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Hermite-Hadamard Inequalities Involving Riemann-Liouville Fractional Integrals via s-convex Functions and Applications to Special Means

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Abstract. In this paper, we establish some new Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals via s-convex functions in the second sense. Meanwhile, we present many useful estimates on these types of new Hermite-Hadamard type inequalities. Some applications to special means of real numbers are given.

1. Introduction

It is well known that the classical Hermite-Hadamard type inequality provides a lower and an upper estimations for the integral average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain. This interesting inequality was firstly discovered by Hermite in 1881 in the journal Mathesis (see Mitrinović and Lacković [12]). However, this beautiful result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (see Pečarić et al. [17]). For more recent results which generalize, improve, and extend this classical Hermite-Hadamard inequality, one can see Dragomir [5], Sarikaya et al. [19], Xiao et al. [27], Bessenyei [4], Tseng et al. [24], Niculescu [13], Bai et al. [3], Li and Qi [11], Tunç [25], Srivastava et al. [23], and references therein. In particular, let us note that Professor Srivastava et al. present some interesting refinements and extensions of the Hermite-Hadamard inequalities in n variables (see Theorems 6-12, [23]) and compare with some various inequalities in earlier works.

Fractional calculus have recently proved to be a powerful tool for the study of dynamical properties of many interesting systems in physics, chemistry, and engineering. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more recent development on fractional calculus, one can see the monographs of Kilbas et al. [9], Lakshmikantham et al. [10], and Podlubny [18].

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Due to the widely application of Hermite-Hadamard type inequality and fractional calculus, it is natural to offer to study Hermite-Hadamard type inequalities involving fractional integrals. It is remarkable that Sarikaya et al. [20] begin to study inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals and give the following new Hermite-Hadamard type inequalities.

Theorem 1.1. *(Theorem 2, [20]) Let $f : [a, b] \to R$ be a positive function with $0 \le a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequality for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)] \le \frac{f(a)+f(b)}{2}.$$

Here, the symbol $J^\alpha_{a^+}f$ and $J^\alpha_{b^-}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R_+ := [0, \infty)$ are defined by

$$(J^\alpha_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ (0 \le a < x \le b),\ (J^\alpha_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ (0 \le a < x \le b),$$

respectively. Here $\Gamma(\cdot)$ is the Gamma function.

Lemma 1.2. *(Lemma 2, [20]) Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)] = \frac{b-a}{2}\int_0^1 [(1-t)^\alpha - t^\alpha]f'(ta+(1-t)b)dt.$$

Theorem 1.3. *(Theorem 3, [20]) Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following equality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \le \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)|f'(a) + f'(b)|.$$

In addition to the classical convex functions, Hudzik and Maligranda [8] introduced the definition of s-convex functions in the second sense as following.

Definition 1.4. *A function $f : I \subseteq R_+ \to R_+$ is said to be s-convex on I if the inequality $f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$ holds for all $x, y \in I$ and $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.*

Applying the tool of s-convex functions in the second sense, Set [22] gives some new inequalities of Ostrowski type involving fractional integrals. The author generalizes the interesting and useful Ostrowski inequality (see [14]) and other numerous extensions and variants of Ostrowski inequalities (see for example [1, 7, 15, 21]).

The authors [2, 6] study the inequalities of Hermite-Hadamard type via s-convex functions in the second sense, to the best of our knowledge, Hadamard type inequalities involving Riemann-Liouville fractional integrals via s-convex functions have not been studied extensively. Motivated by the recent results given in [2, 6, 20, 22, 26], we will establish here some new Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals via s-convex functions in the second sense. Meanwhile, we present many useful estimates on these types of new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals.

2. Hermite-Hadamard Inequality for Fractional Integrals via s-convex Functions

An important Hermite-Hadamard inequality involving Riemann-Liouville fractional integrals (with $\alpha \ge 1$) can be represented as follows.

Theorem 2.1. *Let $\alpha \geq 1$ and $f : [a, b] \to R$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a s-convex function on $[a, b]$, then the following inequality for fractional integrals hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] \leq \frac{[f(a)+f(b)]}{2}\left[\frac{1}{\alpha+s} + \frac{2}{\alpha+s}\left(1-\frac{1}{2^{\alpha+s}}\right)\right]. \tag{1}$$

Proof. Since f is a s-convex function on $[a, b]$, we have for $x, y \in [a, b]$ with $\lambda = \frac{1}{2}$ $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2^s} f(x) + \frac{1}{2^s} f(y)$, i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$2^s f\left(\frac{a+b}{2}\right) \leq f(ta + (1-t)b) + f((1-t)a + tb). \tag{2}$$

Multiplying both sides of (2) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)]$$

and the first inequality in (1) is proved.

Because f is a s-convex function, we have

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq [t^s + (1-t)^s][f(a) + f(b)]. \tag{3}$$

Then multiplying both sides of (3) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain the right-sided inequality in (1). The proof is completed. □

Remark 2.2. *One can follow the same ideas to construct fractional version $F_n^{(1)}, F_n^{(2)}, F_n^{(3)}$ (see (3.4)-(3.6), [23]) try to extend to study fractional Hermite-Hadamard inequalities in n variables based on these fundamental results. We shall study such interesting problems in the forthcoming works.*

3. Hermite-Hadamard Type Inequalities for Fractional Integrals via s-convex Functions

Now we are ready to state the first result in this section.

Theorem 3.1. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$, such that $f \in L[a, b]$. If f' is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)]\right| \leq \frac{b-a}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right)(|f'(a)| + |f'(b)|).$$

Proof. Using Lemma 1.2 and the s-convexity of f, we have

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)]\right| \\ \leq\ & \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}} [(1-t)^{\alpha} - t^{\alpha}][t^s|f'(a)| + (1-t)^s|f'(b)|]dt + \int_{\frac{1}{2}}^1 [t^{\alpha} - (1-t)^{\alpha}][t^s|f'(a)| + (1-t)^s|f'(b)|]dt\right\} \\ :=\ & \frac{b-a}{2}(K_1 + K_2). \end{aligned} \tag{4}$$

Calculating K_1 and K_2, we have

$$\begin{aligned} K_1 &\leq |f'(a)|\left(\int_0^{\frac{1}{2}} (1-t)^{\alpha+s} ds - \int_0^{\frac{1}{2}} t^{\alpha+s} ds\right) + |f'(b)|\left(\int_0^{\frac{1}{2}} (1-t)^{\alpha+s} ds - \int_0^{\frac{1}{2}} t^{\alpha+s} ds\right) \\ &= \frac{|f'(a)| + |f'(b)|}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right), \end{aligned} \tag{5}$$

and

$$\begin{aligned} K_2 &\le |f'(a)|\left[\int_{\frac{1}{2}}^{1} t^{\alpha+s}ds - \int_{\frac{1}{2}}^{1}(1-t)^{\alpha+s}ds\right] + |f'(b)|\left[\int_{\frac{1}{2}}^{1} t^{\alpha+s}ds - \int_{\frac{1}{2}}^{1}(1-t)^{\alpha+s}ds\right] \\ &= \frac{|f'(a)|+|f'(b)|}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right). \end{aligned} \tag{6}$$

Thus if we use (5) and (6) in (4), we obtain the result. The proof is completed. □

The second theorem gives a new upper bound of the left-Hadamard inequality for s-convex mappings.

Theorem 3.2. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$, such that $f\in L[a,b]$. If $|f'|^q (q>1)$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]\right| \\ \le &\left(\frac{b-a}{2}\right)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ &\times\left[(|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q)^{\frac{1}{q}}+((2^{s+1}-1)|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}\right], \end{aligned} \tag{7}$$

where $\frac{1}{p}=1-\frac{1}{q}$.

Proof. Using Lemma 1.2 and Hölder inequality and the s-convexity of $|f'|^q (q>1)$, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]\right| \\ \le\ &\frac{b-a}{2}\left(\frac{1}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}(t^s|f'(a)|^q+(1-t)^s|f'(b)|^q)dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(t^s|f'(a)|^q+(1-t)^s|f'(b)|^q)dt\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

We note that

$$\int_0^{\frac{1}{2}} t^s|f'(a)|^q+(1-t)^s|f'(b)|^q dt = \frac{1}{(s+1)2^{(s+1)}}|f'(a)|^q+\frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)|f'(b)|^q \tag{8}$$

and

$$\int_{\frac{1}{2}}^1 (t^s|f'(a)|^q+(1-t)^s|f'(b)|^q)dt = \frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)|f'(a)|^q+\frac{1}{(s+1)2^{(s+1)}}|f'(b)|^q. \tag{9}$$

Note that (8) and (9), we get our result. The proof is completed. □

It is not difficult to see that Theorem 3.2 can be extended to the following result.

Corollary 3.3. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$, such that $f\in L[a,b]$. If $|f'|^q,(q>1)$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]\right| \\ \le &\left(\frac{b-a}{2}\right)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\times\left(1+(2^{s+1}-1)^{\frac{1}{q}}\right)(|f'(a)|+|f'(b)|), \end{aligned}$$

where $\frac{1}{p}=1-\frac{1}{q}$.

Proof. We consider inequality (7), and we let $a_1=|f'(a)|^q$, $b_1=(2^{s+1}-1)|f'(b)|^q$, $a_2=(2^{s+1}-1)|f'(a)|^q$, $b_2=|f'(b)|^q$. Here, $0<\frac{1}{q}<1$ for $q>1$. Using the fact $\sum_{i=1}^n(a_i+b_i)^r\le\sum_{i=1}^n a_i^r+\sum_{i=1}^n b_i^r$ for $0<r<1, a_1,a_2,\cdots,a_n>0$ and $b_1,b_2,\cdots,b_n>0$, we obtain the required result. This completes the proof. □

The following theorem is the third main result in this section.

Theorem 3.4. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$,such that $f \in L[a,b]$. If $|f'|^q, (q > 1)$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality for fractional integrals holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \\
\leq\ & (b-a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(1-\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\left(1-\frac{1}{2^{\alpha+s}}\right)^{\frac{1}{q}} \times (|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.
\end{aligned} \tag{10}$$

Proof. Using Lemma 1.2 and the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \\
\leq\ & \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}}[(1-t)^\alpha - t^\alpha]|f'(ta+(1-t)b)|dt + \int_{\frac{1}{2}}^1[t^\alpha-(1-t)^\alpha]|f'(ta+(1-t)b)|dt\right\} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(1-\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^{\frac{1}{2}}[(1-t)^\alpha - t^\alpha]|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1[t^\alpha-(1-t)^\alpha]|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Because the s-convex of $|f'|^q, (q > 1)$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}[(1-t)^\alpha - t^\alpha]|f'(ta+(1-t)b)|^q dt \\
\leq\ & \int_0^{\frac{1}{2}}(1-t)^\alpha t^s dt|f'(a)|^q + \int_0^{\frac{1}{2}}(1-t)^{\alpha+s}dt|f'(b)|^q - \int_0^{\frac{1}{2}}t^{\alpha+s}dt|f'(a)|^q - \int_0^{\frac{1}{2}}t^\alpha(1-t)^s dt|f'(b)|^q \\
\leq\ & \frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right)(|f'(a)|^q + |f'(b)|^q),
\end{aligned} \tag{11}$$

and similarly

$$\int_{\frac{1}{2}}^1[t^\alpha-(1-t)^\alpha]|f'(ta+(1-t)b)|^q dt \leq \frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right)(|f'(a)|^q + |f'(b)|^q). \tag{12}$$

A combination of (11) and (12), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \\
\leq\ & (b-a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(1-\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\left(1-\frac{1}{2^{\alpha+s}}\right)^{\frac{1}{q}}(|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.5. *Let f be as in Theorem 3.4, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \\
\leq\ & (b-a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(1-\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\left(1-\frac{1}{2^{\alpha+s}}\right)^{\frac{1}{q}}(|f'(a)| + |f'(b)|).
\end{aligned}$$

Proof. Using the technique in the proof of Corollary 3.3, by considering inequality (10), one can obtain the result. □

To end this section, we give the following Hermite-Hadamard type inequality for concave mapping.

Theorem 3.6. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$, such that $f \in L[a, b]$. If $|f'|^q (q > 1)$ is concave on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)]\right| \leq \frac{b-a}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right|\right].$$

Proof. Using Theorem 3.1 and Hölder inequality and the s-convex of $|f'|^q, (q > 1)$, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)]\right| \\ \leq\ & \frac{b-a}{2}\left(\frac{1}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}} |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right\}, \end{aligned}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

We note that $|f'|^q$ is concave on $[a, b]$, and use the Jensen integral inequality, we have

$$\int_0^{\frac{1}{2}} |f'(ta+(1-t)b|^q dt \quad \leq \quad \left(\int_0^{\frac{1}{2}} t^s dt\right)\left|f'\left(\frac{\int_0^{\frac{1}{2}}(ta+(1-t)b)dt}{\int_0^{\frac{1}{2}} t^s dt}\right)\right|^q \leq \frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q,$$

and analogously

$$\int_{\frac{1}{2}}^1 |f'(ta+(1-t)b)|^q dt \quad \leq \quad \left(\int_{\frac{1}{2}}^1 t^s dt\right)\left|f'\left(\frac{\int_{\frac{1}{2}}^1(ta+(1-t)b)dt}{\int_{\frac{1}{2}}^1 t^s dt}\right)\right|^q \leq \frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q.$$

Combining all obtained inequalities, we get the required result. □

4. Applications to Some Special Means

Consider the following special means (see Pearce and Pečarić [16]) for arbitrary real numbers $\alpha, \beta, \alpha \neq \beta$ as follows:

$$\begin{aligned} H(\alpha,\beta) &= \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},\ \alpha,\ \beta \in R \setminus \{0\}, \\ A(\alpha,\beta) &= \frac{\alpha+\beta}{2},\ \alpha, \beta \in R, \\ L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|},\ |\alpha| \neq |\beta|,\ \alpha\beta \neq 0, \\ L_n(\alpha,\beta) &= \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\ n \in Z \setminus \{-1, 0\},\ \alpha, \beta \in R,\ \alpha \neq \beta. \end{aligned}$$

Now, using the obtained results in Section 3, we give some applications to special means of real numbers.

Proposition 4.1. *Let $a, b \in R_+$, $a < b$, $s \in (0, 1]$ and $q > 1$. Then*

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b)| \leq \begin{cases} \frac{2(b-a)}{s+2}\left(1-\frac{1}{2^{1+s}}\right)A(a^{-2}, b^{-2}), \\ (b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^p}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ \times\left(1+(2^{s+1}-1)^{\frac{1}{q}}\right)A(a^{-2}, b^{-2}), \\ 2(b-a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}}\left(1-\frac{1}{2^{1+s}}\right)^{\frac{1}{q}} A(a^{-2}, b^{-2}), \end{cases} \tag{13}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof. Applying Theorem 3.1, Corollary 3.3, and Corollary 3.5 respectively, for $f(x) = \frac{1}{x}$ and $\alpha = 1$, one can obtain the result immediately. □

Proposition 4.2. *Let $a, b \in R_+$, $a < b$, $n \in Z$, $|n| \geq 2$, $s \in (0,1]$ and $q > 1$. Then*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq \begin{cases} \frac{2|n|(b-a)}{s+2}\left(1 - \frac{1}{2^{1+s}}\right) A(a^{n-1}, b^{n-1}), \\ |n|(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(1 - \frac{1}{2^p}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ \times\left(1 + (2^{s+1} - 1)^{\frac{1}{q}}\right) A(a^{n-1}, b^{n-1}), \\ 2|n|(b-a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}}\left(1 - \frac{1}{2^{1+s}}\right)^{\frac{1}{q}} A(a^{n-1}, b^{n-1}), \end{cases} \tag{14}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof. Applying Theorem 3.1, Corollary 3.3, and Corollary 3.5 respectively, for $f(x) = x^n$ and $\alpha = 1$, one can obtain the result immediately. □

Proposition 4.3. *Let $a, b \in R_+$, $(a < b)$, $a^{-1} > b^{-1}$. For $n \in Z$, $|n| \geq 2$, $s \in (0,1]$ and $q > 1$, we have*

$$|H^{-1}(b^{-1}, a^{-1}) - L^{-1}(b^{-1}, a^{-1})| \leq \begin{cases} \frac{2(b-a)}{ab(s+2)}\left(1 - \frac{1}{2^{1+s}}\right) H^{-1}(a^{-2}, b^{-2}), \\ \frac{b-a}{ab}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(1 - \frac{1}{2^p}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ \times\left(1 + (2^{s+1} - 1)^{\frac{1}{q}}\right) H^{-1}(a^{-2}, b^{-2}), \\ \frac{2(b-a)}{ab}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}}\left(1 - \frac{1}{2^{1+s}}\right)^{\frac{1}{q}} H^{-1}(a^{-2}, b^{-2}), \end{cases} \tag{15}$$

$$|H^{-1}(b^n, a^n) - L_n^n(b^{-1}, a^{-1})| \leq \begin{cases} \frac{2|n|(b-a)}{ab(s+2)}\left(1 - \frac{1}{2^{1+s}}\right) H^{-1}(a^{n-1}, b^{n-1}), \\ \frac{|n|(b-a)}{ab}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(1 - \frac{1}{2^p}\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ \times\left(1 + (2^{s+1} - 1)^{\frac{1}{q}}\right) H^{-1}(a^{n-1}, b^{n-1}), \\ \frac{2|n|(b-a)}{ab}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}}\left(1 - \frac{1}{2^{1+s}}\right)^{\frac{1}{q}} H^{-1}(a^{n-1}, b^{n-1}), \end{cases} \tag{16}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof. Making the substitutions $a \to b^{-1}$, $b \to a^{-1}$ in the inequalities (13) and (14), one can obtain desired inequalities (15) and (16) respectively. □

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