Filomat 30:5 (2016), 1161–1174
DOI 10.2298/FIL1605161D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Comparison of Spectral Invariants in Lagrangian and Hamiltonian Floer Theory

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Abstract. We compare spectral invariants in periodic orbits and Lagrangian Floer homology case, for a closed symplectic manifold P and its closed Lagrangian submanifolds L, when $\omega|_{\pi_2(P,L)} = 0$, and $\mu|_{\pi_2(P,L)} = 0$. We define a product $HF_*(H) \otimes HF_*(H : L) \to HF_*(H : L)$ and prove subadditivity of invariants with respect to this product.

1. Introduction

In [15, 16] Oh defined spectral invariants for the case of cotangent bundle $P = T^*M$ (and the canonical Liouville symplectic form), where the action functional

$$a_H(x) := \int_x \theta - \int_0^1 H(x(t), t)dt$$

is well defined. Let $HF_*(H : O_M)$ denote Lagrangian Floer homology of the pair $(O_M, \phi_H^1(O_M))$, where ϕ_H^1 is the time-one map generated by a Hamiltonian H. Denote by $HF_*^\lambda(H : O_M)$ the filtered homology defined via filtered Floer complex:

$$CF_*^\lambda(H : O_M) := \mathbb{Z}_2\langle\{x \in \mathrm{Crit}\, a_H \mid a_H(x) < \lambda\}\rangle.$$

These homology groups are well defined since the boundary map preserves the filtration:

$$\partial : CF_*^\lambda(H : O_M) \to CF_*^\lambda(H : O_M),$$

due to well defined action functional that decreases along its gradient flow lines. For a singular homology class α define

$$\sigma(\alpha, H) := \inf\{\lambda \in \mathbb{R} \mid F_H(\alpha) \in \mathrm{Im}(\iota_*^\lambda)\}$$

where

$$\iota_*^\lambda : HF_*^\lambda(H : O_M) \to HF_*(H : O_M)$$

2010 *Mathematics Subject Classification*. Primary 53D40; Secondary 53D12, 57R58, 57R17
Keywords. Spectral invariants, Lagrangian submanifolds, homology product
Received: 19 March 2014; Accepted: 20 December 2014
Communicated by Vladimir Dragović
This work is partially supported by Ministry of Education and Science of Republic of Serbia Project #ON174034
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is the homomorphism induced by inclusion and

$$F_H : H_*(M) \to HF_*(H : O_M)$$

is an isomorphism between singular and Floer homology groups. The construction of spectral invariants is done in [15] in the case of conormal bundle boundary condition, and in [16] for cohomology classes. This construction is based on Viterbo's idea for generating functions defined in the case of cotangent bundle (see [21]).

It turned out that Oh's and Viterbo's invariants are in fact the same, see [9, 10].

Oh proved in [15] that these invariants are independent both of the choice of almost complex structure J (which enters the definition of Floer homology) and, after a certain normalization, on the choice of H as long as $\phi_H^1(L_0) = L_1$. Using these invariants $\sigma(\alpha, L_1) := \sigma(\alpha, H)$, Oh derived the non–degeneracy of Hofer's metric for Lagrangian submanifolds.

Lagrangian spectral invariants σ were also used in [11, 12] in the characterization of geodesics in Hofer's metric for Lagrangian submanifolds of the cotangent bundle via quasi–autonomous Hamiltonians.

In [8], Leclercq constructed spectral invariants for Lagrangian Floer theory without the assumption $\omega = -d\theta$. He considered the case when L is a closed submanifold of a compact (or convex at infinity) symplectic manifold P and

$$\omega|_{\pi_2(P,L)} = 0, \quad \mu|_{\pi_2(P,L)} = 0,$$

where μ stands for Maslov index. He used a module structure of Floer homology over a Morse homology ring and Albers' Piunikhin–Salamon–Schwarz (we will also use the abbreviation PSS) isomorphisms (see below or [2]) to prove that, after a certain normalization, the spectral invariants only depend on L and $L' := \phi_H^1(L)$.

Schwarz defined similar invariants in the case of Floer theory for contractible periodic orbits in [20]. If (P, ω) is a symplectic manifold with $\omega|_{\pi_2(P)} = 0$, then the action functional is well defined as:

$$\mathcal{A}_H(a) := \int_{D^2} \bar{a}^*\omega - \int_0^1 H(a(t), t)dt,$$

where $\bar{a} : D^2 \to P$ is any extension of a to the unit disc. For both $\omega|_{\pi_2(P)} = 0$ and $c_1|_{\pi_2(P)} = 0$ Schwarz defined symplectic invariants as:

$$\rho(\alpha, H) := \inf\{\lambda \in \mathbb{R} \mid \mathrm{PSS}(\alpha) \in \mathrm{Im}(\iota_*^\lambda)\}.$$

Here $\alpha \in H^*_{\mathrm{sing}}(P)$ is a nonzero cohomology class, PSS is the Piunikhin–Salamon–Schwarz isomorphism and ι_*^λ is the homomorphism induced by inclusion

$$\iota^\lambda : CF_*^\lambda(H) \to CF_*(H),$$

where $CF_*(H)$ and $CF_*^\lambda(H)$ are (filtered) Floer chain complexes for Hamiltonian periodic orbits. For each nonzero cohomology class α, $\rho(\alpha, \cdot)$ is a section of the action spectrum bundle

$$\begin{array}{l} \Sigma := \bigcup_{\widetilde{\Phi}\in\widetilde{\mathrm{Ham}}(P)} \{\widetilde{\Phi}\} \times \left\{\mathcal{A}_H(x) \mid x \in \mathrm{Fix}^0\left(\Phi_H^1\right), \Phi_H^1 \in \widetilde{\Phi}\right\} \\ \downarrow \\ \mathrm{Ham}(P, \omega) \end{array}$$

which is continuos with respect to Hofer's metric, and which carries certain properties (see [20] for details and also [5]).

In his paper [2], Albers constructed PSS morphisms for the Lagrangian case and showed that, in certain dimensions, these morphisms are isomorphisms (see also [7] for the case of cotangent bundle $P = T^*M$). These are the isomorphisms that Leclercq used in his already mentioned paper [8] to define the Lagrangian invariants σ (see Subsection 3.2 below). Albers considered the case of a closed, monotone Lagrangian submanifold L with minimal Maslov index $\Sigma_L \geq 2$. He also constructed morphisms $\chi : HF_*(H : L) \to HF_*(H)$ and $\tau : HF_*(H) \to HF_*(H : L)$ where $HF_*(H)$ denotes Floer homology for periodic Hamiltonian orbits and

$HF_*(H : L)$ denotes Lagrangian Floer homology of the pair $(L, \phi_H^1(L))$. This construction is based on counting the numbers of "chimneys" and it was independently considered by Abbondandolo and Schwarz in [1]. Using these homomorphisms, Albers proved the commutativity of certain diagrams (see (8) below).

In this paper, we consider the case of a closed symplectic manifold (P, ω) and its closed smooth Lagrangian submanifold L with topological assumptions

$$\omega|_{\pi_2(P,L)} = 0, \quad \mu|_{\pi_2(P,L)} = 0$$

and symplectic invariants ρ for periodic orbits, and σ for Lagrangian case (see Subsection 3.2 below).

We will use the homomorphisms constructed by "chimneys" to compare these spectral invariants. Similar comparison was made by Monzner, Vichery and Zapolsky in a different context (see [13]). Further, we define the product $\circ$ using perturbed pseudoholomorphic curves that connect Hamiltonian periodic orbits and Hamiltonian paths with Lagrangian boundary conditions. This product was previously defined, for example by Hu and Lalonde [6], in the more general context of monotone Lagrangians. The main result of the paper is the following.

Theorem 1.1. *Let P be a closed symplectic manifold and $L \subset P$ its closed Lagrangian submanifold such that $\omega|_{\pi_2(P,L)} = 0$, $\mu|_{\pi_2(P,L)} = 0$. Let $H_j : P \times [0, 1] \to \mathbb{R}$ be three (time dependent) Hamiltonians, for $j = 1, 2, 3$. Then there exists a product*

$$\circ : HF_*(H_1) \otimes HF_*(H_2 : L) \to HF_*(H_3 : L)$$

which, in the case when $H_2 = H_3$, turns the Lagrangian Floer homology $HF_*(H_2 : L)$ into a module over Floer homology for periodic orbits $HF_*(H_1)$. For $H_3 = H_1 \sharp H_2$, and $a \in HF_*(H_1)$, $b \in HF_*(H_2 : L)$, it holds:

$$\sigma(\mathrm{PSS}^{-1}(a \circ b), H_1 \sharp H_2) \leq \rho(\mathrm{PSS}^{-1}(a), H_1) + \sigma(\mathrm{PSS}^{-1}(b), H_2). \tag{1}$$

The proof of Theorem 1.1 is given in Section 4.

In Section 2 we recall the construction of Floer homology and PSS −type isomorphisms and their properties that we will use in the paper.

In Section 3 we construct spectral invariants in periodic orbits and Lagrangian case, and prove that Lagrangian spectral invariants do not depend on H as long as $\phi_H^1(L)$ is fixed, up to a constant. Besides, we prove certain inequalities between these two types of invariants (Theorem 3.5 and Theorem 3.6).

We would like to thank Rémi Leclerq for pointing to us an error in the previous version of the paper.

We would like to thank the anonymous referee for many valuable suggestions and corrections.

2. Recalls and Preliminaries

Throughout the paper we will assume that (P, ω) is a closed symplectic manifold and L is its closed Lagrangian submanifold.

2.1. *Floer homology and* PSS *(iso)morphisms*

Let us first briefly sketch the construction of Floer homology and PSS isomorphisms for periodic orbits. For a smooth (generic) Hamiltonian $H : P \times S^1 \to \mathbb{R}$, the Floer complex $CF_*(H)$ is defined as a vector space over $\mathbb{Z}_2$ with the generators

$$\mathcal{P}(H) := \{a \in C^\infty(S^1, P) \mid \dot{a}(t) = X_H(a(t)), [a] = 0 \in \pi_1(P)\}$$

and it is graded by the Conley–Zehnder index (see [19], for example). Floer differential is defined as

$$\delta(a) := \sum_b n(a, b) b,$$

where $n(a,b)$ is the number (modulo 2) of elements of the following set:

$$\mathcal{M}(a,b;H,J) := \left\{ u : \mathbb{R}\times S^1 \to P \;\middle|\; \begin{array}{l} \partial_s u + J(\partial_t u - X_H(u)) = 0 \\ u(-\infty,t) = a(t) \\ u(+\infty,t) = b(t) \end{array} \right\}$$

modulo $\mathbb{R}$−action $(\tau,u) \mapsto u(\cdot+\tau,\cdot)$. Here X_H is the Hamiltonian vector field, i.e. $\omega(X_H,\cdot) = dH(\cdot)$. Floer homology $HF_*(H)$ and Morse homology $HM_*(P,f)$ for Morse function $f : P \to \mathbb{R}$ are isomorphic. One way to prove this is via the PSS isomorphisms. Define two cut-off functions

$$\rho_R(s) = \begin{cases} 1, & s \ge R+1, \\ 0, & s \le R \end{cases}, \quad \tilde{\rho}_R(s) := \rho_R(-s).$$

Let p be a critical point of f. Let g be a Riemannian metric such that the pair (f,g) is Morse–Smale. Define

$$\mathrm{PSS}(p) := \sum_x n(p,x)x$$

and extend it on the chain level by linearity. Here $n(p,x)$ is a number (modulo 2) of pairs (γ,u):

$$\gamma : (-\infty,0] \to P, \quad u : [0,+\infty)\times S^1 \to P$$

that satisfy

$$\begin{cases} \frac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ \partial_s u + J(\partial_t u - X_{\rho_R H}(u)) = 0 \\ \gamma(-\infty) = p, \; u(+\infty,t) = a(t) \\ \gamma(0) = u(0,\frac{1}{2}), \end{cases}$$

where ∇f denotes the gradient of f with respect to g. This mapping commutes with the differentials, that is

$$\mathrm{PSS}\circ\partial_{\mathrm{Morse}} = \delta\circ\mathrm{PSS},$$

where $\partial_{\mathrm{Morse}}$ is Morse differential, so we have $\mathrm{PSS} : HM_*(P,f) \to HF_*(H)$ (we keep the same notation). It is proven in [18] that, under our assumptions, PSS is actually an isomorphism, and its inverse, PSS^{-1} is defined by counting the "reverse" mixed objects, i.e. the pairs (u,γ) that satisfy:

$$\begin{cases} u : (-\infty,0]\times S^1 \to P, \quad \gamma : [0,+\infty) \to P \\ \partial_s u + J(\partial_t u - X_{\tilde{\rho}_R H}(u)) = 0 \\ \frac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ u(-\infty,t) = a(t), \; \gamma(+\infty) = p \\ u(0,\frac{1}{2}) = \gamma(0) \end{cases}$$

(see Figure 1).

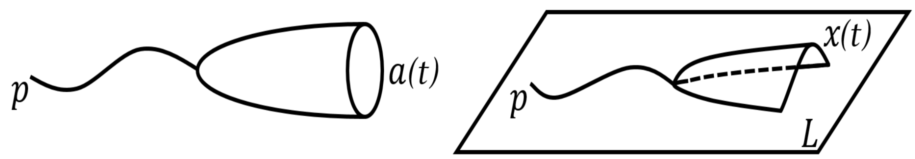

Figure 1: Mixed type objects that define PSS isomorphisms in the case of periodic orbits and in Lagrangian intersections case

Now let us recall the construction of Floer homology and PSS −type morphisms in the Lagrangian case. Suppose that P and L are as above and that H is non degenerate, i.e. that $L \pitchfork \phi_H^1(L)$. For our purposes, we will set

$$\omega|_{\pi_2(P,L)} = 0, \quad \mu|_{\pi_2(P,L)} = 0 \tag{2}$$

where μ is Maslov index. (Of course, Floer homology for Lagrangian intersections can be defined in a more general context.) Let $H : P \times S^1 \to \mathbb{R}$ be a smooth Hamiltonian function. Floer complex $CF_*(H : L)$ is defined as a vector space over $\mathbb{Z}_2$ with the generators

$$\mathcal{P}(H, L) := \{x \in C^\infty([0, 1], P) \mid \dot{x}(t) = X_H(x(t)), x(0), x(1) \in L, [x] = 0 \in \pi_1(P, L)\}.$$

The grading is given by relative Maslov index, which is well defined, since $\mu|_{\pi_2(P,L)} = 0$ (see, for example, [14] for details). Floer differential is defined by counting the pseudo–holomorphic tunnels, i.e.

$$\partial(x) = \sum_y n(x, y)y,$$

where $n(x, y)$ is the number (modulo 2) of elements of the set

$$\mathcal{M}(x, y; H, J) := \left\{ u : \mathbb{R} \times [0, 1] \to P \;\middle|\; \begin{array}{l} \partial_s u + J(\partial_t u - X_H(u)) = 0 \\ u(s, i) \in L, \ i \in \{0, 1\} \\ u(-\infty, t) = x(t) \\ u(+\infty, t) = y(t) \end{array} \right\}$$

modulo $\mathbb{R}$−action $(\tau, u) \mapsto u(\cdot + \tau, \cdot)$. As usual, we denote this quotient space by

$$\widehat{\mathcal{M}}(x, y; H, J) := \mathcal{M}(x, y; H, J)/\mathbb{R}.$$

The Albers' PSS −type morphisms are well defined in more general cases than (2), that is when L is monotone and minimal Maslov number N_L is at least 2 (see [2]). Let us recall this construction. For critical point p of Morse function $f : L \to \mathbb{R}$, define

$$\Phi : CM_*(L, f) \to CF_*(H : L), \quad \Phi(p) := \sum_x n(p, x)x$$

where $n(p, x)$ is a number (modulo 2) of pairs (γ, u), that satisfy

$$\begin{cases} \gamma : (-\infty, 0] \to L, \quad u : [0, +\infty) \times [0, 1] \to P \\ \frac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ \partial_s u + J(\partial_t u - X_{\rho_R H}(u)) = 0 \\ u(s, 0), u(s, 1), u(0, t) \in L \\ \gamma(-\infty) = p, \ u(+\infty, t) = x(t) \\ \gamma(0) = u(0, \frac{1}{2}) \end{cases} \tag{3}$$

(see Figure 1). We denote the set of solutions of (3) by $\mathcal{M}_{p,x}^{f,H}$. The set $\mathcal{M}_{p,x}^{f,H}$ is a $(m_f(p) - \mu(x))$−dimensional manifold where $m_f(p)$ is the Morse index of a critical point p (note that this requires a particular choice of the reference of the Maslov index).

The map Φ turns out to be well defined in the homology level, and under our assumption (2), an isomorphism between Morse and Floer homologies in all dimensions. We will denote this isomorphism of homology groups again by Φ. Its inverse Ψ is defined on the generators of Floer complex as

$$\Psi : CF_*(H_1 : L) \to CM_*(L, f), \quad \Psi(x) := \sum_p n(x, p)p$$

where $n(x,p)$ is a number (modulo 2) of pairs (u,γ) that solve the equations:

$$\begin{cases} u:(-\infty,0]\times[0,1]\to P, \quad \gamma:[0,+\infty)\to L\\ \partial_s u + J(\partial_t u - X_{\tilde\rho_R H}(u)) = 0\\ \frac{d\gamma}{ds} = -\nabla f(\gamma(s))\\ u(s,0), u(s,1), u(0,t)\in L\\ u(-\infty,t) = x(t),\ \gamma(+\infty) = p\\ u(0,\frac{1}{2}) = \gamma(0). \end{cases}$$

The proofs of the above facts are usually based on the analysis of certain moduli spaces, especially in dimensions zero and one, and their boundaries as well. The description of these boundaries and the proof of compactness in zero dimension case use Gromov compactness and gluing theorems. Bubbling is controlled due to topological assumptions (2).

For the sake of simplicity, we will denote these isomorphisms also by PSS, whenever there is no risk of confusion. More precisely

$$\mathrm{PSS} := \Phi, \quad \mathrm{PSS}^{-1} = \Psi.$$

3. Spectral Invariants and Their Comparison

3.1. Action functionals

In this subsection we will recall the constructions of two action functionals – for contractible loops and for contractible paths with the ends on a Lagrangian submanifold.

In the case of periodic orbits, we will suppose that $\omega|_{\pi_2(P)} = 0$, which is true if the second equality in (2) holds. We define the action functional on the space of smooth contractible loops

$$\Omega_0(P) := \{a\in C^\infty(S^1,P) \mid [a] = 0\in\pi_1(P)\}$$

in a standard way:

$$\mathcal{A}_H(a) := -\iint_{D^2}\tilde a^*\omega - \int_{S^1} H(a(t),t)\,dt, \tag{4}$$

where $\tilde a$ is any map from a disc with $\tilde a|_{S^1} = \gamma$. This map exists since a is contractible and the first integral in (4) does not depend on the choice of $\tilde a$ when the condition (2) is fulfilled. One easily checks that the critical points of $\mathcal{A}_H$ are Hamiltonian periodic orbits.

Let us now define the action functional for the Lagrangian case. Let P, L and H be as above and suppose that

$$\omega|_{\pi_2(P,L)} = 0. \tag{5}$$

The second condition in (2) does not have to be fulfilled in order to define the action functional. For the domain of the action functional a_H we choose:

$$\Omega_0(P,L) := \{x\in C^\infty([0,1],P) \mid x(0),x(1)\in L,\ [x] = 0\in\pi_1(P,L)\}.$$

Set:

$$a_H(x,h) := -\iint_{D^2_+} h^*\omega - \int_0^1 H(x(t),t)\,dt,$$

where h is any map from the upper half-disc D^2_+ to P that restricts to x on the upper half-circle. Since $\omega|_{\pi_2(P,L)} = 0$, the first integral does not depend on h, so we denote $a_H(x) := a_H(x,h)$.

We compute the differential of a_H. For any variation $x_\varepsilon(t)$ of $x(t)$ with

$$x_\varepsilon(0), x_\varepsilon(1)\in L$$

let $h_\varepsilon(s,t)$ be any smooth map from D^2_+ that satisfies

$$\begin{aligned} &h_\varepsilon(0,t) \in L, \text{ for } t \in [-1,1],\\ &h_\varepsilon(\cos(\pi\tau), \sin(\pi\tau)) = x_\varepsilon(\tau) \text{ for } \tau \in [0,1]. \end{aligned} \tag{6}$$

Denote by

$$\xi(t) := \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} x_\varepsilon(t), \quad \zeta(s,t) := \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} h_\varepsilon(s,t).$$

Using Cartan's and Stokes' formula, and the boundary conditions (6), one easily gets

$$\begin{aligned} da_H(x)(\xi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} a_H(x_\varepsilon) = -\iint_{D^2_+} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} h^*_\varepsilon\omega - \int_0^1 dH(\xi)dt =\\ -\iint_{D^2_+} d(i(\zeta)\omega) - \int_0^1 dH(\xi)dt = -\int_{\partial(D^2_+)} i(\zeta)\omega - \int_0^1 dH(\xi)dt =\\ -\int_{S^1_+} \omega\left(\xi, \frac{dx}{dt}\right) - \int_0^1 dH(\xi)dt = -\int_0^1 \omega\left(\xi, \frac{dx}{dt} - X_H\right)dt, \end{aligned}$$

so the critical points of a_H are Hamiltonian paths with ends in L.

Remark 3.1. If $y \in \Omega_0(P,L) \cap \Omega_0(P)$, i.e. $y(0) = y(1) \in L$ and $[y] = 0 \in \pi_1(P)$, then $a_H(y) = \mathcal{A}_H(y)$. □

3.2. Invariants

Now let us recall the definition of spectral invariants. We will start with periodic orbits case. If (2) holds, we have well defined action functional $\mathcal{A}_H$. Denote by

$$CF^\lambda_*(H) := \left\{\sum c_a a \in CF_*(H) \mid c_a = 0 \text{ for } \mathcal{A}_H(a) \geq \lambda\right\}.$$

Note that the Floer differential δ preserves filtrations given by $\mathcal{A}_H$, and define

$$\delta_\lambda := \delta|_{CF^\lambda_*(H)}, \quad HF^\lambda_*(H) := H_*(CF^\lambda_*(H), \delta_\lambda).$$

Denote by

$$\iota^\lambda_* : HF^\lambda_*(H) \to HF_*(H)$$

the homomorphism induced by the inclusion map ι^λ. For $\alpha \in HM_*(P,f)$, define

$$\rho(\alpha, H) := \inf\left\{\lambda \mid \mathrm{PSS}(\alpha) \in \mathrm{Im}\left(\iota^\lambda_*\right)\right\}. \tag{7}$$

The above definition is also valid in the case when α is a singular homology class, since Morse and singular homologies are isomorphic (in the rest of the paper we will also sometimes identify Morse and singular homologies).

Now we consider the Lagrangian case. Suppose that P and L are closed and that they satisfy the condition (2). Suppose also that Hamiltonian paths with the ends in L belong to Ω_0 (i.e. are zero in $\pi_1(P,L)$). Since the action functional a_H is well defined and since the differential ∂ preserves the filtration given by a_H, we can set

$$\begin{aligned} &CF^\lambda_*(H:L) := \left\{\sum c_x x \in CF_*(H:L) \mid c_x = 0 \text{ for } a_H(x) \geq \lambda\right\}\\ &\partial_\lambda := \partial|_{CF^\lambda_*(H:L)}\\ &HF^\lambda_*(H:L) := H_*(CF^\lambda_*(H:L), \partial_\lambda). \end{aligned}$$

Denote by

$$\jmath^\lambda_* : HF^\lambda_*(H:L) \to HF_*(H:L)$$

the homomorphism induced by the inclusion map $\jmath^\lambda$. For given singular (or Morse) homology class $\alpha \in HM_*(L,f)$, define

$$\sigma(\alpha, H) := \inf\left\{\lambda \mid \mathrm{PSS}(\alpha) \in \mathrm{Im}\left(\jmath^\lambda_*\right)\right\}.$$

Theorem 3.2. *If $\phi_H^1(L) = \phi_K^1(L)$, then*

$$\sigma(\alpha, H) - \sigma(\alpha, K) = C = C(H, K).$$

Proof: Denote by $L_1 = \phi_H^1(L) = \phi_K^1(L)$. Let $x, y \in L \cap L_1$. Let $c(s)$ be any smooth path in L connecting x and y (s.t. $c(0) = x$, $c(1) = y$). Denote by

$$\begin{aligned} &\gamma_{H,x}(t) := \phi_H^t(\phi^{-1}(x)), \quad \gamma_{K,x}(t) := \phi_K^t(\phi^{-1}(x)),\\ &\gamma_H^s(t) := \phi_H^t(\phi^{-1}(c(s))), \quad \gamma_K^s(t) := \phi_K^t(\phi^{-1}(c(s))),\\ &u_H(s,t) := \gamma_H^s(t), \quad u_K(s,t) := \gamma_K^s(t). \end{aligned}$$

Define $f(s)$ to be

$$f(s) := \iint_{D_s} \omega - \left(\int_{\gamma_H^s} Hdt - \int_{\gamma_K^s} Kdt \right),$$

where D_s is the surface consisting of the union of $u_H(\tau, t)$, $u_K(\tau, t)$ $(\tau, t) \in [0, s] \times [0, 1]$, and the two half discs with the boundaries on γ_H and L (respectively γ_K and L), which exists due to the assumption $[\gamma_{H,x}] = [\gamma_{K,x}] = 0 \in \pi_1(P, L)$. Now, using Stokes' and Cartan's formula, as well as the condition $\pi_2(P, L) = 0$, one easily derives $f'(s) = 0$. Obviously, $f(0) = a_H(\gamma_{H,x}) - a_K(\gamma_{K,x})$. Since $\pi_2(P, L) = 0$, it holds $f(1) = a_H(\gamma_{H,y}) - a_K(\gamma_{K,y})$. This means that $f \equiv const$, i. e.

$$a_H(\gamma_{H,x}) - a_K(\gamma_{K,x}) = a_H(\gamma_{H,y}) - a_K(\gamma_{K,y}).$$

We now proceed as in [15], namely, we switch to the geometric, instead of dynamic version of Floer homology. More precisely, there is a transformation between $L \cap L_1$ and Hamiltonian paths with ends on L, as well as perturbed holomorphic discs with boundary on L on one side, and *holomorphic* discs but with the conditions $u(s, 0) \in L$, $u(s, 1) \in L_1$, so we have an one-to-one correspondence between the generators and the boundary operators in two versions of Floer homology. Therefore, the set of elements of Floer homology that participate in the definitions on invariants are the same in two versions, so the claim follows. □

3.3. Chimneys and comparison between invariants

Let us first recall Albers' construction of a homomorphism between Floer homology for periodic Hamiltonian orbits and Lagrangian Floer homology. We assume that the Hamiltonian $H : P \times [0, 1] \to \mathbb{R}$ is admissible in the sense of [2], meaning that there are no constant contractible periodic orbits. For

$$\Sigma := \mathbb{R} \times [0, 1]/\sim, \quad \text{where } (s, 0) \sim (s, 1) \text{ for } s \leq 0,$$

$a \in CF_*(H)$ and $x \in CF_*(H : L)$ define the manifold of chimneys as:

$$\mathcal{M}(a, x) := \left\{ u : \Sigma \to P \;\middle|\; \begin{array}{l} \partial_s u + J(\partial_t u - X_H \circ u) = 0 \\ u(s, 0), u(s, 1) \in L, \text{ for } s \geq 0 \\ u(-\infty, t) = a(t),\, u(+\infty, t) = x(t) \end{array} \right\}$$

(see Figure 2). For $a \in CF_*(H)$, define

$$\tau(a) := \sum n(a, x)\, x$$

where $n(a, x)$ stands for the number (mod 2) of zero-dimensional component of $\mathcal{M}(a, x)$. This homomorphism descends to $HF_*(H)$, namely, since $\tau \circ \delta = \partial \circ \tau$, it is well defined as a map:

$$\tau : HF_*(H) \to HF_*(H : L).$$

The following diagram commutes:

$$\begin{tikzcd} HF_*(H) \arrow[r, "\tau"] \arrow[d, "\mathrm{PSS}^{-1}"'] & HF_*(H : L) \arrow[d, "\mathrm{PSS}^{-1}"'] \\ H_*(P) \arrow[r, "\imath_!"] & H_*(L) \end{tikzcd} \tag{8}$$

where $\imath_! = \mathrm{PD}^{-1} \circ \imath_* \circ \mathrm{PD}$ is a homomorphism defined by Poincaré duality map and the inclusion and $H_*(P)$ and $H_*(L)$ are singular or Morse homologies (see [2] for the details).

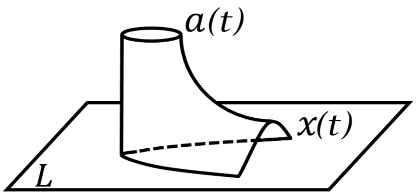

Figure 2: A "chimney" that defines the homomorphism τ

Proposition 3.3. *The homomorphism τ induces a homomorphism τ_λ on filtered homology groups:*

$$\tau_\lambda : HF_*^\lambda(H) \to HF_*^\lambda(H : L).$$

Proof: Let $a \in CF_*^\lambda(H)$ and $x \in CF_*^\lambda(H : L)$ such that there exists $u \in \mathcal{M}(a, x)$. Denote by $y(t) := u(0, t)$. Since $a \in \Omega_0(P)$ and there exists u connecting y and a, we have $y \in \Omega_0(P)$. Similarly, from $x \in \Omega_0(P, L)$ and the existence of u connecting y and x, we conclude $y \in \Omega_0(P, L)$. Since $y \in \Omega_0(P, L) \cap \Omega_0(P)$, it holds $a_H(y) = \mathcal{A}_H(y)$, so we have:

$$\begin{aligned}
&a_H(x) - \mathcal{A}_H(a) = a_H(x) - a_H(y) + \mathcal{A}_H(y) - \mathcal{A}_H(a) = \\
&\int_0^{+\infty} \frac{d}{ds} a_H(u(s, \cdot))ds + \int_{-\infty}^0 \frac{d}{ds}\mathcal{A}_H(u(s, \cdot))ds = \\
&- \int_{-\infty}^{\infty} \int_0^1 \omega\left(\partial_s u, \partial_t u - X_H \circ u\right) dtds = \\
&- \int_{-\infty}^{+\infty} \int_0^1 \|\partial_s u\|_{g_J}^2 \, dtds \leq 0.
\end{aligned} \tag{9}$$

Hence, if $a \in CF_*^\lambda(H)$, then $\tau(a) \in CF_*^\lambda(H : L)$. So, we can define

$$\tau_\lambda := \tau|_{CF_*^\lambda(H)} : CF_*^\lambda(H) \to CF_*^\lambda(H : L).$$

Let a be in $\mathrm{Im}(\delta_\lambda)$, i.e. $a = \delta b$, with $b \in CF_*(H)$ and $\mathcal{A}_H(b) < \lambda$. We know that $\mathcal{A}_H(a) < \lambda$ also, since the action functional decreases along perturbed holomorphic strips that define the differential δ. From (9) we have that $a_H(\tau(b)) < \lambda$, so

$$\tau_\lambda(\delta_\lambda b) = \tau_\lambda(a) = \tau(a) = \tau(\delta b) = \partial \tau(b) = \partial_\lambda \tau_\lambda(b),$$

since τ commutes with the differentials. This implies that τ_λ descends to the homology level. □

Remark 3.4. It is obvious that the diagram

$$\begin{tikzcd}
HF_*^\lambda(H) \arrow[r, "\tau_\lambda"] \arrow[d, "\imath_*^\lambda"'] & HF_*^\lambda(H : L) \arrow[d, "\jmath_*^\lambda"'] \\
HF_*(H) \arrow[r, "\tau"] & HF_*(H : L)
\end{tikzcd} \tag{10}$$

commutes. □

Theorem 3.5. *If $\alpha \in H_*(P)$ is a singular (or Morse) homology class, then $\rho(\alpha, H) \geq \sigma(\mathrm{PSS}(\imath_!(\alpha)), H)$.*

Proof: Consider the following commutative diagrams:

$$\begin{tikzcd} HF_*^\lambda(H) \arrow[r, "\tau_\lambda"] \arrow[d, "\imath_*^\lambda"'] & HF_*^\lambda(H:L) \arrow[d, "\jmath_*^\lambda"'] \\ HF_*(H) \arrow[r, "\tau"] \arrow[d, "\mathrm{PSS}^{-1}"'] & HF_*(H:L) \arrow[d, "\mathrm{PSS}^{-1}"'] \\ H_*(P) \arrow[r, "\imath_!"] & H_*(L) \end{tikzcd} \tag{11}$$

The upper diagram is (10) and the lower is Albers' (8). For given $\alpha \in H_*(P)$ and $\beta \in H_*(L)$, let us define the sets:

$$A_H(\alpha) := \{\lambda \mid \mathrm{PSS}(\alpha) \in \mathrm{Im}(\imath_*^\lambda)\}$$
$$A_{H:L}(\beta) := \{\lambda \mid \mathrm{PSS}(\beta) \in \mathrm{Im}(\jmath_*^\lambda)\}.$$

Let $\lambda \in A_H(\alpha)$. There exists $a \in HF_*^\lambda(H)$ such that $\mathrm{PSS}^{-1}(\imath_*^\lambda(a)) = \alpha$. Since both diagrams (11) commute, this implies that $\mathrm{PSS}^{-1}(\jmath_*^\lambda(\tau_\lambda(a))) = \imath_!(\alpha)$, so $\mathrm{PSS}(\imath_!(\alpha)) \in \mathrm{Im}(\jmath_*^\lambda)$. This means that $\lambda \in A_{H:L}(\imath_!(\alpha))$ i.e.

$$A_H(\alpha) \subset A_{H:L}(\imath_!(\alpha)).$$

Since

$$\rho(\alpha, H) = \inf A_H(\alpha), \quad \sigma(\beta, H) = \inf A_{H:L}(\beta)$$

the claim follows. $\square$

In the same way, considering Albers' commutative diagram

$$\begin{tikzcd} HF_*(H:L) \arrow[r, "\chi"] \arrow[d, "\mathrm{PSS}^{-1}"'] & HF_*(H) \arrow[d, "\mathrm{PSS}^{-1}"'] \\ H_*(L) \arrow[r, "\imath_*"] & H_*(P) \end{tikzcd}$$

where χ is also defined using chimneys, but in opposite direction (see [2]), one can prove the following

Theorem 3.6. *If $\beta \in H_*(L)$ is a singular (or Morse) homology class, then $\rho(\imath_*(\beta), H) \leq \sigma(\beta, H)$.* $\square$

4. Proof of Theorem 1.1

The product

$$\circ : HF_*(H_1) \otimes HF_*(H_2 : L) \to HF_*(H_3 : L)$$

is define by counting of a sort of pair-of-pants objects. More precisely, consider the disjoint union

$$\mathbb{R} \times [-1, 0] \sqcup \mathbb{R} \times [0, 1]$$

and identify $(s, 0^-)$ with $(s, 0^+)$ for all $s \geq 0$ as well as $(s, 0^+)$ with $(s, 1)$, for $s \leq 0$ (see figure below). Denote the obtained Riemannian surface with boundary by Σ. Denote by Σ_1, Σ_2, Σ_3 the two "incoming" and one "outgoing" ends, such that

$$\Sigma_1 \approx S^1 \times (-\infty, 0],$$
$$\Sigma_2 \approx [0, 1] \times (-\infty, 0],$$
$$\Sigma_3 \approx [0, 1] \times [0, +\infty),$$
$$\Sigma_0 := \Sigma \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$$

and by $u_j := u|_{\Sigma_j}$.

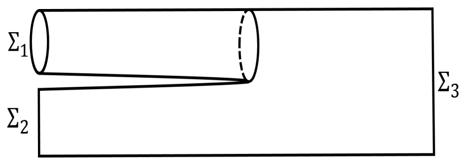

Figure 3: Riemannian surface Σ

Let $\rho_j : \mathbb{R} \to [0,1]$ denote the smooth cut-off functions such that

$$\rho_1(s) = \rho_2(s) = \begin{cases} 1, & s \leq -2, \\ 0, & s \geq -1 \end{cases} \qquad \rho_3(s) := \rho_1(-s).$$

For $a \in CF_*(H_1)$, $x \in CF_*(H_2 : L)$ and $y \in CF_*(H_3 : L)$, denote by $\mathcal{M}(a, x; y)$ the set of all u that satisfy

$$\begin{cases} u : \Sigma \to P, \\ \partial_s u_j + J(\partial_t u_j - X_{\rho_j H_j} \circ u_j) = 0, \ j = 1, 2, 3, \\ \partial_s u + J\partial_t u = 0, \text{ on } \Sigma_0, \\ u(s, -1), u(s, 0^-) \in L, \ s \leq 0, \\ u(s, -1), u(s, 1) \in L, \ s \geq 0, \\ u_1(-\infty, t) = a(t), \\ u_2(-\infty, t) = x(t), \\ u_3(+\infty, t) = y(t). \end{cases}$$

For generic choices, the set $\mathcal{M}(a, x; y)$ is a smooth manifold of dimension $\mu^{CZ}_{H_1}(a) + \mu_{H_2}(x) - \mu_{H_3}(y) + n$, where μ^{CZ} denotes the Conley–Zehnder index, and μ_{H_j} the (corresponding) Maslov index.

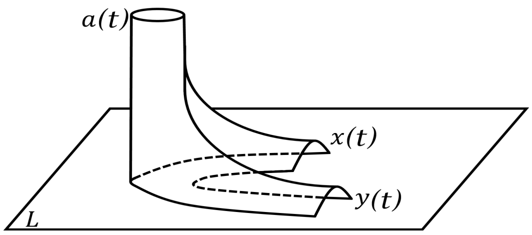

Figure 4: Moduli space $\mathcal{M}(a, x; y)$

Let $n(a, x; y)$ denote the number (modulo 2) of $\mathcal{M}(a, x; y)$ in dimension zero. Then we define the map product

$$HF_*(H_1) \otimes HF_*(H_2 : L) \to HF_*(H_3 : L)$$

by

$$a \circ x := \sum_y n(a, x; y) y$$

on generators and extend it by (bi)linearity on $HF_*(H_1) \otimes HF_*(H_2 : L)$. Using standard cobordism arguments, one can show that $\circ$ descends to the homology level and, when $H_2 = H_3$, it defines the product that makes $HF_*(H_2 : L)$ a module over $HF_*(H_1)$.

4.1. Proof of Theorem 1.1

In order to prove the inequality (1) and the Theorem 1.1, we consider, as in [17] and [20], the bundle $\widetilde{P} \to \Sigma$ whose fiber is isomorphic to (P, ω) and we fix the trivializations

$$\varphi_j : \widetilde{P}_j := \widetilde{P}|_{\Sigma_j} \to \Sigma_j \times P$$

for $j = 1, 2, 3$. On each $\widetilde{P}_j$ let

$$\tilde{\omega}_j := \varphi_j^*(\omega + d(\rho_j H_j dt)).$$

We will use the following theorem by Entov:

Theorem 4.1. *[3] There exists a closed two form $\tilde{\omega}$ such that*

(1) $\tilde{\omega}|_{\Sigma_j} = \tilde{\omega}_j$;

(2) $\tilde{\omega}$ *restricts to* ω *at each fiber;*

(3) $\tilde{\omega}^{\wedge(n+1)} = 0$.

Let $\tilde{\omega}$ be as in Theorem 4.1 and let

$$\Omega_\lambda := \tilde{\omega} + \lambda\omega_\Sigma,$$

where ω_Σ is an area form on Σ such that $\int_\Sigma \omega_\Sigma = 1$. Choose an almost complex structure $\widetilde{J}$ on $\widetilde{P}$ such that

(1) $\widetilde{J}$ is $\tilde{\omega}$ compatible on each fiber, so it preserves the vertical tangent space;

(2) the projection $\pi : \widetilde{P} \to \Sigma$ is $(\widetilde{J}, i)$ pseudoholomorphic, i.e. $d\pi \circ \widetilde{J} = i \circ d\pi$, where i is the complex structure on Σ;

(3) $(\varphi_j)_*\widetilde{J} = i \oplus J_j$, where $J_j(s,t,x) := (\phi^t_{\rho_j H_j})^* J$.

With such a choice, we get that the $\widetilde{J}$–holomorphic sections $\tilde{u}$ over Σ_j (or some shorter cylindrical ends, i.e., diffeomorphic to $(-\infty, K_j] \times S^1$, etc.) are precisely the solutions of

$$\partial_s u + J\left(\partial u_s - X_{(\rho_j H_j)} \circ u\right) = 0. \tag{12}$$

As in [20] or [3] we obtain, for $a \in CF_*(H_1)$, $x \in CF_*(H_2 : L)$ and $y \in CF_*(H_1 \sharp H_2 : L)$

$$\int \tilde{u}^*\tilde{\omega} = \mathcal{A}_{H_1}(a) + a_{H_2}(x) - a_{H_1\sharp H_2}(y), \tag{13}$$

whenever there exists a $\widetilde{J}$–holomorphic section $\tilde{u} : \Sigma \to \widetilde{P}$ that satisfies (12) on fibers. Since $\widetilde{J}$ is Ω_λ–compatible, it holds

$$0 \le \int \tilde{u}^*\Omega_\lambda = \int \tilde{u}^*\omega + \lambda \int \tilde{u}^*\omega_\Sigma = \int \tilde{u}^*\omega + \lambda \int \omega_\Sigma = \int \tilde{u}^*\omega + \lambda.$$

Now we use the Entov's result again, that enables us to choose, for any $\delta > 0$, a closed two form $\tilde{\omega}$ such that Ω_λ is symplectic for all $\lambda \ge \delta$ (see [3] Theorems 3.6.1 and 3.7.4).

Let $\delta > 0$, $a \in HF_*(H_1)$, $b \in HF_*(H_2 : L)$. Let $\tilde{a}$ and x be representatives of the classes a and b respectively, such that

$$\mathcal{A}_{H_1}(\tilde{a}) \le \rho(\mathrm{PSS}^{-1}(a), H_1) + \delta, \quad a_{H_2}(x) \le \sigma(\mathrm{PSS}^{-1}(b), H_2) + \delta.$$

For any $y \in a \circ b$, there exists $u \in \mathcal{M}(\tilde{a}, x; y)$, so we have

$$\begin{aligned} a_{H_1 \sharp H_2}(y) &\leq \mathcal{A}_{H_1}(\tilde{a}) + a_{H_2}(x) + \delta \\ &\leq \rho(\mathrm{PSS}^{-1}(a), H_1) + \delta + \sigma(\mathrm{PSS}^{-1}(b), H_2) + \delta + \delta \\ &= \rho(\mathrm{PSS}^{-1}(\alpha), H_1) + \sigma((\mathrm{PSS}^{-1}(b), H_2) + 3\delta. \end{aligned}$$

Since the above inequality is true for all $\delta > 0$ and y, we conclude

$$\sigma((\mathrm{PSS}^{-1}(a \circ b), H_1 \sharp H_2) \leq \rho(\mathrm{PSS}^{-1}(a), H_1) + \sigma((\mathrm{PSS}^{-1}(b), H_2),$$

so the Theorem 1.1 follows.

Remark 4.2. For a smooth submanifold L of P and three Morse functions

$$f_1 : P \to \mathbb{R}, \quad f_2, f_3 : L \to \mathbb{R}$$

one can define a Morse homology product

$$\cdot : HM_*(P, f_1) \otimes HM_*(L, f_2) \to HM_*(L, f_3)$$

as follows. Let p_j be critical points of f_j, for $j = 1, 2, 3$. The set $\mathcal{M}(p_1, p_2; p_3)$ is defined as the set of all trees $\gamma := (\gamma_1, \gamma_2, \gamma_3)$ such that

$$\begin{cases} \gamma_1 : (-\infty, 0] \to P, \ \gamma_2 : (-\infty, 0] \to L, \ \gamma_3 : [0, +\infty) \to L, \\ \dot{\gamma}_j = -\nabla f_j(\gamma_j), \ j = 1, 2, 3, \\ \gamma_1(-\infty) = p_1, \ \gamma_2(-\infty) = p_2, \ \gamma_3(+\infty) = p_3, \\ \gamma_1(0) = \gamma_2(0) = \gamma_3(0). \end{cases}$$

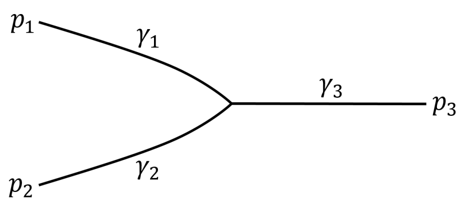

Figure 5: The set of trees $\mathcal{M}(p_1, p_2; p_3)$

For generic choices the set $\mathcal{M}(p_1, p_2; p_3)$ is a smooth manifold of dimension

$$m_{f_1}(p_1) + m_{f_2}(p_2) - m_{f_3}(p_3) - \dim P$$

where m_f is the corresponding Morse index. If $n(p_1, p_2; p_3)$ denotes the number of elements of zero-dimensional component, then the product $\cdot$ is defined as:

$$p_1 \cdot p_2 = \sum_{p_3} n(p_1, p_2; p_3) p_3$$

on generators.

Using the standard cobordism arguments, one can prove that the product $\circ$ and $\cdot$ commute with PSS type isomorphisms, more precisely, for $\alpha \in HM_*(P)$, $\beta \in HM_*(L)$, it holds:

$$\mathrm{PSS}(\alpha \cdot \beta) = \mathrm{PSS}(\alpha) \circ \mathrm{PSS}(\beta), \tag{14}$$

where PSS denote both types are PSS-type isomorphisms. Let

$$a := \mathrm{PSS}(\alpha), \quad b := \mathrm{PSS}(\beta).$$

From (1) and (14) we get

$$\begin{aligned}&\sigma(\alpha\cdot\beta, H_1\sharp H_2) = \sigma(\mathrm{PSS}^{-1}(a\circ b), H_1\sharp H_2) \leq \\ &\rho(\mathrm{PSS}^{-1}(a), H_1) + \sigma(\mathrm{PSS}^{-1}(b), H_2) = \rho(\alpha, H_1) + \sigma(\beta, H_2),\end{aligned}$$

so

$$\sigma(\alpha\cdot\beta, H_1\sharp H_2) \leq \rho(\alpha, H_1) + \sigma(\beta, H_2).$$

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