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Representation of the Fourier Transform of Distributions in $K'_{p,k}, k < 0$.

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Abstract. In this note we give a structure theorem of the distributions in the space $K'_{p,k}, k < 0$, which is a subspace of the space of distributions which grow no faster than $e^{|x|^p}, p > 1$, and use this structure theorem to give a representation of the Fourier transform of the distributions in these spaces.

The Fourier transform of members of several spaces of distributions has been studied by several authors. Gonzalez and Negrin [5] studied the Fourier transform over the spaces $S'_k, k \in Z, k < 0$ of tempered distributions introduced by Horvath [2]. They have shown that the Fourier transform maps each of the spaces $S'_k, k \in Z, k < 0$ onto itself, and proved a representation theorem for the usual Fourier transform of members of these spaces. Hayek, Gonzalez and Negrin [3] proved an inversion formula for the distributional Fourier transform on the spaces $S'_k, k \in Z, k < 0$. They applied their results to obtain a representation on S' for any distribution of S'_k as limit of a sequence of ordinary functions. Gonzalez [4], established a structure theorem of the members of the spaces S'_k and gave a representation of the Fourier transform of these members. Sohn and Pahk [6] introduced the spaces $\mathcal{K}'_{p,k}, k \in Z, k < 0, p > 1$, of distributions of exponential growth. Among other things they studied the Fourier transform of members of these spaces and gave an inversion formula for the elements of the spaces. In this work, along the lines of Barrose-Neto [1, proof of Theorem 6.2], we establish a structure theorem for the distributions in the spaces $\mathcal{K}'_{p,k}, k \in Z, k < 0, p > 1$, then we use this structure theorem to get a representation of the Fourier transform of the elements of these spaces.

1. Preliminaries

We use the standard notations and terminology of Horvath [2] for spaces of functions and distributions. The space $\mathcal{K}_{p,k}, k \in Z, k < 0, p > 1$ of test functions and its dual $\mathcal{K}'_{p,k}, k \in Z, k < 0, p > 1$ are as given by Sohn and Pahk [6]. The space $\mathcal{K}_p$ of functions of exponential decay consists of all functions $\varphi \in C^\infty(\mathcal{R}^n)$ such that

$$v_k(\varphi) = \sup_{\substack{|\alpha| \le k \\ x \in \mathfrak{R}^n}} e^{k|x|^p} \mid D^\alpha \varphi(x) \mid < \infty; \ k = 1, 2, 3, ... \tag{1}$$

The space $\mathcal{K}_p$ with semi-norms $v_k, k = 1, 2, 3, ...$is a Frechet space and the space $\mathcal{D}$ of test functions of compact support is dense in $\mathcal{K}_p$. As in Sohn and Pahk [6], the spaces $\mathcal{K}_{p,k}$ consist of all functions φ in $C^\infty(\mathcal{R}^n)$ such

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that for any $\alpha \in \mathcal{N}^n$ and $\epsilon > 0$, there exists a constant $C = C(f, \alpha, \epsilon) > 0$ such that

$$e^{k|x|^p} \mid D^\alpha \varphi(x) \mid \leq \epsilon \quad \text{for} \quad \mid x \mid > C. \tag{2}$$

We provide $\mathcal{K}_{p,k}$ with the topology defined by the family of seminorms

$$q_{k,\alpha}(\varphi) = \sup_{x \in \mathcal{R}^n} e^{k|x|^p} \mid D^\alpha \varphi(x) \mid, \quad \alpha \in \mathcal{N}^n. \tag{3}$$

It turnes out that $\mathcal{K}_{p,k}$ is a locally convex space which contains $\mathcal{D}$ as a dense subspace. Its strong dual is denoted by $\mathcal{K}'_{p,k}$.

Sohn and Pahk [6] define convolution between elements of $\mathcal{K}'_{p,k}, k < 0, k \in \mathcal{Z}$. If $S, T \in \mathcal{K}'_{p,2^p k}$ and $\varphi \in \mathcal{K}_{p,k}$ the convolution $S * T$ of S and T is defined by

$$\langle S * T, \varphi \rangle = \left\langle S_x, \left\langle T_y, \varphi(x + y) \right\rangle \right\rangle, \tag{4}$$

where the right hand side is understood as the application of the distribution S to the function $\left\langle T_y, \varphi(x + y) \right\rangle \in \mathcal{K}_{p,2^p k}$. It turnes out that $S * T \in \mathcal{K}'_{p,k}$.

Let $T \in \mathcal{K}'_{p,k}, k \in Z, k < 0$. The Fourie transform of T is represented, for each $y \in \mathcal{R}^n$, by

$$(\mathcal{F}T)(y) = \left\langle T_x, e^{ixy} \right\rangle. \tag{5}$$

It follows that $\mathcal{F}T$ is in $\mathcal{K}'_{p,k}, k \in Z, k < 0$, and the Parseval equality

$$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle ; \tag{6}$$

holds true, where $\mathcal{F}\varphi$ is the classical Fourier transorm of $\varphi \in \mathcal{K}_{p,k}$ (see [6]).

2. The Results

Theorem 2.1. *Let $k \in Z$, and $T \in \mathcal{K}'_{p,k}, k < 0$. Then there exist $m \in N$ and $(g_q)_{|q| \leq m}$, $q \in N^n$ continuous functions such that*

$$T = \sum_{|q| \leq m} \partial^q g_q \tag{7}$$

over $\mathcal{K}_{p,k}$, where $\mid g_q(x) \mid \leq M_q e^{(k+n)|x|^p}$, for all $x \in R^n$, and $M_q > 0$ for all $\mid q \mid \leq m$.

Proof. Since $T \in \mathcal{K}'_{p,k}$ it is continuous on $\mathcal{K}_{p,k}$, hence there exist a positive constant C and a nonnegative integer j such that

$$\mid \langle T, \varphi \rangle \mid \leq C \sup_{\substack{\mid \eta \mid \leq j \\ x \in R^n}} e^{k|x|^p} \mid \partial^\eta \varphi(x) \mid ; \quad \forall \varphi \in \mathcal{K}_{p,k} \tag{8}$$

Moreover, for any $\varphi \in \mathcal{K}_{p,k}$, $x = (x_1, x_2, ..., x_n) \in R^n$ and all $\alpha \in N^n$ one has

$$\mid e^{k|x|^p} \partial^\alpha \varphi(x) \mid \leq \int_{-\infty}^{x_1} dt_1 \int_{-\infty}^{x_2} dt_2 ... \int_{-\infty}^{xn} \mid \frac{\partial^n}{\partial t_1 \partial t_2 ... \partial t_n} \{ e^{k|t|^p} \partial^q \varphi(t) \} \mid dt_n. \tag{9}$$

Taking $\beta = (1,1,1,...,1)$ it follows from Leibniz formula that

$$\partial^{\beta}\{e^{k|t|^p}\partial^q\varphi(t)\} = \sum_{\alpha\le\beta}\partial^{\alpha}(e^{k|t|^p})\partial^{\beta-\alpha}(\partial^q\varphi(t)), \tag{10}$$

(because $\binom{\beta}{\alpha} = 1$). Now, for all $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in N^n, \alpha \le \beta$, one has

$$|\, \partial^{\alpha}(e^{k|t|^p}) \,| \le C_{\alpha,k,p}e^{(k+1)|t|^p} \quad (k < 0).$$

Thus , for all $\alpha \in N^n$ with $\alpha \le \beta$, it follows that

$$|\, \partial^{\beta}\{e^{k|t|^p}\partial^q\varphi(t)\} \,| \le \sum_{\alpha\le\beta} C_{\alpha,k,p}e^{(k+1)|t|^p}\partial^{\beta-\alpha}(\partial^q\varphi(t)) \,| \,. \tag{11}$$

Therefore (by continuity of T) there exists a positive constant C_1 such that

$$|\, \langle T,\varphi\rangle \,| \le C_1 \sup_{|\eta|\le j+n} \| e^{(k+1)|x|^p}\partial^{\eta}\varphi(x) \|_1; \quad \forall\varphi \in \mathcal{K}_{p,k}. \tag{12}$$

Set $l = j + n$ and let m be the number of n-tuples $q \in N^n$ which satisfy $|\, q \,| \le l$. Consider the product space $(L^1(R^n)^m = L^1(R^n) \times L^1(R^n) \times ... \times L^1(R^n)$ (m copies) provided with the product topology, and the injection

$$J : \mathcal{K}_{p,k} \to (L^1(\mathcal{R}^n))^m$$

$$J(\varphi)(x) = (e^{(k+1)|x|^p}\partial^{q_1}\varphi(x), e^{(k+1)|x|^p}\partial^{q_2}\varphi(x), ..., e^{(k+1)|x|^p}\partial^{q_m}\varphi(x)), \tag{13}$$

where $q_1, q_2, ..., q_m$ are all members of N^n with $|\, q_j \,| \le l, 1 \le j \le m$.
Define the map $\mathcal{L}_T : J(\mathcal{K}_{p,k}) \to C$ by

$$\mathcal{L}_T(e^{(k+1)|x|^p}\partial^{q_1}\varphi(x), e^{(k+1)|x|^p}\partial^{q_2}\varphi(x), ..., e^{(k+1)|x|^p}\partial^{q_m}\varphi(x)) = \langle T,\varphi\rangle \tag{14}$$

It follows from inequality (2.6) that $\mathcal{L}_T$ is a continuous linear functional. It follows from the Hahn-Banach theorem that we can extend it as a continuous linear functional on all of $(L^1(R^n)^m$ with the same norm. Since the dual of $L^1(R^n)$ is $L^{\infty}(R^n)$, it follows from the Riesz representation theorem that there exist m measurable functions $\phi_q \in L^{\infty}(R^n), |\, q \,| \le l$, such that

$$\mathcal{L}_T(\psi_{q_1}, \psi_{q_2}, ..., \psi_{q_m}) = \sum_{|q|\le l}\int_{\mathcal{R}^n}\phi_q(t)\psi_q(t)dt, \tag{15}$$

for all $(\psi_{q_1}, \psi_{q_2}, ..., \psi_{q_m}) \in (L^1(R^n)^m$. In particular,

$$\mathcal{L}_T(J(\varphi)) = \langle T,\varphi\rangle = \sum_{|q|\le l}\int_{\mathcal{R}^n}\phi_q(t)e^{(k+1)|t|^p}\partial^q\varphi(t)dt \,, \forall\varphi \in \mathcal{K}_{p,k}. \tag{16}$$

Hence

$$T = \sum_{|q|\le l}(-1)^{|q|}\partial^q[e^{(k+1)|t|^p}\phi_q(t)] \,, \text{ over } \mathcal{K}_{p,k}. \tag{17}$$

Put $h_q(t) = (-1)^{|q|}[e^{(k+1)|t|^p}\phi_q(t)]$, $|q| \le l$. Since $e^{-(k+1)|t|^p}h_q \in L^\infty(R^n)$ for all $|q| \le l$, it follows that

$$T = \sum_{|q|\le l} \partial^q h_q \quad \text{over } \mathcal{K}_{p,k}. \tag{18}$$

For $q \in N^n$ with $|q| \le l$ define the function θ_q on R^n by

$$\theta_q(x) = \int_0^{x_1} dt_1 \int_0^{x_2} dt_2 ... \int_0^{x_n} e^{-(k+1)|t|^p} h_q(t)dt_n, \quad x = (x_1, x_2, ..., x_n). \tag{19}$$

Since $e^{-(k+1)|t|^p}h_q(t) \in L^\infty(R^n)$ it follows that $h_q \in L^1_{loc}(R^n)$ and θ_q are continuous functions on R^n (because the partial derivatives exist and they are continuous). Moreover, for $\beta = (1, 1, ...1)$ one has

$$\partial^\beta \theta_q(x) = e^{-(k+1)|x|^p} h_q \text{ a.e..}$$

Thus

$$\begin{aligned} |\theta_q(x)| &= \left| \int_0^{x_1} dt_1 \int_0^{x_2} dt_2 ... \int_0^{x_n} e^{-(k+1)|t|^p} h_q(t)dt_n \right| \\ &\le \| e^{-(k+1)|t|^p} h_q \|_\infty \left| \int_0^{x_1} dt_1 \int_0^{x_2} dt_2 ... \int_0^{x_n} dt_n \right| \\ &\le | x_1 x_2 ... x_n | \| e^{-(k+1)|t|^p} h_q \|_\infty, \end{aligned} \tag{20}$$

for all $x = (x_1, x_2, ..., x_n) \in R^n$ and all $q \in N^n$ with $|q| \le l$. Using differentiation formulas one has

$$e^{k|x|^p} \partial^\beta \theta_q(x) = \sum_{\alpha\le\beta} (-1)^{|\alpha|} \partial^{\beta-\alpha}[\{\partial^\alpha e^{k|x|^p}\}\theta_q(x)]. \tag{21}$$

Also, for all $\alpha \in N^n$, it follows that

$$\partial^\alpha(e^{k|x|^p}) \le e^{k|x|^p} \sum_{\gamma\le\alpha} M_{\alpha,\gamma} |x|^{r(p,\gamma)} \le e^{(k+1)|x|^p}, \tag{22}$$

where $r(p, \gamma)$ is a function of p and γ. It follows from (2.12), (2.14), (2.15) and (2.16) that,

$$\begin{aligned} T &= \sum_{|q|\le l} \partial^q h_q = \sum_{|q|\le l} \partial^q [e^{(k+1)|x|^p} \partial^\beta \theta_q(x)] \\ &= \sum_{|q|\le l} \partial^q [\sum_{\alpha\le\beta} (-1)^{|\alpha|} \partial^{\beta-\alpha} \{(\partial^\alpha e^{(k+1)|x|^p})\theta_q(x)\}] \\ &= \sum_{|q|\le l} \sum_{\alpha\le\beta} \partial^q \partial^{\beta-\alpha}((-1)^{|\alpha|}\{[\sum_{\gamma\le\alpha} M_{\alpha,\gamma} x^{r(p,\gamma)}]e^{(k+1)|x|^p}\theta_q(x)\} \\ &= \sum_{|\nu|\le l+n} \partial^\nu g_\nu(x) = \sum_{|\nu|\le m} \partial^\nu g_\nu(x), \end{aligned} \tag{23}$$

where

$$g_\nu(x) = e^{(k+1)|x|^p}\Big(\sum_{\gamma\le\alpha}(-1)^{|\alpha|}M_{\alpha,\gamma}x^{r(p,\gamma)}\theta_q(x)\Big), \tag{24}$$

for $\nu = q + \beta - \gamma$, and $g_\nu(x) = 0$ otherwise, and

$$| g_\nu(x) |\le e^{(k+1)|x|^p}M_\nu \mid x_1x_2...x_n \mid^{|\gamma|}\mid x\mid^{r(p,\gamma)} \le M_\nu e^{(k+n)|x|^p} \tag{25}$$

This completes the proof of the theorem. □

Theorem 2.2. *Let $T \in K'_{p,k}, 2k + 3n < 0, k \in Z$, be given by the representation*

$$T = \sum_{|q|\le m}\partial^q g_q; \tag{26}$$

where $(g_q)_{|q|\le m}$, $q \in N^n$ as in theorem 1. Then $\hat{T}$, the Fourier transform of T, is given by

$$\hat{T}(y) = \left\langle T_x, e^{ixy}\right\rangle = \sum_{|q|\le m}(-iy)^q\hat{g}_q(y); \quad y \in \mathcal{R}^n, \tag{27}$$

where $\hat{g}_q(y) = \int_{\mathcal{R}^n} g_q(x)\, e^{ixy}dx$ is the classical Fourier transform of g_q.

Proof. Since g_q decreases very rapidly it follows that the integral is convergent and $\hat{g}_q$ exists. Using (2.19) and polar coordinates one gets

$$\int_{\mathcal{R}^n} \mid g_q(x)\mid dx \le M_q\int_{\mathcal{R}^n} e^{(k+n)|x|^p}dx;$$

$$= M_q\int_0^{2\pi}\int_0^{2\pi}...\int_0^{2\pi}\int_0^{\infty} d\theta_1 d\theta_2...d\theta_{n-1}(1 + r^2)^{k+n}r^{n-1}dr \tag{28}$$

which converges for $2k + 3n < 0$.By the Parseval equality it follows that for any $T \in K'_{p,k}, k + n < 0$, one has

$$\left\langle T,\overset{\wedge}{\varphi}\right\rangle = \left\langle \hat{T},\varphi\right\rangle = \int_{\mathcal{R}^n}\left\langle T_x, e^{ixy}\right\rangle\varphi(y)dy \tag{29}$$

for all $\varphi \in K_p$.

It follows from theorem 1 that

$$\begin{aligned}\left\langle T,\overset{\wedge}{\varphi}\right\rangle &= \sum_{|q|\le m}\left\langle \partial^q g_q(x),\overset{\wedge}{\varphi}(x)\right\rangle = \sum_{|q|\le m}\left\langle g_q(x),\partial^q\overset{\wedge}{\varphi}(x)\right\rangle\\ &= \sum_{|q|\le m}\left\langle g_q(x),\widehat{(-ix)^q\varphi(x)}\right\rangle\\ &= \sum_{|q|\le m}\left\langle \widehat{g_q}(x),(-ix)^q\varphi(x)\right\rangle = \sum_{|q|\le m}\left\langle (-ix)^q\widehat{g_q}(x),\varphi(x)\right\rangle\end{aligned} \tag{30}$$

Substituting in the left hand side of (2.23), one gets

$$\sum_{|q|\leq m}\left\langle(-ix)^q\widehat{g_q}(x),\varphi(x)\right\rangle=\int_{\mathcal{R}^n}\sum_{|q|\leq m}(-ix)^q\widehat{g_q}(x)\varphi(x)dx$$

$$=\int_{\mathcal{R}^n}\left\langle T_y,e^{ixy}\right\rangle\varphi(x)dx; \tag{31}$$

for all $\varphi\in K_p$. Since this is true for all $\varphi\in K_p$ and the functions $\sum_{|q|\leq m}(-ix)^q\widehat{g_q}(x)$, $\left\langle T_y,e^{ixy}\right\rangle$ are continuous on R^n, it follows that

$$\left\langle T_y,e^{ixy}\right\rangle=\sum_{|q|\leq m}(-ix)^q\widehat{g_q}(x). \tag{32}$$

□

Remark 2.3. *It follows from the above theorems that, if* $T_1,T_2\in\mathcal{K}'_{p,k}, k+n<0, k\in Z$ *then* $\widehat{T_1*T_2}=\widehat{T_1}.\widehat{T_2}$.

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