambda) = -1$,

$$\psi_1(\lambda) = -(2 - \lambda + q_1), \quad \psi_0(\lambda) = -(2 - \lambda + q_0)(2 - \lambda + q_1) + 1.$$

Then

$$\begin{aligned} \Omega(\lambda) &= -(2 - \lambda + q_0)(2 - \lambda + q_1) + 1 \\ &= -\lambda^2 + (4 + q_0 + q_1)\lambda + 3 + 2q_0 + 2q_1 + q_0q_1 \\ &= -(\lambda - \lambda_1)(\lambda - \lambda_2), \end{aligned}$$

where the two eigenvalues are

$$\lambda_1 = \frac{4 + q_0 + q_1 - q}{2} \quad \text{and} \quad \lambda_2 = \frac{4 + q_0 + q_1 + q}{2}$$

with

$$q = \sqrt{4 + (q_0 - q_1)^2}.$$

Thus we have

$$\begin{aligned} &\varphi_0(\lambda_1) = 0, \quad \varphi_1(\lambda_1) = 1, \quad \varphi_2(\lambda_1) = \frac{q_0 - q_1 + q}{2}, \quad \varphi_3(\lambda_1) = 0, \\ &\varphi_0(\lambda_2) = 0, \quad \varphi_1(\lambda_2) = 1, \quad \varphi_2(\lambda_2) = \frac{q_0 - q_1 - q}{2}, \quad \varphi_3(\lambda_2) = 0, \\ &\psi_0(\lambda_1) = 0, \quad \psi_1(\lambda_1) = \frac{q_0 - q_1 - q}{2}, \quad \psi_2(\lambda_1) = -1, \quad \psi_3(\lambda_1) = 0, \\ &\psi_0(\lambda_2) = 0, \quad \psi_1(\lambda_2) = \frac{q_0 - q_1 + q}{2}, \quad \psi_2(\lambda_2) = -1, \quad \psi_3(\lambda_2) = 0. \end{aligned}$$

Note that $\varphi(\lambda_1)$ and $\psi(\lambda_1)$ have no generalized zero in $(0, 3)$, while $\varphi(\lambda_2)$ and $\psi(\lambda_2)$ each has one generalized zero in $(0, 3)$. Next, we have

$$\alpha_1 = \frac{q^2 + q(q_0 - q_1)}{2}, \quad \alpha_2 = \frac{q^2 - q(q_0 - q_1)}{2}$$

and

$$\beta_1 = \frac{q_0 - q_1 - q}{2}, \quad \beta_2 = \frac{q_0 - q_1 + q}{2}.$$

The matrices occurring in Section 2 take the form

$$\Phi = \begin{pmatrix} 1 & 1 \\ \frac{q_0-q_1+q}{2} & \frac{q_0-q_1-q}{2} \end{pmatrix}, \quad \Phi^T\Phi = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$
$$\mathcal{K} = \begin{pmatrix} 1 & 0 \\ q_0 & 1 \end{pmatrix}, \quad \mathcal{K}^T\mathcal{K} = \begin{pmatrix} 1 + q_0^2 & q_0 \\ q_0 & 1 \end{pmatrix},$$

and since

$$S_0(\lambda) = 0, \quad S_1(\lambda) = 1, \quad S_2(\lambda) = 2 - \lambda, \quad S_3(\lambda) = 3 - 4\lambda + \lambda^2,$$

we obtain

$$\mathcal{S} = \begin{pmatrix} 1 & 1 \\ \frac{-q_0-q_1+q}{2} & \frac{-q_0-q_1-q}{2} \end{pmatrix} \quad \text{and} \quad \mathcal{K}\mathcal{S} = \begin{pmatrix} 1 & 1 \\ \frac{q_0-q_1+q}{2} & \frac{q_0-q_1-q}{2} \end{pmatrix} = \Phi.$$

## 5. Remarks

1. Although this was not needed in order to obtain the results of this paper, it will be of importance to discuss the *transformation operator K* in great detail. In fact, the recursion (7) may be stated as a partial difference equation in the form

$$\Delta_1^2 K(i, j+1) - \Delta_2^2 K(i+1, j) = q(i)K(i+1, j+1),$$

where

$$\Delta_1 K(i, j) = K(i+1, j) - K(i, j) \quad \text{and} \quad \Delta_2 K(i, j) = K(i, j+1) - K(i, j).$$

It is easy to establish the following identities for $i \in \mathbb{N}_0$:

$$K(i+1, i) = \sum_{k=0}^{i-1} q_k =: Q_i,$$
$$K(i+2, i) = \sum_{k=1}^{i} q_k Q_k =: \tilde{Q}_i,$$
$$K(i+3, i) = Q_{i+1} - q_0 + \sum_{k=2}^{i+1} q_k \tilde{Q}_{k-1}.$$

2. Corresponding results of this paper may also be given when the Dirichlet boundary conditions (4) are replaced by

$$\Delta y_0 - h y_0 = \Delta y_{N+1} + H y_{N+1} = 0,$$

where $h, H \in \mathbb{R}$ are parameters, and then the objective is to use the spectral data to not only determine the potential but also the coefficients $h$ and $H$ of the boundary conditions.

3. It will be interesting to obtain corresponding results for problems governed by dynamic equations on time scales (see [2]) which include the continuous and discrete cases within one theory, extending it also to other cases “in between” such as, for example, $q$-difference equations. In this setting, one would consider the Sturm–Liouville dynamic equation (see [2])

$$-y^{\Delta\Delta}(t) + q(t)y(\sigma(t)) = \lambda y(\sigma(t)), \quad t \in \mathbb{T}.$$

## References


[1] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan. *Discrete Oscillation Theory*. Hindawi Publishing Corporation, 2005.

[2] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser, Boston, 2001.

[3] G. Borg. Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math.*, 78:1–96, 1946.

[4] G. Freiling and V. A. Yurko. *Inverse Sturm–Liouville problems and their applications*. Nova Science Publishers Inc., Huntington, NY, 2001.

[5] O. H. Hald. Discrete inverse Sturm-Liouville problems. I. Uniqueness for symmetric potentials. *Numer. Math.*, 27(2):249–256, 1976/77.

[6] H. Hochstadt. The inverse Sturm–Liouville problem. *Comm. Pure Appl. Math.*, 26:715–729, 1973. Collection of articles dedicated to Wilhelm Magnus.

[7] W. G. Kelley and A. C. Peterson. *Difference Equations: An Introduction with Applications*. Academic Press, San Diego, second edition, 2001.

[8] H. Koyunbakan. Inverse problem for a quadratic pencil of Sturm–Liouville operator. *J. Math. Anal. Appl.*, 378(2):549–554, 2011.

[9] W. Kratz. *Quadratic Functionals in Variational Analysis and Control Theory*, volume 6 of *Mathematical Topics*. Akademie Verlag, Berlin, 1995.

[10] N. Levinson. The inverse Sturm–Liouville problem. *Mat. Tidsskr. B.*, 1949:25–30, 1949.

[11] B. M. Levitan. *Inverse Sturm–Liouville problems*. VSP, Zeist, 1987. Translated from the Russian by O. Efimov.

[12] V. A. Marčenko. Some questions of the theory of one-dimensional linear differential operators of the second order. I. *Trudy Moskov. Mat. Obšč.*, 1:327–420, 1952.

[13] V. A. Yurko. *Inverse spectral problems for differential operators and their applications*, volume 2 of *Analytical Methods and Special Functions*. Gordon and Breach Science Publishers, Amsterdam, 2000.