Filomat 30:6 (2016), 1465–1474
DOI 10.2298/FIL1606465C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Some Applications of the First-Order Differential Subordinations

**Nak Eun Cho[a], Hyo Jeong Lee[a], Ji Hyang Park[a], Rekha Srivastava[b]**

[a] *Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*

**Abstract.** The object of the present paper is to give some applications of the first-order differential subordinations. We also extend and improve several previously known results.

## 1. Introduction

Let $\mathcal{A}$ denote the class of all functions $f$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the usual normalization given by

$$f(0) = f'(0) - 1 = 0.$$

If $f$ and $g$ are analytic in $\mathbb{U}$, then we say that the function $f$ is subordinate to $g$ if there exists a Schwarz function $w$ analytic in $\mathbb{U}$, with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have (cf. [5])

$$f \prec g \quad \Longleftrightarrow \quad f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f \in \mathbb{U}$ is said to be strongly starlike of order $\eta$ $(0 < \eta \leqq 1)$ if and only if

2010 *Mathematics Subject Classification*. Primary 30C80; Secondary 30C45
*Keywords*. Analytic functions; Differential subordination; Univalent functions; Strongly starlike function
Received: 21 April 2014; Accepted: 12 Jun 2014
Communicated by Hari M. Srivastava
This work was supported by a Research Grant of Pukyong National University(2016 year).
*Email addresses:* necho@pknu.ac.kr (Nak Eun Cho), hjlj.lee@gmail.com (Hyo Jeong Lee), jihyang1022@naver.com (Ji Hyang Park), rekhas@math.uvic.ca (Rekha Srivastava)

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\eta} \qquad (z \in \mathbb{U}). \tag{1.1}$$

We note that the conditions (1.1) can be written by

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}[\eta]$ the subclass of $\mathcal{A}$ consisting of all strongly starlike functions of order $\eta$ $(0 < \eta \leqq 1)$. We also note that $\mathcal{S}[1] \equiv \mathcal{S}^*$ is the well-known class of all normalized starlike functions in $\mathbb{U}$. The class $\mathcal{S}[\eta]$ and the related classes have been extensively studied by Mocanu [6] and Nunokawa [7].

If $\psi$ is analytic in a domain $\mathbb{D} \subset \mathbb{C}^2$, $h$ is univalent in $\mathbb{U}$ and $p$ is analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in \mathbb{D}$ for $z \in \mathbb{U}$, then $p$ is said to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z) \ (z \in \mathbb{U}). \tag{1.2}$$

The univalent function $q$ is said to be a dominant of the differential subordination (1.2) if $p \prec q$ for all $p$ satisfying (1.2). If $\tilde{q}$ is a dominant of (1.2) and $\tilde{q} \prec q$ for all dominants of (1.2), then $\tilde{q}$ is said to be the best dominant of the differential subordination (1.2). The general theory of the first-order differential subordinations, with many interesting applications, especially in the theory of univalent functions, was developed by Miller and Mocanu ([4]; see also [5]). For several applications of the principle of differential subordinations in the investigations of various interesting subclasses of analytic and univalent functions, we refer the reader to the recent works [11], [12], [13], [14] and [15].

In the present paper, we propose to derive some applications of the first-order differential subordinations. We also extend and improve the results proven earlier by Cho and Kim [1], Miller *et al.* [3], and Nunokawa *et al.* [7, 8, 9, 10].

## 2. The First Main Result

In proving our results, we shall need the following lemma due to Miller and Mocanu [4].

**Lemma.** *Let $q$ be univalent in $\mathbb{U}$ and let $\theta$ and $\varphi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with*

$$q(\omega) \neq 0 \qquad \textit{when} \qquad \omega \in q(\mathbb{U}).$$

*Set*

$$Q(z) = zq'(z)\varphi(q(z)), \ h(z) = \theta(q(z)) + Q(z)$$

*and suppose that*

(i) *$Q$ is starlike in $\mathbb{U}$*
(ii) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{U}).$

*If $p$ is analytic in $\mathbb{U}$ with*

$$p(0) = q(0), \qquad p(\mathbb{U}) \subset \mathbb{D}$$

*and*

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) \qquad (z \in \mathbb{U}), \tag{2.1}$$

*then*

$$p(z) \prec q(z) \qquad (z \in \mathbb{U})$$

*and $q$ is the best dominant of (2.1).*

With the help of the above Lemma, we now derive the following Theorem 1.

**Theorem 1.** *Let $p$ be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|\arg\left(\beta p^{\gamma}(z) + \alpha z p'(z) p^{\gamma-2}(z)\right)\right| < \frac{\pi}{2}\delta(\alpha,\beta,\gamma,\eta) \tag{2.2}$$

$$(\alpha, \beta > 0;\ 0 \leqq \gamma \leqq 1;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*where*

$$\delta(\alpha,\beta,\gamma,\eta) = \eta\gamma + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\eta\cos\frac{\pi}{2}\eta}{\beta(1+\eta)^{\frac{1+\eta}{2}}(1-\eta)^{\frac{1-\eta}{2}} + \alpha\eta\sin\frac{\pi}{2}\eta}\right), \tag{2.3}$$

*then*

$$|\arg p(z)| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}). \tag{2.4}$$

*Proof.* Let

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\eta}, \quad \theta(\omega) = \beta\omega^{\gamma} \quad \text{and} \quad \varphi(\omega) = \alpha\omega^{\gamma-2}$$

in the above Lemma. Then $q$ is univalent(convex) in $\mathbb{U}$ and

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}) \qquad \text{and} \qquad \varphi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U})).$$

It follows that

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2\alpha\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\eta(\gamma-1)}$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \beta\left(\frac{1+z}{1-z}\right)^{\eta\gamma} + \frac{2\alpha\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\eta(\gamma-1)}. \end{aligned}$$

Therefore, we have

$$\Re\left\{\frac{zQ'(z)}{Q(z)}\right\} = \Re\left\{\frac{1+z^2+2\eta(\gamma-1)z}{1-z^2}\right\} > 0 \qquad (z \in \mathbb{U}),$$

which implies that $Q$ is starlike in $\mathbb{U}$ and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\beta}{\alpha}q(z) + \frac{zQ'(z)}{Q(z)}\right\} > 0.$$

We note that $h(0) = \beta$ and

$$\begin{aligned} h(e^{i\theta}) &= \left(i\cot\frac{\theta}{2}\right)^{\eta\gamma}\left(\beta + i\,\frac{\alpha\eta}{\sin\theta}\left(i\cot\frac{\theta}{2}\right)^{-\eta}\right) \\ &= \left|\cot\frac{\theta}{2}\right|^{\eta\gamma} e^{\pm\frac{\pi}{2}\eta\gamma}\left(\beta + i\,\frac{\alpha\eta}{\sin\theta|\cot\frac{\theta}{2}|^{\eta}e^{\pm\frac{\pi}{2}\eta}}\right). \end{aligned} \tag{2.5}$$

where we take " + " for $0 < \theta < \pi$, and " − " for $-\pi < \theta < 0$. From the previous relation (2.5), we can see that the real and the imaginary part of $h(e^{i\theta})$ is an even and odd function of $\theta$, respectively. Without loss of generality, we suppose that $0 < \theta < \pi$. Then we get

$$\begin{aligned}\arg h(e^{i\theta}) &= \frac{\pi}{2}\eta\gamma + \arg\left(\beta + \frac{\alpha\eta e^{i\frac{\pi}{2}(1-\eta)}}{\sin\theta\left|\cot\frac{\theta}{2}\right|^{\eta}}\right)\\ &= \frac{\pi}{2}\eta\gamma + \arg\left(\beta + \alpha\eta e^{i\frac{\pi}{2}(1-\eta)}\frac{t^2+1}{2t^{\eta+1}}\right),\end{aligned}$$

where

$$t = \cot\frac{\theta}{2} \qquad (0 < t < \infty).$$

Since the function

$$g(t) = \frac{t^2+1}{2t^{\eta+1}} \qquad (0 < t < \infty)$$

has the minimum value at

$$t_0 = \left(\frac{1+\eta}{1-\eta}\right)^{1/2},$$

we have

$$\begin{aligned}\arg h(e^{i\theta}) &\geqq \frac{\pi}{2}\eta\gamma + \tan^{-1}\left(\frac{\alpha\eta\cos\frac{\pi}{2}\eta}{\beta(1+\eta)^{\frac{1+\eta}{2}}(1-\eta)^{\frac{1-\eta}{2}} + \alpha\eta\sin\frac{\pi}{2}\eta}\right)\\ &= \frac{\pi}{2}\delta(\alpha,\beta,\gamma,\eta),\end{aligned}$$

where $\delta(\beta,\alpha,\gamma,\eta)$ is given by (2.3). Therefore, we conclude that the condition (2.2) implies

$$\beta p^{\gamma}(z) + \alpha z p'(z) p^{\gamma-2}(z) \prec h(z) \qquad (z \in \mathbb{U})$$

Then, by the above Lemma, we have

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\eta} \qquad (z \in \mathbb{U}),$$

or, equivalently,

$$|\arg p(z)| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 1.

**Remark 1.** If we take $\gamma = 1$ in Theorem 1, then it is noted that $p(z) \neq 0$ for $z \in \mathbb{U}$. In fact, if $p$ has a zero $z_0 \in \mathbb{U}$ of order $m$, then we may write

$$p(z) = (z - z_0)^m p_1(z) \qquad (m \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

where $p_1$ is analytic in $\mathbb{U}$ with $p_1(z_0) \neq 0$. Then

$$\beta p(z) + \alpha\frac{zp'(z)}{p(z)} = \beta p(z) + \alpha\frac{zp_1{}'(z)}{p_1(z)} + \frac{\alpha m z}{z - z_0}. \tag{2.6}$$

Thus, choosing $z \to z_0$, suitably the argument of the right-hand of (2.6) can take any value between 0 and $2\pi$, which contradicts the hypothesis (2.2).

## 3. Further Results and Their Applications

If we take

$$\alpha = \beta = 1 \qquad \text{and} \qquad p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U})$$

in Theorem 1, we have the following result.

**Corollary 1.** *Let $f \in \mathcal{A}$ with $zf(z)/f(z) \neq 0$ in $\mathbb{U}$. If*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)^{\gamma-1}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\delta(1, 1, \gamma, \eta)$$

$$(0 \leqq \gamma \leqq 1;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*where $\delta(1, 1\gamma, \eta)$ is given by (2.3) with $\alpha = \beta = 1$, then $f \in \mathcal{S}[\eta]$.*

Taking $\gamma = 1$ in Theorem 1, we have the following result by Nunokawa and Owa [8].

**Corollary 2.** *Let $p$ be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|\arg\left(\beta p(z) + \alpha\frac{zp'(z)}{p(z)}\right)\right| < \frac{\pi}{2}\delta \qquad (\alpha, \beta > 0;\ 0 < \delta \leqq 1;\ z \in \mathbb{U}),$$

*then*

$$|\arg\ p(z)| < \frac{\pi\delta}{2} \qquad (z \in \mathbb{U}).$$

**Remark 2**. For $\alpha = \beta = \delta = 1$, Corollary 2 is the result obtained by Miller *et al.* [3].

Applying Theorem 1, we have the following result by Cho and Kim [1].

**Corollary 3.** *If*

$$\left|\arg\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{\phi(f(z))}\right) + \beta\left(\frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right\}\right| < \frac{\pi}{2}\delta(\alpha, \beta, 1, \eta)$$

$$(\alpha, \beta > 0;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*where $\phi(\omega)$ is analytic in $f(\mathbb{U})$, $\phi(0) = \phi'(0) - 1 = 0$, $\phi(\omega) \neq 0$ in $f(\mathbb{U})\backslash\{0\}$ and $\delta(\alpha, \beta, 1, \eta)$ is given by (2.3) with $\gamma = 1$, then*

$$\left|\arg\ \frac{zf'(z)}{\phi(f(z))}\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

*Proof.* Letting

$$p(z) = \frac{zf'(z)}{\phi(f(z))} \qquad (z \in \mathbb{U}),$$

we see that

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{\phi(f(z))}\right) + \beta\left(\frac{z(\phi(f(z)))'}{\phi(f(z))}\right) = \beta p(z) + \alpha\frac{zp'(z)}{p(z)}.$$

Therefore, by using Theorem 1 with $\gamma = 1$, we have Corollary 3.

If we set

$$\beta = 1, \qquad \phi(\omega) = \omega \qquad \text{and} \qquad p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U})$$

in Corollary 3, we have the following result.

**Corollary 4.** *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\alpha\left\{1 + \frac{zf''(z)}{f'(z)}\right\} + (1-\alpha)\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\delta(\alpha, 1, 1, \eta)$$
$$(\alpha > 0;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*where $\delta(\alpha, 1, 1, \eta)$ is given by (2.3) with $\beta = \gamma = 1$. Then $f \in \mathcal{S}[\eta]$.*

**Remark 3.** For $\alpha = 1$, Corollary 4 is the result obtained by Nunokawa [7] and Nunokawa and Thomas [10].

If we take

$$\gamma = 1 \qquad \text{and} \qquad p(z) = \frac{f(z)}{z} \quad (z \in \mathbb{U})$$

in Theorem 1, we have the the following Corollary 5.

**Corollary 5.** *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\beta\frac{zf'(z)}{f(z)} + \alpha\left\{\frac{zf'(z)}{f(z)} - 1\right\}\right)\right| < \frac{\pi}{2}\delta(\alpha, \beta, 1, \eta)$$
$$(\alpha, \beta > 0;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}),$$

*where $\delta(\alpha, \beta, 1, \eta)$ is given by (2.3) with $\gamma = 1$.*

Next, applying the above Lemma, we prove the following Theorem 2 below.

**Theorem 2.** *Let $p$ be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|\arg\left(\beta p^{\gamma}(z) + \alpha z p'(z) p^{\gamma-1}(z)\right)\right| < \frac{\pi}{2}\delta(\alpha, \beta, \eta, \gamma) \tag{3.1}$$
$$(\alpha, \beta > 0;\ \gamma \geqq 0;\ 0 < \eta \leqq 1; z \in \mathbb{U}),$$

*where $\delta(\alpha, \beta, \eta, \gamma)$ $(0 < \delta(\alpha, \beta, \eta, \gamma) < 1)$ is the solution of the equation:*

$$\delta(\alpha, \beta, \eta, \gamma) = \gamma\eta + \frac{2}{\pi}\tan^{-1}\frac{\alpha\eta}{\beta}, \tag{3.2}$$

*then*

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

*Proof.* Let

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\eta}, \quad \theta(\omega) = \beta\omega^{\gamma} \quad \text{and} \quad \varphi(\omega) = \alpha\omega^{\gamma-1}$$

in the above Lemma. Then $q$ is univalent(convex) in $\mathbb{U}$ and

$$\Re\{q(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Further, $\theta$ and $\varphi$ are analytic in $q(\mathbb{U})$ and

$$\varphi(\omega) \neq 0 \qquad (\omega \in q(\mathbb{U})).$$

Set

$$Q(z) = zq'(z)\varphi(q(z)) = \left(\frac{1+z}{1-z}\right)^{\eta\gamma} \frac{2\alpha\eta z}{1-z^2}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \left(\frac{1+z}{1-z}\right)^{\eta\gamma} \left(\beta + \frac{2\alpha\eta z}{1-z^2}\right).$$

Then we can see easily that the conditions (i) and (ii) of the above Lemma are satisfied. We also note that $h(0) = \beta$ and

$$\begin{aligned} h(e^{i\theta}) &= \left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^{\eta\gamma} \left(\beta + \frac{2\alpha\eta e^{i\theta}}{1-e^{2i\theta}}\right) \\ &= \left(i\cot\frac{\theta}{2}\right)^{\eta\gamma} \left(\beta + i\frac{\alpha\eta}{\sin\theta}\right) \\ &= \left|\cot\frac{\theta}{2}\right| e^{\pm\frac{\pi\eta}{2}} \left(\beta + i\frac{\alpha\eta}{\sin\theta}\right), \end{aligned} \tag{3.3}$$

where we take " + " for $0 < \theta < \pi$, and " − " for $-\pi < \theta < 0$. From the previous relation (3.3), we can see that the real and imaginary part of $h(e^{i\theta})$ is an even and odd function of $\theta$, respectively. Without loss of generality, we suppose that $0 < \theta < \pi$. Hence, from (3.3), we have

$$\begin{aligned} \arg h(e^{i\theta}) &= \frac{\pi}{2}\eta\gamma + \arg\left(\beta + i\frac{\alpha\eta}{\sin\theta}\right) \\ &= \frac{\pi}{2}\eta\gamma + \tan^{-1}\frac{\alpha\eta}{\beta\sin\theta} \\ &\geqq \frac{\pi}{2}\eta\gamma + \tan^{-1}\frac{\alpha\eta}{\beta} \\ &= \frac{\pi}{2}\delta(\alpha,\beta,\eta,\gamma), \end{aligned}$$

where $\delta(\alpha,\beta,\eta,\gamma)$ is the solution of the equation given by (3.2). Therefore, we conclude that the condition (3.1) implies that

$$\beta p^{\gamma}(z) + \alpha z p'(z) p^{\gamma-1}(z) \prec h(z) \qquad (z \in \mathbb{U}).$$

Then, by the above Lemma, we have

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\eta} \qquad (z \in \mathbb{U}),$$

or equivalently,

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 2.

**Remark 4.** If we take $\gamma = 0$ in Theorem 2, then we also note that $p(z) \neq 0$ in $\mathbb{U}$ as done in Remark 1.

Taking

$$\alpha = 1 \qquad \text{and} \qquad \gamma = 0$$

in Theorem 2, we have the following result by Nunokawa *et al.* [9].

**Corollary 6.** *Let $p$ be analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|\arg\left(\beta + \frac{zp'(z)}{p(z)}\right)\right| < \tan^{-1}\frac{\eta}{\beta} \qquad (\beta > 0;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*then*

$$|\arg\ p(z)| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

Letting

$$\beta = 1 \qquad \text{and} \qquad p(z) = \frac{f(z)}{z} \quad (z \in \mathbb{U})$$

in Corollary 6, we have the following result.

**Corollary 7.** *Let $f \in \mathcal{A}$. If*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \tan^{-1}\eta \qquad (0 < \eta \leq 1;\ z \in \mathbb{U}),$$

*then*

$$\left|\arg\frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

Making

$$\alpha = \beta = 1 \qquad \text{and} \qquad p(z) = \frac{f(z)}{z} \quad (z \in \mathbb{U})$$

in Theorem 2, we have the following corollary.

**Corollary 8.** *Let $f \in \mathcal{A}$. If*

$$\left|\arg\frac{zf'(z)f^{\gamma-1}(z)}{z^{\gamma}}\right| < \frac{\pi}{2}\delta(\eta,\gamma) \qquad (\gamma \geqq 0;\ 0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*where $\delta(\eta,\gamma)$ $(0 < \delta(\eta,\gamma) < 1)$ is the solution of the equation*:

$$\delta(\eta,\gamma) = \eta\gamma + \frac{2}{\pi}\tan^{-1}\eta, \tag{3.4}$$

*then*

$$\left|\arg\frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}).$$

**Remark 5.** If we take

$$\gamma = 2 \qquad \text{and} \qquad \delta(\eta, 2) = 1,$$

in Corollary 8, then we have the result obtained by Lee and Nunokawa [2].

Taking $\gamma = 1$ in Corollary 8, we have the following result.

**Corollary 9.** *Let* $f \in \mathcal{A}$. *If*

$$|\arg f'(z)| < \frac{\pi}{2}\delta(\eta) \qquad (0 < \eta \leqq 1;\ z \in \mathbb{U}),$$

*where* $\delta(\eta)$ *is the solution* $\delta(\eta, 1)$ *of the equation given by* (3.4) *with* $\gamma = 1$, *then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\eta \quad (z \in \mathbb{U}).$$

Applying Corollary 9, we have the following result immediately.

**Corollary 10.** *Let* $f \in \mathcal{A}$. *If*

$$|\arg f'(z)| < \frac{\pi}{2}\delta(\eta) \qquad (0 < \eta \leqq 1;\ \in \mathbb{U}),$$

*where* $\delta(\eta)$ *is given by Corollary 9, then*

$$\left|\arg F'(z)\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}),$$

*where F is defined by*

$$F(z) = \int_0^z \frac{f(t)}{t} dt \qquad (z \in \mathbb{U}).$$

Furthermore, from Theorem 2, we have the following result.

**Corollary 11** *Let* $f \in \mathcal{A}$. *If*

$$\left|\arg \frac{zf'(z)f^{\gamma-1}(z)}{z^{\gamma}}\right| < \frac{\pi}{2}\delta(\eta, \gamma, c) \qquad (0 < \eta \leqq 1;\ c > -\gamma;\ \gamma > 0;\ z \in \mathbb{U}),$$

*where* $\delta(\eta, \gamma, c)$ $(0 < \delta(\alpha, \gamma, c) < 1)$ *is the solution of the equation:*

$$\delta(\eta, \gamma, c) = \eta\gamma + \frac{2}{\pi}\tan^{-1}\frac{\eta}{c + \gamma},$$

*then*

$$\left|\arg \frac{zF'(z)F^{\gamma-1}(z)}{z^{\gamma}}\right| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U}),$$

*where F is the integral operator defined by*

$$F(z) = \left(\frac{c+\gamma}{z^c}\int_0^z t^{c-1} f^{\gamma}(t) dt\right)^{1/\gamma} \qquad (z \in \mathbb{U}).$$

*Proof.* It follows from the definition of $F$ that

$$cF^{\gamma}(z) + \gamma z F'(z)F^{\gamma-1}(z) = (c + \gamma)f^{\gamma}(z).$$

Let

$$p(z) = \frac{zF'(z)F^{\gamma-1}(z)}{z^{\gamma}} \qquad (z \in \mathbb{U}).$$

Then, after a simple calculation, we find that

$$(c+\gamma)p(z) + zp'(z) = (c+\gamma)\frac{zf'(z)f^{\gamma-1}(z)}{z^{\gamma}}.$$

Hence, by applying Theorem 2, we have Corollary 11.

## References


[1] N. E. Cho and J. A Kim, On a sufficient condition and an angular estimation for $\phi$-like functions, Taiwanese J. Math. 2 (1998) 397–403.

[2] S. K. Lee and M. Nunokawa, On angular estimate of $f(z)/z$ certain analytic functions, Math. Japon. 45 (1997) 47–49.

[3] S. S. Miller, P. T. Mocanu and M. O. Reade, All alpha-convex functions are starlike, Rev. Roumaine Math. Pures Appl. 17 (1972) 1395–1397.

[4] S. S. Miller and P. T. Mocanu, On some classes of first-order differential subordinations, Michigan Math. J. 32 (1985) 185–195.

[5] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. 225, Marcel Dekker Incorporated, New York and Basel 2000.

[6] P. T. Mocanu, Alpha-convex integral operators and strongly starlike functions, Studia Univ. Babeş-Bolyai Math. 34 (1989) 18–24.

[7] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, Proc. Japan Acad. Ser. A Math. Sci. 69 (1993) 234–237.

[8] M. Nunokawa and S. Owa, On certain differential subordinations, Pan Amer. Math. J. 3 (1993) 35–38.

[9] M. Nunokawa, S. Owa, H. Saitoh and S. Fukui, Arguments of certain analytic functions, Chinese J. Math. 23 (1995) 41–48.

[10] M. Nunokawa and D. K. Thomas, On convex and starlike functions in a sector, J. Austral. Math. Soc. Ser. A 60 (1996) 363–368.

[11] H. Shiraishi, S. Owa, T. Hayami, K. Kuroki and H. M. Srivastava, Starlike problems for certain analytic functions concerned with subordinations, Pan Amer. Math. J. 21 (2011) 63–77.

[12] H. M. Srivastava and S. S. Eker, Some applications of a subordination theorem for a class of analytic functions, Appl. Math. Lett. 21 (2008) 394–399.

[13] H. M. Srivastava, D.-G. Yang and N-E. Xu, Subordinations for multivalent analytic functions associated with the Dziok-Srivastava operator, Integral Transforms Spec. Funct. 20 (2009) 581–606.

[14] Q.-H. Xu, C.-B. Lv, N.-C. Luo and H. M. Srivastava, Sharp coefficient estimates for a certain general class of spirallike functions by means of differential subordination, Filomat 27 (2013) 1351–1356.

[15] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, Some applications of differential subordination and the Dziok-Srivastava convolution operator, Appl. Math. Comput. 230 (2014) 496–508.