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Generalized Analytic Fourier-Feynman Transforms with Respect to Gaussian Processes on Function Space

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Abstract. In this article, we introduce a generalized analytic Fourier-Feynman transform and a multiple generalized analytic Fourier-Feynman transform with respect to Gaussian processes on the function space $C_{a,b}[0,T]$ induced by generalized Brownian motion process. We derive a rotation formula for our multiple generalized analytic Fourier-Feynman transform.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is, the space of all real-valued continuous functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m_w denote Wiener measure. Then $(C_0[0,T], \mathcal{M}, m_w)$ is a complete measure space.

The concept of the '*analytic*' Feynman integral on the Wiener space $C_0[0,T]$ was initiated by Cameron [2]. The foundation of the definition of the analytic Feynman integral also can be found in [1, 3]. There has been a tremendous amount of papers on the analytic Feynman integral theory. Furthermore, the concept of the analytic Fourier-Feynman transform on $C_0[0,T]$ has been developed in the literature. For an elementary introduction of the analytic Feynman integral and the analytic Fourier-Feynman transform, see [14] and the references cited therein.

The concepts of the analytic $\mathcal{Z}_h$-Wiener integral (the Wiener integral with respect to Gaussian paths $\mathcal{Z}_h$) and the analytic $\mathcal{Z}_h$-Feynman integral (the analytic Feynman integral with respect to Gaussian paths $\mathcal{Z}_h$) on $C_0[0,T]$ were introduced by Chung, Park and Skoug in [12], and further developed in [4, 10, 13]. In [4, 10, 12, 13], the $\mathcal{Z}_h$-Wiener integral is defined by the Wiener integral $\int_{C_0[0,T]} F(\mathcal{Z}_h(x,\cdot))dm_w(x)$ where $\mathcal{Z}_h(x,\cdot)$ is the Gaussian path given by the stochastic integral $\mathcal{Z}_h(x,t) = \int_0^t h(s)dx(s)$ with $h \in L^2[0,T]$.

On the other hand, in [5, 7–9], the authors studied a generalized analytic Fourier-Feynman transform and a generalized integral transform on the very general function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$, induced by generalized Brownian motion process, was introduced by J. Yeh [15, 16] and was used extensively in [5–9, 11].

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In this article, we introduce the generalized analytic $\mathcal{Z}_h$-Feynman integral, the generalized analytic $\mathcal{Z}_h$-Fourier-Feynman transform, and the multiple generalized analytic Fourier-Feynman transform with respect to Gaussian paths on the function space $C_{a,b}[0,T]$. We also derive a rotation formula involving the two transforms.

2. Preliminaries

In this section, we briefly list some of the preliminaries from [5, 7, 8, 15] that we will need to establish the results in this paper.

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$, $a'(t)\in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. The generalized Brownian motion process Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. By Theorem 14.2 [16, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence, $(C_{a,b}[0,T],\ \mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. We then complete this function space to obtain $(C_{a,b}[0,T],\mathcal{W}(C_{a,b}[0,T]),\mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

We note that the coordinate process defined by $e_t(x)=x(t)$ on $C_{a,b}[0,T]\times[0,T]$ is also the generalized Brownian motion process determined by $a(t)$ and $b(t)$. The function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$, considered in papers [1–4, 10, 12, 13] if and only if $a(t)\equiv 0$ and $b(t)=t$ for all $t\in[0,T]$. For more detailed studies about this function space $C_{a,b}[0,T]$, see [5–9, 11, 15].

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\rho B\in\mathcal{W}(C_{a,b}[0,T])$ for all $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\ \cdot)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho>0$. If two functionals F and G are equal s-a.e., we write $F\approx G$.

Let $L^2_{a,b}[0,T]$ be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T]:=\left\{v:\int_0^T v^2(s)db(s)<+\infty \text{ and } \int_0^T v^2(s)d|a|(s)<+\infty\right\}$$

where $|a|(\cdot)$ denotes the total variation function of the function $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b}:=\int_0^T u(t)v(t)dm_{|a|,b}(t)\equiv\int_0^T u(t)v(t)d[b(t)+|a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue-Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$. Note that $\|u\|_{a,b}\equiv\sqrt{(u,u)_{a,b}}=0$ if and only if $u(t)=0$ a.e. on $[0,T]$ and that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$, then $L^2_{a,b}[0,T]=L^2[0,T]$. In fact

$$(L^2_{a,b}[0,T],\|\cdot\|_{a,b})\subset(L^2_{0,b}[0,T],\|\cdot\|_{0,b})=(L^2[0,T],\|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

For each $v\in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v,x\rangle$ is given by the formula

$$\langle v,x\rangle:=\lim_{n\to\infty}\int_0^T\sum_{j=1}^n(v,\phi_j)_{a,b}\phi_j(t)dx(t)$$

for μ-a.e. $x \in C_{a,b}[0,T]$ where $\{\phi_j\}_{j=1}^{\infty}$ is a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that $(\phi_j, \phi_k)_{a,b} = \delta_{jk}$, the Kronecker delta. If v is of bounded variation on $[0,T]$, then the PWZ stochastic integral $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_{a,b}[0,T]$. Furthermore, for each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, \cdot\rangle : C_{a,b}[0,T] \to \mathbb{R}$ is a Gaussian random variable with mean $\int_0^T v(t)da(t) = \int_0^T v(t)a'(t)dt$ and variance $\int_0^T v^2(t)db(t) = \int_0^T v^2(t)b'(t)dt$. For more details, see [5, 7, 8].

3. Gaussian Processes

For any $h \in L^2_{a,b}[0,T]$ with $\|h\|_{a,b} > 0$, let $\mathcal{Z}_h(x,t)$ denote the PWZ stochastic integral

$$\mathcal{Z}_h(x,t) := \langle h\chi_{[0,t]}, x\rangle, \tag{3.1}$$

let $\beta_h(t) := \int_0^t h^2(u)db(u)$, and let $\gamma_h(t) := \int_0^t h(u)da(u)$. Then $\mathcal{Z}_h : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ is a Gaussian process with mean function

$$\int_{C_{a,b}[0,T]} \mathcal{Z}_h(x,t)d\mu(x) = \int_0^t h(u)da(u) = \gamma_h(t)$$

and covariance function

$$\int_{C_{a,b}[0,T]} \Big(\mathcal{Z}_h(x,s) - \gamma_h(s)\Big)\Big(\mathcal{Z}_h(x,t) - \gamma_h(t)\Big)d\mu(x) = \int_0^{\min\{s,t\}} h^2(u)db(u) = \beta_h(\min\{s,t\}).$$

In addition, by [16, Theorem 21.1], $\mathcal{Z}_h(\cdot,t)$ is stochastically continuous in t on $[0,T]$. Of course if $h(t) \equiv 1$, then $\mathcal{Z}_1(x,t) = x(t)$. Furthermore, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and the Gaussian process (3.1) with $h(t) \equiv 1$ is an ordinary Wiener process.

For $h_1, h_2 \in L^2_{a,b}[0,T]$ with $\|h_j\|_{a,b} > 0$, $j \in \{1,2\}$, let $\mathcal{Z}_{h_1}$ and $\mathcal{Z}_{h_2}$ be the Gaussian processes given by (3.1) with h replaced with h_1 and h_2 respectively. Then the process

$$\mathfrak{Z}_{h_1,h_2} : C_{a,b}[0,T] \times C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$$

given by

$$\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t) := \mathcal{Z}_{h_1}(x_1,t) + \mathcal{Z}_{h_2}(x_2,t) \tag{3.2}$$

is also a Gaussian process with mean $\mathfrak{m}_{h_1,h_2}(t) = \gamma_{h_1}(t) + \gamma_{h_2}(t)$ and variance $\mathfrak{v}_{h_1,h_2}(t) = \beta_{h_1}(t) + \beta_{h_2}(t)$. More precisely, the covariance of the process $\mathfrak{Z}_{h_1,h_2}$ is given by

$$\begin{aligned}&\int_{C^2_{a,b}[0,T]} \Big(\mathfrak{Z}_{h_1,h_2}(x_1,x_2,s) - \mathfrak{m}_{h_1,h_2}(s)\Big)\Big(\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t) - \mathfrak{m}_{h_1,h_2}(t)\Big)d(\mu\times\mu)(x_1,x_2)\\ &= \beta_{h_1}(\min\{s,t\}) + \beta_{h_2}(\min\{s,t\})\\ &= \mathfrak{v}_{h_1,h_2}(\min\{s,t\}).\end{aligned}$$

Let h_1 and h_2 be elements of $L^2_{a,b}[0,T]$. Then there exists a function $\mathbf{s} \in L^2_{a,b}[0,T]$ such that

$$\mathbf{s}^2(t) = h_1^2(t) + h_2^2(t) \tag{3.3}$$

for $m_{|a|,b}$-a.e. $t \in [0,T]$. Note that the function '$\mathbf{s}$' satisfying (3.3) is not unique. We will use the symbol $\mathbf{s}(h_1,h_2)$ for the functions '$\mathbf{s}$' that satisfy (3.3) above.

We consider a stochastic process associated with the process $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}$. Define a process

$$\mathcal{R}_{h_1,h_2} : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$$

by

$$\mathcal{R}_{h_1,h_2}(x,t) := \mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,t) + \int_0^t \Big(h_1(u) + h_2(u) - \mathbf{s}(h_1,h_2)(u)\Big)da(u). \tag{3.4}$$

Then $\mathcal{R}_{h_1,h_2}$ is a Gaussian process with mean

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \mathcal{R}_{h_1,h_2}(x,t)d\mu(x)\\
&= \int_{C_{a,b}[0,T]} \mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,t)d\mu(x) + \int_0^t \Big(h_1(u) + h_2(u) - \mathbf{s}(h_2,h_2)(u)\Big)da(u)\\
&= \gamma_{h_1}(t) + \gamma_{h_2}(t) = \mathfrak{m}_{h_1,h_2}(t)
\end{aligned}$$

and covariance

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \Big(\mathcal{R}_{h_1,h_2}(x,s) - \mathfrak{m}_{h_1,h_2}(s)\Big)\Big(\mathcal{R}_{h_1,h_2}(x,t) - \mathfrak{m}_{h_1,h_2}(t)\Big)d\mu(x)\\
&= \int_0^{\min\{s,t\}} \mathbf{s}^2(h_1,h_2)(u)db(u) = \int_0^{\min\{s,t\}} \Big(h_1^2(u) + h_2^2(u)\Big)db(u)\\
&= \beta_{h_1}(\min\{s,t\}) + \beta_{h_2}(\min\{s,t\})\\
&= \mathfrak{v}_{h_1,h_2}(\min\{s,t\}).
\end{aligned}$$

Also, $\mathcal{R}_{\mathbf{s}(h_1,h_2)}(\cdot,t)$ is stochastically continuous in t on $[0,T]$.

From these facts, one can see that $\mathfrak{Z}_{h_1,h_2}$ and $\mathcal{R}_{h_1,h_2}$ have the same distribution and that for any random variable F on $C_{a,b}[0,T]$,

$$\int_{C^2_{a,b}[0,T]} F\Big(\mathfrak{Z}_{h_1,h_2}(x_1,x_2,\cdot)\Big)d(\mu\times\mu)(x_1,x_2) \overset{*}{=} \int_{C_{a,b}[0,T]} F\Big(\mathcal{R}_{h_1,h_2}(x,\cdot)\Big)d\mu(x), \tag{3.5}$$

where by $\overset{*}{=}$ we mean that if either side exists, both sides exist and equality holds.

Remark 3.1. *In [11], the authors investigated a rotation property of the function space measure μ. The result is summarized as follows: for a measurable functional F and every nonzero real p and q,*

$$\begin{aligned}
&\int_{C^2_{a,b}[0,T]} F(px_1 + qx_2)d(\mu\times\mu)(x_1,x_2)\\
&\overset{*}{=} \int_{C_{a,b}[0,T]} F\bigg(\sqrt{p^2+q^2}x + \Big(p + q - \sqrt{p^2+q^2}\Big)a\bigg)d\mu(x).
\end{aligned}$$

But, by the observation presented above, we also obtain an alternative result such that

$$\begin{aligned}
&\int_{C^2_{a,b}[0,T]} F(px_1 + qx_2)d(\mu\times\mu)(x_1,x_2)\\
&\overset{*}{=} \int_{C_{a,b}[0,T]} F\bigg(-\sqrt{p^2+q^2}x + \Big(p + q + \sqrt{p^2+q^2}\Big)a\bigg)d\mu(x).
\end{aligned}$$

4. A Rotation Theorem for Generalized Analytic Feynman Integrals with Respect to Gaussian Processes

Let $\mathbb{C}$ denote the set of complex numbers. Let $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let $\mathcal{G}$ be a stochastically continuous Gaussian process on $C_{a,b}[0,T] \times [0,T]$. We define the $\mathcal{G}$-function space integral (the function space integral with respect to the Gaussian process $\mathcal{G}$) for functionals F on $C_{a,b}[0,T]$ by the formula

$$I_{\mathcal{G}}[F] \equiv I_{\mathcal{G},x}[F(\mathcal{G}(x,\cdot))] := \int_{C_{a,b}[0,T]} F\Big(\mathcal{G}(x,\cdot)\Big)d\mu(x)$$

whenever the integral exists.

Let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_{a,b}[0,T]$ such that

$$J_F(\mathcal{G};\lambda) := I_{\mathcal{G},x}[F(\lambda^{-1/2}\mathcal{G}(x,\cdot))]$$

exists and is finite for all $\lambda > 0$. If there exists a function $J_F^*(\mathcal{G};\lambda)$ analytic on $\mathbb{C}_+$ such that $J_F^*(\mathcal{G};\lambda) = J_F(\mathcal{G};\lambda)$ for all $\lambda > 0$, then $J_F^*(\mathcal{G};\lambda)$ is defined to be the analytic $\mathcal{G}$-function space integral (the analytic function space integral with respect to the process $\mathcal{G}$) of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$I_{\mathcal{G}}^{\mathrm{an}_\lambda}[F] \equiv I_{\mathcal{G},x}^{\mathrm{an}_\lambda}[F(\mathcal{G}(x,\cdot))] \equiv \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F\Big(\mathcal{G}(x,\cdot)\Big)d\mu(x) := J_F^*(\mathcal{G};\lambda).$$

Let q be a nonzero real number and let Γ_q be a connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that $\Gamma_q \cap (0,+\infty)$ is an open interval. Let F be a measurable functional whose analytic $\mathcal{G}$-function space integral exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic $\mathcal{G}$-Feynman integral (the generalized analytic Feynman integral with respect to the process $\mathcal{G}$) of F with parameter q and we write

$$I_{\mathcal{G}}^{\mathrm{anf}_q}[F] \equiv I_{\mathcal{G},x}^{\mathrm{anf}_q}[F(\mathcal{G}(x,\cdot))] := \lim_{\lambda\to -iq} I_{\mathcal{G},x}^{\mathrm{an}_\lambda}[F(\mathcal{G}(x,\cdot))], \tag{4.1}$$

where λ approaches $-iq$ through values in Γ_q.

In the case of the generalized analytic $\mathcal{Z}_h$-Feynman integral, if we choose $h \equiv 1$ on $[0,T]$, then the definition of the generalized analytic $\mathcal{Z}_1$-Feynman integral agrees with the previous definitions of the generalized analytic Feynman integral [5, 8, 9].

Now we will establish a rotation formula of our generalized analytic Feynman integral.

Lemma 4.1. *Given $h_j \in L_{a,b}^2[0,T]$, $j \in \{1,2\}$, with $\|h_j\|_{a,b} > 0$, let $\mathcal{Z}_{h_j}$ be the Gaussian processes given by* (3.1) *with h replaced with h_j, and let $\mathcal{R}_{h_1,h_2}$ be the Gaussian process given by* (3.4)*. Let F be a scale-invariant measurable functional that the analytic function space integrals $I_{\mathcal{Z}_{h_1}}^{\mathrm{an}_\lambda}[F]$, $I_{\mathcal{Z}_{h_2}}^{\mathrm{an}_\lambda}[F]$ and $I_{\mathcal{R}_{h_1,h_2}}^{\mathrm{an}_\lambda}[F]$ exist for every $\lambda \in \mathbb{C}_+$. Furthermore assume that $I_{\mathcal{Z}_{h_2},x_2}^{\mathrm{an}_{\lambda_2}}[I_{\mathcal{Z}_{h_1},x_1}^{\mathrm{an}_{\lambda_1}}[F(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot))]]$ exists for every $(\lambda_1,\lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$. Then for each $\lambda \in \mathbb{C}_+$,*

$$I_{\mathcal{Z}_{h_2},x_2}^{\mathrm{an}_\lambda}\Big[I_{\mathcal{Z}_{h_1},x_1}^{\mathrm{an}_\lambda}\Big[F\Big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big] = I_{\mathcal{R}_{h_1,h_2},x}^{\mathrm{an}_\lambda}\Big[F\Big(\mathcal{R}_{h_1,h_2}(x,\cdot)\Big)\Big]. \tag{4.2}$$

Proof. In view of the definition of the analytic function space integral with respect to the Gaussian process, we first note that the existences of the generalized analytic integrals

$$I_{\mathcal{Z}_{h_1}}^{\mathrm{an}_\lambda}[F],\ I_{\mathcal{Z}_{h_2}}^{\mathrm{an}_\lambda}[F],\ I_{\mathcal{R}_{h_1,h_2}}^{\mathrm{an}_\lambda}[F],\ \text{ and } I_{\mathcal{Z}_{h_2},x_2}^{\mathrm{an}_{\lambda_2}}\Big[I_{\mathcal{Z}_{h_1},x_1}^{\mathrm{an}_{\lambda_1}}\Big[F\Big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big]$$

guarantee that the function space integrals

$$I_{\mathcal{Z}_{h_1},x}\Big[F\Big(\lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)\Big)\Big],\ I_{\mathcal{Z}_{h_2},x}\Big[F\Big(\lambda^{-1/2}\mathcal{Z}_{h_2}(x,\cdot)\Big)\Big],\ I_{\mathcal{R}_{h_1,h_2},x}\Big[F\Big(\lambda^{-1/2}\mathcal{R}_{h_1,h_2}(x,\cdot)\Big)\Big],$$

$$I_{\mathcal{Z}_{h_2},x_2}\Big[I_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(\lambda_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)+\lambda_2^{-1/2}\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big],\ \text{ and } I_{\mathcal{Z}_{h_2},x_2}\Big[I_{\mathcal{Z}_{h_1},x_1}^{\mathrm{an}_{\zeta_1}}\Big[F\Big(\mathcal{Z}_{h_1}(x_1,\cdot)+\zeta_2^{-1/2}\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big]$$

all exist for any $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\zeta_1 \in \mathbb{C}_+$, and $\zeta_2 > 0$.

Next, the existence of the analytic function space integral

$$J_1^*(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda_1, \lambda_2) \equiv I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2}, x_2}\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big], \quad (\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+, \tag{4.3}$$

also ensure that the analytic function space integral

$$J_1^*(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda, \lambda) = I^{\mathrm{an}_{\lambda}}_{\mathcal{Z}_{h_2}, x_2}\Big[I^{\mathrm{an}_{\lambda}}_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big]$$

is well-defined for all $\lambda \in \mathbb{C}_+$. In equation (4.3) above, $J_1^*(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda_1, \lambda_2)$ means the analytic function space integral, which is the analytic continuation of the function space integral

$$I_{\mathcal{Z}_{h_2}, x_2}\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1}}\Big[F\Big(\mathcal{Z}_{h_1}(x_1, \cdot) + \lambda_2^{-1/2}\mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big], \quad (\lambda_1, \lambda_2) \in \mathbb{C}_+ \times (0, +\infty).$$

On the other hand, using the Fubini theorem, (3.2) and (3.5), it follows that for all $\lambda > 0$,

$$\begin{aligned}
&I_{\mathcal{Z}_{h_2}, x_2}\Big[I_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\lambda^{-1/2}\mathcal{Z}_{h_1}(x_1, \cdot) + \lambda^{-1/2}\mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big] \\
&= I_{\mathcal{Z}_{h_2}, x_2}\Big[I_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\lambda^{-1/2}[\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)]\Big)\Big]\Big] \\
&= I_{\mathcal{Z}_{h_2}, x_2}\Big[I_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\lambda^{-1/2}\mathfrak{Z}_{h_1, h_2}(x_1, x_2, \cdot)\Big)\Big]\Big] \\
&= I_{\mathcal{R}_{h_1, h_2}, x}\Big[F\Big(\lambda^{-1/2}\mathcal{R}_{h_1, h_2}(x, \cdot)\Big)\Big].
\end{aligned}$$

We now use the analytic continuation to obtain our desired conclusion. □

Remark 4.2. *Let*

$$J_{\lambda_1}(\mathcal{Z}_{h_2}; \lambda_2) := I_{\mathcal{Z}_{h_2}, x_2}\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\mathcal{Z}_{h_1}(x_1, \cdot) + \lambda_2^{-1/2}\mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big], \quad (\lambda_1, \lambda_2) \in \mathbb{C}_+ \times (0, +\infty),$$

$$\begin{aligned}
J_{\lambda_2}(\mathcal{Z}_{h_1}; \lambda_1) &:= I_{\mathcal{Z}_{h_1}, x_1}\Big[I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2}, x_2}\Big[F\Big(\lambda_1^{-1/2}\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big] \\
&= I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2}, x_2}\Big[I_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\lambda_1^{-1/2}\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big], \quad (\lambda_1, \lambda_2) \in (0, +\infty) \times \mathbb{C}_+,
\end{aligned}$$

and

$$J(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda_1, \lambda_2) = I_{\mathcal{Z}_{h_2}, x_2}\Big[I_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\lambda_1^{-1/2}\mathcal{Z}_{h_1}(x_1, \cdot) + \lambda_2^{-1/2}\mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big], \quad (\lambda_1, \lambda_2) \in (0, +\infty) \times (0, +\infty).$$

Also, let $J^*_{\lambda_1}(\mathcal{Z}_{h_2}; \lambda_2)$, $\lambda_2 \in \mathbb{C}_+$, denote the analytic continuation of $J_{\lambda_1}(\mathcal{Z}_{h_2}; \lambda_2)$, let $J^*_{\lambda_2}(\mathcal{Z}_{h_1}; \lambda_1)$, $\lambda_1 \in \mathbb{C}_+$, denote the analytic continuation of $J_{\lambda_2}(\mathcal{Z}_{h_1}; \lambda_1)$, and let $J^{**}(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \cdot, \cdot)$ denote the analytic continuation on $\mathbb{C}_+ \times \mathbb{C}_+$ of the function $J(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \cdot, \cdot)$. Clearly, $J^*_{\lambda_1}(\mathcal{Z}_{h_2}; \lambda_2) = J_1^*(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda_1, \lambda_2)$ where $J_1^*(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \cdot, \cdot)$ is the analytic function on $\mathbb{C}_+ \times \mathbb{C}_+$ given in (4.3) *above.*

From the assumptions in Lemma 4.1, one can see that the three analytic function space integrals $J^*_{\lambda_1}(\mathcal{Z}_{h_2}; \lambda_2)$, $J^*_{\lambda_2}(\mathcal{Z}_{h_2}; \lambda_2)$, and $J^{**}(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda_1, \lambda_2)$ all exist, and

$$J^*_{\lambda_1}(\mathcal{Z}_{h_2}; \lambda_2) = J^*_{\lambda_2}(\mathcal{Z}_{h_1}; \lambda_1) = J^{**}(\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}; \lambda_1, \lambda_2)$$

for all $(\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$.

Theorem 4.3. *Let $\mathcal{Z}_{h_1}$, $\mathcal{Z}_{h_2}$, $\mathcal{R}_{h_1, h_2}$, and F be as in Lemma 4.1. Then for a real $q \in \mathbb{R} \setminus \{0\}$,*

$$I^{\mathrm{anf}_q}_{\mathcal{Z}_{h_2}, x_2}\Big[I^{\mathrm{anf}_q}_{\mathcal{Z}_{h_1}, x_1}\Big[F\Big(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)\Big)\Big]\Big] \overset{*}{=} I^{\mathrm{anf}_q}_{\mathcal{R}_{h_1, h_2}, x}\Big[F\Big(\mathcal{R}_{h_1, h_2}(x, \cdot)\Big)\Big]. \tag{4.4}$$

Proof. To obtain equation (4.4), one may establish that

$$\lim_{\substack{\lambda_1,\lambda_2\to -iq\\ \lambda_1,\lambda_2\in\Gamma}} I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big] = I^{\mathrm{anf}_q}_{\mathcal{R}_{h_1,h_2},x}\Big[F\Big(\mathcal{R}_{h_1,h_2}(x,\cdot)\Big)\Big].$$

But, as shown in the proof of Lemma 4.1, the assumption that the analytic function space integrals $I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_1}}[F]$, $I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_2}}[F]$, and $I^{\mathrm{an}_\lambda}_{\mathcal{R}_{h_1,h_2}}[F]$ exist for every $\lambda\in\mathbb{C}_+$, and the analytic function space integral $I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2},x_2}[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1},x_1}[F(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot))]]$ exists for every $(\lambda_1,\lambda_2)\in\mathbb{C}_+\times\mathbb{C}_+$, says the fact that $I^{\mathrm{an}_\lambda}_{\mathcal{R}_{h_1,h_2}}[F]$ is analytic on $\mathbb{C}_+$, as a function of λ, and $I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2},x_2}[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1},x_1}[F(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot))]]$ is analytic on $\mathbb{C}_+\times\mathbb{C}_+$, as a function of (λ_1,λ_2). Thus, to establish equation (4.4), it will suffice to show that

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma}} I^{\mathrm{an}_{\lambda}}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{an}_{\lambda}}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big] = I^{\mathrm{anf}_q}_{\mathcal{R}_{h_1,h_2},x}\Big[F\Big(\mathcal{R}_{h_1,h_2}(x,\cdot)\Big)\Big].$$

Using equation (4.2) and the analytic continuation, we obtain the desired result. □

5. Multiple Generalized Analytic Fourier-Feynman Transform with Respect to Gaussian Processes

We begin this section with the definitions of the generalized analytic Fourier-Feynman transform with respect to Gaussian process and the multiple generalized analytic Fourier-Feynman transform with respect to Gaussian processes of functionals on $C_{a,b}[0,T]$. Let F be a scale-invariant measurable functional on $C_{a,b}[0,T]$ and let $\mathcal{G}$ be a stochastically continuous Gaussian process on $C_{a,b}[0,T]\times[0,T]$. For $\lambda\in\mathbb{C}_+$ and $y\in C_{a,b}[0,T]$, let

$$T_{\lambda,\mathcal{G}}(F)(y) := I^{\mathrm{an}_\lambda}_{\mathcal{G},x}\Big[F\Big(y+\mathcal{G}(x,\cdot)\Big)\Big] \tag{5.1}$$

denote the analytic function space transform of F. Let q be a nonzero real number and let Γ_q be a connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that $\Gamma_q\cap(0,+\infty)$ is an open interval. We define the L_1 generalized analytic Fourier-Feynman transform with respect to the process $\mathcal{G}$, $T^{(1)}_{q,\mathcal{G}}(F)$ of F, by the formula (if it exists)

$$T^{(1)}_{q,\mathcal{G}}(F)(y) := \lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma}} T_{\lambda,\mathcal{G}}(F)(y) \tag{5.2}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

We note that $T^{(1)}_{q,\mathcal{G}}(F)$ is defined only s-a.e.. We also note that if $T^{(1)}_{q,\mathcal{G}}(F)$ exists and if $F\approx G$, then $T^{(1)}_{q,\mathcal{G}}(G)$ exists and $T^{(1)}_{q,\mathcal{G}}(G)\approx T^{(1)}_{q,\mathcal{G}}(F)$. From equations (5.1), (5.2), and (4.1), it follows that

$$T^{(1)}_{q,\mathcal{G}}(F)(y) \stackrel{*}{=} I^{\mathrm{anf}_q}_{\mathcal{G}}[F(y+\cdot)] \tag{5.3}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

Next, let $\mathcal{G}_j$, $j\in\{1,\dots,n\}$, be stochastically continuous Gaussian processes on $C_{a,b}[0,T]\times[0,T]$. For $\lambda>0$ and $y\in C_{a,b}[0,T]$, define a transform $\mathcal{M}_{\lambda,(\mathcal{G}_1,\dots,\mathcal{G}_n)}(F)(y)$ as follows:

$$\begin{aligned}
&\mathcal{M}_{\lambda,(\mathcal{G}_1,\dots,\mathcal{G}_n)}(F)(y)\\
&:= \int_{C^n_{a,b}[0,T]} F\Big(y+\lambda^{-1/2}\sum_{j=1}^n \mathcal{G}_j(x_j,\cdot)\Big)d\mu^n(x_1,\dots,x_n)\\
&\equiv I_{\mathcal{G}_n,x_n}\Big[I_{\mathcal{G}_{n-1},x_{n-1}}\Big[\cdots\Big[I_{\mathcal{G}_1,x_1}\Big[F\Big(y+\lambda^{-1/2}\sum_{j=1}^n\mathcal{G}_j(x_j,\cdot)\Big)\Big]\Big]\cdots\Big]\Big].
\end{aligned}$$

Let $\mathcal{M}_{\lambda,(\mathcal{G}_1,\ldots,\mathcal{G}_n)}(F)(y)$ also denote an analytic extension of $\mathcal{M}_{\lambda,(\mathcal{G}_1,\ldots,\mathcal{G}_n)}(F)(y)$ as a function of $\lambda \in \mathbb{C}_+$. Given $q \in \mathbb{R}\setminus\{0\}$ and a connected neighborhood Γ_q of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that $\Gamma_q \cap (0,+\infty)$ is an open interval, we define the L_1 multiple generalized analytic Fourier-Feynman transforms with respect to the Gaussian processes $(\mathcal{G}_1,\ldots,\mathcal{G}_n)$, $\mathcal{M}^{(1)}_{q,(\mathcal{G}_1,\ldots,\mathcal{G}_n)}(F)$ of F, by the formula (if it exists)

$$\mathcal{M}^{(1)}_{q,(\mathcal{G}_1,\ldots,\mathcal{G}_n)}(F)(y) := \lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma}} \mathcal{M}_{\lambda,(\mathcal{G}_1,\ldots,\mathcal{G}_n)}(F)(y) \tag{5.4}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Clearly, we have that $\mathcal{M}_{\lambda,(\mathcal{G})}(F) = T_{\lambda,\mathcal{G}}(F)$ for all $\lambda \in \mathbb{C}_+$, and $\mathcal{M}^{(1)}_{q,(\mathcal{G})}(F) = T^{(1)}_{q,\mathcal{G}}(F)$ for any nonzero real q if the transforms exist.

Theorem 5.1. *Let $\mathcal{Z}_{h_1}$, $\mathcal{Z}_{h_2}$, and $\mathcal{R}_{h_1,h_2}$ be as in Lemma 4.1. Let F be a scale-invariant measurable functional that the analytic transforms*

$$T_{\lambda,\mathcal{Z}_{h_1}}(F)(y) \equiv I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_1}}[F(y+\cdot)],\ T_{\lambda,\mathcal{Z}_{h_2}}(F)(y) \equiv I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_2}}[F(y+\cdot)],\ \text{ and }\ T_{\lambda,\mathcal{R}_{h_1,h_2}}(F)(y) \equiv I^{\mathrm{an}_\lambda}_{\mathcal{R}_{h_1,h_2}}[F(y+\cdot)]$$

exist for every $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_{a,b}[0,T]$. Furthermore assume that the analytic function space integral

$$I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(y+\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big]$$

exists for every $(\lambda_1,\lambda_2) \in \mathbb{C}_+\times\mathbb{C}_+$ and s-a.e. $y \in C_{a,b}[0,T]$. Then for a real $q \in \mathbb{R}\setminus\{0\}$,

$$\mathcal{M}^{(1)}_{q,(\mathcal{Z}_{h_1},\mathcal{Z}_{h_2})}(F)(y) \overset{*}{=} T^{(1)}_{q,\mathcal{R}_{h_1,h_2}}(F)(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$.

Proof. First, proceedings as in the proofs of Lemma 4.1 and Theorem 4.3, we conclude that

$$I^{\mathrm{anf}_q}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{anf}_q}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(y+\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big] \overset{*}{=} I^{\mathrm{anf}_q}_{\mathcal{R}_{h_1,h_2},x}\Big[F\Big(y+\mathcal{R}_{h_1,h_2}(x,\cdot)\Big)\Big]. \tag{5.5}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Next, in view of equation (5.4) and under the assumption, it follows that for s-a.e. $y \in C_{a,b}[0,T]$,

$$\begin{aligned}
\mathcal{M}^{(1)}_{q,(\mathcal{Z}_{h_1},\mathcal{Z}_{h_2})}(F)(y) &= \lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma}} \mathcal{M}_{\lambda,(\mathcal{Z}_{h_1},\mathcal{Z}_{h_2})}(F)(y)\\
&= \lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma}} I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(y+\lambda^{-1/2}[\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)]\Big)\Big]\Big]\\
&= \lim_{\substack{\lambda_1,\lambda_2\to -iq\\ \lambda_1,\lambda_2\in\Gamma}} I^{\mathrm{an}_{\lambda_2}}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(y+\lambda_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)+\lambda_2^{-1/2}\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big]\\
&= I^{\mathrm{anf}_q}_{\mathcal{Z}_{h_2},x_2}\Big[I^{\mathrm{anf}_q}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(y+\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\Big)\Big]\Big].
\end{aligned} \tag{5.6}$$

Finally using equations (5.6), (5.5), and (5.3) with $\mathcal{G}$ replaced with $\mathcal{R}_{h_1,h_2}$, it follows that for s-a.e. $y \in C_{a,b}[0,T]$,

$$\mathcal{M}^{(1)}_{q,(\mathcal{Z}_{h_1},\mathcal{Z}_{h_2})}(F)(y) \overset{*}{=} T^{(1)}_{q,\mathcal{R}_{h_1,h_2}}(F)(y),$$

as desired. □

The following theorem follows by the use of mathematical induction.

Theorem 5.2. *Given $h_j \in L^2_{a,b}[0,T]$, $j \in \{1,\dots,n\}$, with $\|h_j\|_{a,b} > 0$, let $\mathcal{Z}_{h_j}$ be the Gaussian processes given by* (3.1) *with h replaced with h_j, and let $\mathcal{R}_{h_1,\dots,h_n} : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ be the Gaussian process given by*

$$\mathcal{R}_{h_1,\dots,h_n}(x,t) := \mathcal{Z}_{\mathbf{s}(h_1,\dots,h_n)}(x,t) + \int_0^t \Big[\sum_{j=1}^n h_j(u) - \mathbf{s}(h_1,\dots,h_n)(u)\Big] da(u),$$

where $\mathbf{s}(h_1,\dots,h_n)$ is an element of $L^2_{a,b}[0,T]$ which satisfies the condition

$$\mathbf{s}^2(h_1,\dots,h_n) = \sum_{j=1}^n h_j^2$$

for $m_{|a|,b}$-a.e. on $[0,T]$. Let F be a scale-invariant measurable functional that the analytic function space transforms $T_{\lambda,\mathcal{Z}_{h_j}}(F)(y) \equiv I^{\mathrm{an}_\lambda}_{\mathcal{Z}_{h_j}}[F(y+\cdot)]$, $j \in \{1,\dots,n\}$, and $T_{\lambda,\mathcal{R}_{h_1,\dots,h_n}}(F)(y) \equiv I^{\mathrm{an}_\lambda}_{\mathcal{R}_{h_1,\dots,h_n}}[F]$ exist for every $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_{a,b}[0,T]$. Furthermore assume that the analytic function space integral

$$I^{\mathrm{an}_{\lambda_n}}_{\mathcal{Z}_{h_n},x_n}\Big[\cdots\Big[I^{\mathrm{an}_{\lambda_1}}_{\mathcal{Z}_{h_1},x_1}\Big[F\Big(y + \sum_{j=1}^n \mathcal{Z}_{h_j}(x_j,\cdot)\Big)\Big]\Big]\cdots\Big]$$

exists for every $(\lambda_1,\dots,\lambda_n) \in \mathbb{C}_+^n$ and s-a.e. $y \in C_{a,b}[0,T]$. Then, for a real $q \in \mathbb{R} \setminus \{0\}$ and s-a.e. $y \in C_{a,b}[0,T]$,

$$\mathcal{M}^{(1)}_{q,(\mathcal{Z}_{h_1},\dots,\mathcal{Z}_{h_n})}(F)(y) \overset{*}{=} T^{(1)}_{q,\mathcal{R}_{h_1,\dots,h_n}}(F)(y).$$

We note that the hypotheses (and hence the conclusions) of Lemma 4.1, Theorems 4.3, 5.1, and 5.2 above are indeed satisfied by many large classes of functionals. These classes of functionals include:

(a) The Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ defined by Chang and Skoug in [8].

(b) Various spaces of functionals of the form

$$F(x) = f(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle)$$

for appropriate f as discussed in [5, 7, 9].

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