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Steklov Averages as Positive Linear Operators

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Abstract. We introduce a class of positive linear operators defined by Steklov means, investigate their properties and prove that the Weierstrass operators can be approximated in terms of the Steklov averages.

1. Introduction

For $b > 0$ let $L_{n,b} : C(\mathbb{R}) \to C(\mathbb{R})$ be defined by

$$\begin{cases} L_{0,b}f = f \\ L_{n,b}f(x) = \dfrac{1}{2b}\displaystyle\int_{x-b}^{x+b} L_{n-1,b}f(t)dt, \; n \geq 1, \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R})$, $x \in \mathbb{R}$.

Then $(L_{n,b}f)_{n\geq 0}$ are the Steklov averages of f with increment b; see [19, p. 163].Their relations with the theory of C_0-semigroups of operators were investigated in [7] and [8].

Expanding some results presented in [20, Ch.24], we shall give several analytic and probabilistic representations of the positive linear operators $L_{n,b}$.

As consequences, some properties of these operators will be derived. In particular, we give recurrence relations for computing the moments of the operators $L_{n,b}$ and the moments of the associated B-spline functions. Another result is concerned with the convergence of the sequence $(L_{n,b}f)_{n\geq 1}$ when f is periodic.

We shall see also that the Weierstrass operators can be approximated by means of the Steklov averages as well as in terms of the Bernstein operators. Some new interesting results on linear operators are given in [13]and [14].

2. Representations of the Steklov Averages

For $n \geq 1$ and $i = 0, 1, \ldots, n$, let $h_{n,i} = -1 + \dfrac{2i}{n}$. Given $x \in \mathbb{R}$ and $b > 0$, let $B_{n-1}^{x,b}$ be the B-spline function of degree $n-1$ associated to the points

$$x - b = x + bh_{n,0} < x + bh_{n,1} < \ldots < x + bh_{n,n} = x + b. \tag{2.1}$$

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Then $B_{n-1}^{x,b}$ is in $C^{n-2}(\mathbb{R})$ and vanishes outside $[x-b, x+b]$. The divided difference of a function $f \in C^n(\mathbb{R})$ on the nodes (2.1) can be expressed by

$$[x + bh_{n,0}, \dots, x + bh_{n,n} f] = \frac{1}{n!} \int_{-\infty}^{+\infty} f^{(n)}(t) B_{n-1}^{x,b}(t) dt. \tag{2.2}$$

For t and y in $\mathbb{R}$ we have

$$B_{n-1}^{x,b}(t) = B_{n-1}^{x+y,b}(t+y). \tag{2.3}$$

A well-known property of the B-spline functions (see, e.g., [18, Prop. 1.3.9]) asserts that

$$\frac{d}{dt} B_{n-1}^{x,nb}(t) = \frac{1}{2b} (B_{n-2}^{x-b,(n-1)b}(t) - B_{n-2}^{x+b,(n-1)b}(t)),\ n \geq 2. \tag{2.4}$$

Now let $X_i(x, n, b)$, $i = 1, \dots, n$, be independent random variables, uniformly distributed in $[x - nb, x + nb]$. Let

$$Y_n^{x,b} = \frac{1}{n} \sum_{i=1}^{n} X_i(x, n, b).$$

The probability density of $\frac{1}{n} X_i(x, n, b)$ is the function

$$\varphi_{x,n,b}(t) = \begin{cases} \frac{1}{2b}, & t \in \left[\frac{x}{n} - b, \frac{x}{n} + b\right] \\ 0, & \text{otherwise.} \end{cases}$$

It is well-known that the density of $Y_n^{x,b}$ is $B_{n-1}^{x,nb}$, which is also the n-fold convolution of $\varphi_{x,n,b}$.

On the other hand, it is easy to prove that $\varphi_{x,n,b}$ is a Pólya frequency function (see the definition in [11]). Using Proposition 1.5, p. 333 of [11] we conclude that $B_{n-1}^{x,nb}$ is a Pólya frequency function.

Theorem 2.1. *For $f \in C(\mathbb{R})$ and $n \geq 1$, $L_{n,b} f$ is the convolution of f and $B_{n-1}^{0,nb}$. Moreover,*

$$L_{n,b} f(x) = E f(Y_n^{x,b}), \tag{2.5}$$

where E denotes mathematical expectation.

Proof. For a fixed $b > 0$ let $A_n f$ be the convolution of f and $B_{n-1}^{0,nb}$, i.e.,

$$A_n f(x) = \int_{-\infty}^{+\infty} f(t) B_{n-1}^{0,nb}(x-t) dt,\ x \in \mathbb{R}. \tag{2.6}$$

Since $B_{n-1}^{0,nb}$ is an even function we have alternatively

$$A_n f(x) = \int_{-\infty}^{+\infty} f(t) B_{n-1}^{0,nb}(t-x) dt = \int_{-\infty}^{+\infty} f(u+x) B_{n-1}^{0,nb}(u) du. \tag{2.7}$$

Let $f_1 \in C^1(\mathbb{R})$, $f_1' = f$. Then (2.7) yields

$$\frac{d}{dx} A_n f_1(x) = A_n f(x). \tag{2.8}$$

By using (2.4) and (2.3) we obtain

$$A_n f(x) = \frac{d}{dx} \int_{-\infty}^{+\infty} f_1(t) B_{n-1}^{0,nb}(x-t) dt$$

$$= \frac{1}{2b}\int_{-\infty}^{+\infty} f_1(t)(B_{n-2}^{-b,(n-1)b}(x-t) - B_{n-2}^{b,(n-1)b}(x-t))dt$$

$$= \frac{1}{2b}\int_{-\infty}^{+\infty} f_1(t)(B_{n-2}^{0,(n-1)b}(x+b-t) - B_{n-2}^{0,(n-1)b}(x-b-t))dt$$

$$= \frac{1}{2b}(A_{n-1}f_1(x+b) - A_{n-1}f_1(x-b)) = \frac{1}{2b}\int_{x-b}^{x+b} \frac{d}{dt}A_{n-1}f_1(t)dt.$$

Now (2.8) implies

$$A_n f(x) = \frac{1}{2b}\int_{x-b}^{x+b} A_{n-1}f(t)dt.$$

This means that $L_{n,b}$ and A_n satisfy the same recurrence relation. Moreover, from (1.1) and (2.6) we deduce

$$L_{1,b}f(x) = \frac{1}{2b}\int_{x-b}^{x+b} f(t)dt = A_1 f(x).$$

Thus $L_{n,b} = A_n$, $n \geq 1$. In particular,

$$L_{n,b}f(x) = \int_{-\infty}^{+\infty} f(t)B_{n-1}^{x,nb}(t)dt = Ef(Y_n^{x,b})$$

and the proof is finished. □

Remark 2.2. *As a consequence of (2.5) and (2.2) we have also the following representations:*

$$L_{n,b}f(x) = \frac{1}{(2nb)^n}\int_{x-nb}^{x+nb}\cdots\int_{x-nb}^{x+nb} f\left(\frac{t_1+\ldots+t_n}{n}\right)dt_1\ldots dt_n, \tag{2.9}$$

$$L_{n,b}f(x) = n![x+nbh_{n,0},\ldots,x+nbh_{n,n};f_n], \tag{2.10}$$

where f_n is an arbitrary function in $C^n(\mathbb{R})$ with $f_n^{(n)} = f$.

3. Some Properties of the Operators $L_{n,b}$

(I) Let $e_i(t) = t^i$, $t \in \mathbb{R}$, $i = 0, 1, \ldots$. By using (1.1) it is easy to deduce that

$$L_{n,b}e_i = e_i + p_{n,b,i}, \tag{3.1}$$

where $p_{n,b,i}$ is a polynomial of degree $\leq i-1$. In particular,

$$L_{n,b}e_0 = e_0; \quad L_{n,b}e_1 = e_1; \quad L_{n,b}e_2 = e_2 + \frac{nb^2}{3}e_0. \tag{3.2}$$

By using the general theory (see [4]) we can derive various qualitative and quantitative Korovkin-type results involving the positive linear operators $L_{n,b}$. (See also [15] and [16]).

In establishing such results, an important role is played by the moments of the operators $L_{n,b}$, defined for $m = 0, 1, \ldots$ and $x \in \mathbb{R}$ by

$$M_{n,b,m}(x) := \frac{1}{m!}L_{n,b}(e_1 - xe_0)^m(x). \tag{3.3}$$

In fact, we have

$$m!M_{n,b,m}(x) = \int_{-\infty}^{+\infty}(t-x)^m B_{n-1}^{0,nb}(t-x)dt = \int_{-\infty}^{+\infty} u^m B_{n-1}^{0,nb}(u)du.$$

It follows that the moments $M_{n,b,m}$ are constant functions, and the numbers $m!M_{n,b,m}$ are the moments of the function $B_{n-1}^{0,nb}$.

Theorem 3.1. *For each $k = 0, 1, \dots$ we have*

$$M_{n,b,2k+1} = 0, \quad n \geq 1, \tag{3.4}$$

$$M_{n,b,2k} = c_{n,2k} b^{2k}, \quad n \geq 1, \tag{3.5}$$

where

$$c_{1,2k} = \frac{1}{(2k+1)!} \tag{3.6}$$

and

$$c_{n,2k} = \sum_{l=0}^{k} \frac{1}{(2l+1)!} c_{n-1,2k-2l}, \quad n \geq 2. \tag{3.7}$$

Proof. (3.4) is a consequence of the fact that $B_{n-1}^{0,nb}$ is an even function. Let us remark that

$$L_{n,b} = L_{1,b} \circ L_{n-1,b} = \dots = L_{1,b}^{n},$$

and consequently

$$L_{n,b} = L_{n-1,b} \circ L_{1,b}. \tag{3.8}$$

From (3.8) and [9, Theorem 4] we deduce

$$M_{n,b,2k} = \sum_{i=0}^{2k} M_{n-1,b,2k-i} M_{1,b,i}.$$

On the other hand, $M_{1,b,2l+1} = 0$ and

$$M_{1,b,2l} = \frac{1}{(2l+1)!} b^{2l}, \quad l \geq 0. \tag{3.9}$$

It follows that

$$M_{n,b,2k} = \sum_{l=0}^{k} \frac{1}{(2l+1)!} b^{2l} M_{n-1,b,2k-2l}. \tag{3.10}$$

(3.5) for $n = 1$, and (3.6) are consequences of (3.9). Using (3.10) and induction on n, it is easy to prove (3.5) and (3.7). □

Example. From Theorem 3.1 it is easy to obtain

$$c_{n,0} = 1, c_{n,2} = \frac{n}{6}, c_{n,4} = \frac{5n^2 - 2n}{360}, n \geq 1.$$

(II) We have seen that

$$L_{n,b} f(x) = \int_{-\infty}^{+\infty} f(t) B_{n-1}^{0,nb}(x-t) dt$$

and $B_{n-1}^{0,nb}$ is a Pólya frequency function. Consequently,

(a) $L_{n,b}$ has the variation-diminishing properties in the sense of [11], Section 3 of Chapter 1 and Section 4 of Chapter 5. (See also Theorem 4.6, p. 249);

(b) If f is convex with respect to the Tchebycheff system $\{\varphi_1, \dots, \varphi_m\}$, then $L_{n,b} f$ is convex with respect to $\{L_{n,b}\varphi_1, \dots, L_{n,b}\varphi_m\}$. (See [11], Section 4 of Chapter 1).

(III) Due to the probabilistic representation (2.5), we can apply the results of [2] and [3] in order to prove that $L_{n,b}$ preserves global smoothness and diminishes both the ϕ-variation and the fine ϕ-variation. Moreover, if the sequence (nb_n) is decreasing and $f \in C(\mathbb{R})$ is convex, then

$$L_{n,b_n} f \geq L_{n+1,b_{n+1}} f \geq f.$$

The same probabilistic representation allows us to apply to $L_{n,b}$ the Casteljau-type algorithm discussed in [17].

(IV) Let us mention also the following Voronovskaja-type formula established in [10] (see also [1]):

$$\lim_{n\to\infty} n^k \left(L_{n,\frac{1}{n}} f(x) - \sum_{i=0}^{k-1} L_{n,\frac{1}{n}} e_{2i}(0) \frac{f^{(2i)}(x)}{(2i)!} \right) = \frac{1}{k!6^k} f^{(2k)}(x)$$

for $f \in C(\mathbb{R})$ $2k$-times differentiable at x. In particular,

$$\lim_{n\to\infty} n(L_{n,\frac{1}{n}} f(x) - f(x)) = \frac{1}{6} f''(x),$$

$$\lim_{n\to\infty} n\Big(n\Big(n\Big(L_{n,\frac{1}{n}} f(x) - f(x)\Big) - \frac{1}{6} f''(x)\Big) - \frac{1}{72} f^{IV}(x)\Big) = \frac{f^{VI}(x)}{1296} - \frac{f^{IV}(x)}{180}.$$

(V) Let $C_{2\pi} := \{f \in C(\mathbb{R}) : f \text{ is } 2\pi - \text{periodic}\}$, and $b > 0$ fixed.

Theorem 3.2. *For each $f \in C_{2\pi}$ and $b > 0$, $\lim_{n\to\infty} L_{n,b} f = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(t)dt \right) \mathbb{1}$, uniformly on $\mathbb{R}$.*

Proof. Let

$$p(x) = a_0 + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx).$$

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(t)dt$$

and

$$L_{n,b} p(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(t)dt + \sum_{k=1}^{m} \left(\frac{\sin kb}{kb} \right)^n (a_k \cos kx + b_k \sin kx),$$

so that

$$\lim_{n\to\infty} L_{n,b} p = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} p(t)dt \right) \mathbb{1}, \tag{3.11}$$

uniformly on $\mathbb{R}$.

Let $f \in C_{2\pi}$ and $\varepsilon > 0$. Then there exists a trigonometric polynomial p such that $\|f - p\|_\infty \leq \frac{\varepsilon}{3}$ (see [12], p.413).

Consequently we have also

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} p(t)dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt \right| \leq \frac{\varepsilon}{3}.$$

According to (3.3),

$$\exists n_\varepsilon : \left\| L_{n,b} p - \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} p(t)dt \right) \mathbb{1} \right\|_\infty \leq \frac{\varepsilon}{3}, \forall n \geq n_\varepsilon.$$

Now we have

$$\begin{aligned}\left\|L_{n,b}f-\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}f(t)dt\right)\mathbb{1}\right\|_{\infty}&\leq\left\|L_{n,b}(f-p)\right\|+\left\|L_{n,b}p-\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}p(t)dt\right)\mathbb{1}\right\|_{\infty}\\&+\left\|\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}(p(t)-f(t)dt\right)\mathbb{1}\right\|\\&\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\forall n\geq n_{\varepsilon}.\end{aligned}$$

This finishes the proof. □

4. Approximation of the Weierstrass Operator by Steklov Averages and Bernstein Operators

We shall apply a probabilistic technique from [6] in order to approximate the Weierstrass operator defined by

$$\begin{cases}W_0f=f,\\ W_tf(x)=(2\pi t)^{-1/2}\displaystyle\int_{-\infty}^{+\infty}f(u)\exp\left(-\frac{(u-x)^2}{2t}\right)du,\ t>0,\end{cases}\tag{4.1}$$

where $x\in\mathbb{R}$ and f is in the space $CB(\mathbb{R})$ of continuous and bounded functions on $\mathbb{R}$.

Theorem 4.1. *Let $b_n>0$, $n\geq 1$ be such that*

$$\lim_{n\to\infty}nb_n^2=a\geq 0.\tag{4.2}$$

Then for all $f\in CB(\mathbb{R})$ and $x\in\mathbb{R}$ we have

$$\lim_{n\to\infty}L_{n,b_n}f(x)=W_{a/3}f(x).$$

Proof. The probability density of Y_n^{x,b_n} is B_{n-1}^{x,nb_n} and the characteristic function is

$$e^{itx}\left(\frac{\sin b_nt}{b_nt}\right)^n,\ t\in\mathbb{R}.$$

When $n\to\infty$, this sequence of functions converges on $\mathbb{R}$ to the function

$$\exp\left(itx-\frac{at^2}{6}\right),\ t\in\mathbb{R}.$$

From Lévy's convergence theorem we deduce that for $a>0$ and $y\in\mathbb{R}$,

$$\lim_{n\to\infty}\int_{-\infty}^{y}B_{n-1}^{x,nb_n}(t)dt=\left(\frac{3}{2\pi a}\right)^{1/2}\int_{-\infty}^{y}\exp\left(-\frac{3(u-x)^2}{2a}\right)du,\tag{4.3}$$

while for $a=0$ and $y\in\mathbb{R}$,

$$\lim_{n\to\infty}\int_{-\infty}^{y}B_{n-1}^{x,nb_n}(t)dt=\begin{cases}0, & y<x\\ 1/2, & y=x\\ 1, & y>x.\end{cases}\tag{4.4}$$

Let us remark that (4.3) and (4.4) are concrete illustrations of a general result contained in Theorem 5.1, p. 533 of [11].

Now a well-known result from Probability Theory (see, e.g., [5], or Theorem 1 in [6]) yields

$$\lim_{n\to\infty} L_{n,b_n} f(x) = W_{a/3} f(x)$$

for $f \in CB(\mathbb{R})$ and $x \in \mathbb{R}$. This finishes the proof. □

The Weierstrass operator can be also approximated by means of the Bernstein operators. Indeed, let b_n be as in (4.2), $0 < t < 1$, and Z_n a binomial variable with parameters n and t. Then, for a fixed $x \in \mathbb{R}$, the sequence of random variables

$$x + \frac{nb_n}{\sqrt{t(1-t)}}\left(\frac{Z_n}{n} - t\right),\ n \geq 1,$$

converges in law to:

- a normal variable with mean x and variance a, if $a > 0$;

- the constant x, if $a = 0$.

Consequently, if $B_n(g(u); t)$ are the classical Bernstein operators associated to a function $g(u)$, then

$$\lim_{n\to\infty} B_n\left(f\left(x + \frac{nb_n}{\sqrt{t(1-t)}}(u-t)\right); t\right) = W_a f(x),\ f \in CB(\mathbb{R}). \tag{4.5}$$

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