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Extensions of Symmetric Singular Second-Order Dynamic Operators on Time Scales

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Abstract. A space of boundary values is constructed for minimal symmetric singular second-order dynamic operators on semi-infinite and infinite time scales in limit-point and limit-circle cases. A description of all maximal dissipative, maximal accumulative, selfadjoint, and other extensions of such symmetric operators is given in terms of boundary conditions.

1. Introduction and Preliminaries

The theory of extensions of symmetric operators is one of the main branches in operator theory closely related to various fields of mathematics. Historically, J. von Neumann ([15]) pioneered the theory of selfadjoint extensions of densely defined, closed symmetric operators. In operator theory there exists an abstract scheme of constructing maximal dissipative (selfadjoint) extensions of symmetric operators that are parametrized by contraction (unitary) operators (see [5, 7-9, 12, 14, 15-19]). However, regardless of the general scheme, the problem of the description of the maximal dissipative (accumulative), selfadjoint, and other extensions of a given symmetric operator via the boundary conditions is of considerable interest. This problem is particularly interesting in the case of singular operators, because at the singular ends of the interval under consideration the usual boundary conditions are in general meaningless (see [2, 3, 7, 8, 14, 18, 19]).

The study of dynamic equations on time scales is a new area of theoretical exploration in mathematics. Time scale calculus allows us to study more general dynamic operators. In 1990, Hilger ([10]) introduced his theory to unify some results obtained for differential and difference equations. Hilger's results have been developed by many authors (for example see [1, 4, 6, 11, 13, 20]). In this paper, we consider the minimal symmetric singular second-order dynamic operators on semi-infinite and infinite time scales in limit-point and limit-circle cases. We construct a space of boundary values and describe all maximal dissipative (accumulative), selfadjoint and other extensions of minimal symmetric operators in terms of the boundary conditions.

We consider the second-order (or Sturm-Liouville) dynamic expression

$$(\tau x)(t) = \frac{1}{r(t)}(-(p(t)x^{\Delta})^{\nabla} + q(t)x),\ t \in \mathbb{T}_{+} := \mathbf{T} \cap [a, \infty)\,,\ -\infty < a < +\infty, \tag{1.1}$$

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where **T** denotes a time scale which contains the forward jump of a and unbounded above ([4]). We assume that p, q and r are real-valued, and p^{-1}, q and r are locally ∇ integrable functions on $\mathbb{T}_+$, and $r > 0$ on $\mathbb{T}_+$ ([4]). These conditions for p, q and r are minimal; note that there is no sign restriction on the coefficient p.

To pass from the expression (1.1) to operators, we introduce the Hilbert space $\mathcal{L}_r^2(\mathbb{T}_+)$ consisting of all complex valued functions x such that $\int_a^\infty r(t)\,|x(t)|^2\,\nabla t < +\infty$ with the inner product $(x, y) = \int_a^\infty r(t)x(t)\overline{y(t)}\nabla t$.

Denote by $\mathcal{D}_{\max}$ the linear set of all functions $x \in \mathcal{L}_r^2(\mathbb{T}_+)$ such that x is locally Δ absolutely continuous function on $\mathbb{T}_+$ and px^Δ is locally ∇ absolutely continuous functions on $\mathbb{T}_+$ (see [6]), and $\tau x \in \mathcal{L}_r^2(\mathbb{T}_+)$. The expression $x^{[\Delta]} = px^\Delta$ will be called the *first* Δ *quasi-derivative* of x. We define the *maximal* operator $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{L}_{\max}x = \tau x$. For each $x, y \in \mathcal{D}_{\max}$ we define the *Wronski determinant* (or *Wronskian*) $\mathcal{W}_t(x, y) = x(t)y^{[\Delta]}(t) - x^{[\Delta]}(t)y(t)$, $t \in \mathbb{T}_+$.

For two arbitrary functions $x, y \in \mathcal{D}_{\max}$, we have Green's formula

$$\int_a^t r(\xi)(\tau x)(\xi))\overline{y(\xi)}\nabla\xi - \int_a^t r(\xi)x(\xi)\overline{(\tau y)(\xi)}\nabla\xi = [x, y](t) - [x, y](a), \tag{1.2}$$

where $[x, y](t) := \mathcal{W}_t(x, \overline{y})$. It is clear from (1.2) that limit $[x, y](\infty) := \lim_{t\to\infty}[x, y](t)$ exists and is finite for all $x, y \in \mathcal{D}_{\max}$. For any function $x \in \mathcal{D}_{\max}$, $x(a)$ and $x^{[\Delta]}(a)$ can be defined by $x(a) := \lim_{t\to a^+} x(t)$ and $x^{[\Delta]}(a) := \lim_{t\to a^+} x^{[\Delta]}(t)$. These limits exist and are finite (since x is Δ absolutely continuous and $x^{[\Delta]}$ is ∇ absolutely continuous functions on $[a, c] \cap \mathbf{T}_+$, $\forall c \in (a, \infty)$).

In $\mathcal{L}_r^2(\mathbb{T}_+)$, we consider the dense linear set $\mathcal{D}_0$ consisting of smooth, compactly supported functions on $\mathbb{T}_+$. Denote by $\mathcal{L}_0$ the restriction of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_0$. It follows from (1.2) that $\mathcal{L}_0$ is symmetric. Consequently, it admits closure which we denote by $\mathcal{L}_{\min}$. The domain $\mathcal{D}_{\min}$ of $\mathcal{L}_{\min}$ consists of precisely those vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions

$$x(a) = x^{[\Delta]}(a) = 0, [x, y](\infty) = 0, \ \forall y \in \mathcal{D}_{\max}. \tag{1.3}$$

The *minimal* operator $\mathcal{L}_{\min}$ is a closed, symmetric operator with deficiency indices $(1, 1)$ or $(2, 2)$, and $\mathcal{L}_{\max} = \mathcal{L}_{\min}^*$ (see [7-8, 11, 14, 18-20]).

We denote by $\varphi(t)$ and $\psi(t)$ the solutions (real-valued) of the equation

$$\tau x = 0, \ t \in \mathbb{T}_+ \tag{1.4}$$

satisfying the initial conditions

$$\varphi(a) = 1, \varphi^{[\Delta]}(a) = 0, \psi(a) = 0, \psi^{[\Delta]}(a) = 1. \tag{1.5}$$

The Wronskian of the two solutions of (1.4) does not depend on t, and the two solutions of this equation are linearly independent if and only if their Wronskian is non-zero ([4]). It follows from the conditions (1.5) and the constancy of the Wronskian that

$$\mathcal{W}_t(\varphi, \psi) = \mathcal{W}_a(\varphi, \psi) = 1 \ (t \in \mathbb{T}_+). \tag{1.6}$$

Consequently, φ and ψ form a fundamental system of solutions of (1.4).

2. Selfadjoint and Nonselfadjoint Extensions of the Symmetric Operator with One Singular End Point

(a) Let symmetric operator $\mathcal{L}_{\min}$ has deficiency indices $(1, 1)$, so the case of limit-point occurs for dynamic expression τ or $\mathcal{L}_{\min}$. (see [7-8, 11, 14, 18-20]). Then $[x, y](\infty) = 0$ for all $x, y \in \mathcal{D}_{\min}$. The domain $\mathcal{D}_{\min}$ of the symmetric operator $\mathcal{L}_{\min}$ consist of precisely those vectors $x \in \mathcal{D}_{\min}$ satisfying the conditions: $x(a) = x^{[\Delta]}(a) = 0$.

Recall that a linear operator S (with dense domain $\mathcal{D}(S)$) acting in some Hilbert space H is called *dissipative (accumulative)* if $\Im(Sf, f) \geq 0$ ($\Im(Sf, f) \leq 0$) for all $f \in \mathcal{D}(S)$ and *maximal dissipative (maximal accumulative)* if it does not have a proper dissipative (accumulative) extension ([9]).

An important role in the theory of extensions is played by the concept of the space of boundary values of the symmetric operator. The triplet $(\mathcal{H}, \Gamma_1, \Gamma_2)$, where $\mathcal{H}$ is a Hilbert space and Γ_1 and Γ_2 are linear mappings of $\mathcal{D}(A^*)$ into $\mathcal{H}$, is called (see [9, p.152]) a *space of boundary values* of a closed symmetric operator A acting in a Hilbert space H with equal (finite or infinite) deficiency indices if

(i) $(A^*f, g)_H - (f, A^*g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathcal{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathcal{H}}, \forall f, g \in \mathcal{D}(A^*)$, and

(ii) for every $F_1, F_2 \in \mathcal{H}$, there exists a vector $f \in \mathcal{D}(A^*)$ such that $\Gamma_1 f = F_1$ and $\Gamma_2 f = F_2$.

We denote by Γ_1 and Γ_2 the linear mappings of $\mathcal{D}_{\max}$ into $\mathbb{C}$ defined by

$$\Gamma_1 x = -x(a), \ \Gamma_2 x = x^{[\Delta]}(a). \tag{2.1}$$

Then we have

Theorem 2.1. *The triplet $(\mathbb{C}, \Gamma_1, \Gamma_2)$ defined according to (2.1) is a space of boundary values of the operator $\mathcal{L}_{\min}$.*

Proof. The first requirement of the definition of a space of boundary values holds in view $(\mathcal{L}_{\max}x, y) - (x, \mathcal{L}_{\max}y), = -[x, y](a)$ $(\forall x, y \in \mathcal{D}_{\max})$ and

$$(\Gamma_1 x, \Gamma_2 y)_{\mathbb{C}} - (\Gamma_2 x, \Gamma_1 y)_{\mathbb{C}} = -x(a)\, \overline{y}^{[\Delta]}(a) + x^{[\Delta]}(a)\, \overline{y}(a)$$

$$= -[x, y](a) = (\mathcal{L}_{\max}x, y) - (x, \mathcal{L}_{\max}y), \ \forall x, y \in \mathcal{D}_{\max}.$$

The second requirement requirement of the definition of a space of boundary values will be proved as to following

Lemma 2.2. *For any complex numbers α and β there is a function $x \in \mathcal{D}_{\max}$ satisfying the boundary conditions*

$$x(a) = \alpha, \ x^{[\Delta]}(a) = \beta. \tag{2.2}$$

Proof. Let us denote by $\mathcal{L}_{\min,c}$ $(\mathcal{L}_{\max,c})$ the minimal symmetric (maximal) operator generated by τ on the set $[a, c] \cap \mathbf{T}_+$ $(c \in \mathbb{T}_+)$. Let y be an arbitrary vector in $\mathcal{L}_r^2(a, c)$ satisfying

$$(y, \varphi) = \beta, \ (y, \psi) = -\alpha. \tag{2.3}$$

There is such an y even among the linear combination of φ and ψ. Indeed, if we set $y = c_1\varphi + c_2\psi$, then conditions (2.3) are a system of equations for constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent functions φ and ψ and is therefore non-zero. Denote by $x_0(t)$ the solution of $\tau(x) = y(t)$ $(t \in [a, c] \cap \mathbf{T}_+)$ satisfying the initial conditions $x_0(c) = 0$, $x_0^{[\Delta]}(c) = 0$. We first observe that $x_0(t)$ is expressed by

$$x_0(t) = \int_t^c \{\varphi(t)\psi(\xi) - \varphi(\xi)\psi(t)\}\, r(\xi) y(\xi) \nabla\xi.$$

Observing that $\varphi, \psi \in \mathcal{L}_r^2(a, c)$, we have $x_0 \in \mathcal{L}_r^2(a, c)$ and, moreover, $x_0 \in \mathcal{D}_{\max,c}$. Further, applying Green's formula (1.2) to x_0 and φ, we obtain

$$(y, \varphi) = (\tau(x_0), \varphi) = [x_0, \varphi](c) - [x_0, \varphi](a) + (x_0, \tau(\varphi)).$$

But $\tau(\varphi) = 0$, and thus $(x_0, \tau(\varphi)) = 0$. Moreover, since $x_0(c) = 0$, $x_0^{[\Delta]}(c) = 0$, we have

$$-[x_0, \varphi](a) = -x_0(a)\, \varphi^{[\Delta]}(a) + x_0^{[\Delta]}(a)\, \varphi(a) = x_0^{[\Delta]}(a) = \beta.$$

Analogously,

$$-(y, \psi) = -(\tau(x_0), \psi) = -[x_0, \psi](c) + [x_0, \psi](a) - (x_0, \tau(\psi))$$

$$= [x_0, \psi](a) = x_0(a)\, \psi^{[\Delta]}(a) - x_0^{[\Delta]}(a)\, \psi(a) = x_0(a) = \alpha.$$

Therefore

$$x_0(a) = \alpha, x_0^{[\Delta]}(a) = \beta,\ x_0(c) = 0, x_0^{[\Delta]}(c) = 0. \tag{2.4}$$

Now let us define the function

$$x(t) = \begin{cases} x_0(t),\ a \le t \le c,\ t \in \mathbf{T}_+ \\ 0,\ c \le t < \infty,\ t \in \mathbf{T}_+. \end{cases}$$

It is clear that $x \in \mathcal{D}_{\max}$, then lemma is proved. □

Using Theorem 2.1 and [9, Theorem 1.6] we can write the following theorem.

Theorem 2.3. *Every maximal dissipative (accumulative) extension $\mathcal{L}_\theta$ of $\mathcal{L}_{\min}$ is determined by the equality $\mathcal{L}_\theta x = \mathcal{L}_{\max} x$ on the vectors x in $\mathcal{D}_{\max}$ satisfying the boundary condition*

$$x^{[\Delta]}(a) - \theta x(a) = 0, \tag{2.5}$$

where $\Im\theta \ge 0$ or $\theta = \infty$ ($\Im\theta \le 0$ or $\theta = \infty$). *Conversely, for an arbitrary number θ with $\Im\theta \ge 0$ or $\theta = \infty$* ($\Im\theta \le 0$ or $\theta = \infty$), *condition (2.5) determines a maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$. The selfadjoint extensions of $\mathcal{L}_{\min}$ are obtained precisely when* $\Im\theta = 0$ or $\theta = \infty$. *For $\theta = \infty$, the corresponding boundary condition has the form $x(a) = 0$.*

(b) We assume that $\mathcal{L}_{\min}$ has deficiency indices $(2,2)$, so that the limit-circle case holds for the dynamic expression τ or the operator $\mathcal{L}_{\min}$ (see [7-8, 11, 14, 18-20]). Since $\mathcal{L}_{\min}$ has deficiency indices $(2,2)$, $\varphi, \psi \in \mathcal{L}_r^2(\mathbb{T}_+)$ and, moreover, $\varphi, \psi \in \mathcal{D}_{\max}$.

Lemma 2.4. *For arbitrary functions $x, y \in \mathcal{D}_{\max}$, we have the equality* (*the Plücker identity*)

$$[x, y](t) = [x, \varphi](t)[\overline{y}, \psi](t) - [x, \psi](t)[\overline{y}, \varphi](t),\ t \in \mathbb{T}_+ \cup \{\infty\}. \tag{2.6}$$

Proof. Since the functions φ and ψ are real-valued and since $[\varphi, \psi](t) = 1$ $(t \in \mathbb{T}_+ \cup \{\infty\})$, one obtains

$$[x, \varphi](t)[\overline{y}, \psi](t) - [x, \psi](t)[\overline{y}, \varphi](t) = (x\varphi^{[\Delta]} - x^{[\Delta]}\varphi)(t)(\overline{y}\psi^{[\Delta]} - \overline{y}^{[\Delta]}\psi)(t)$$

$$-(x\psi^{[\Delta]} - x^{[\Delta]}\psi)(t)(\overline{y}\varphi^{[\Delta]} - \overline{y}^{[\Delta]}\varphi)(t) = (x\varphi^{[\Delta]}\overline{y}\psi^{[\Delta]} - x\varphi^{[\Delta]}\overline{y}^{[\Delta]}\psi - x^{[\Delta]}\varphi\overline{y}\psi^{[\Delta]}$$

$$+x^{[\Delta]}\varphi\overline{y}^{[\Delta]}\psi - x\psi^{[\Delta]}\overline{y}\varphi^{[\Delta]} + x\psi^{[\Delta]}\overline{y}^{[\Delta]}\varphi + x^{[\Delta]}\psi\overline{y}\varphi^{[\Delta]} - x^{[\Delta]}\psi\overline{y}^{[\Delta]}\varphi)(t)$$

$$= (-x\overline{y}^{[\Delta]} + x^{[\Delta]}\overline{y})(t)(\varphi^{[\Delta]}\psi - \varphi\psi^{[\Delta]})(t) = [x, y](t).$$

The lemma is proved. □

Theorem 2.5. *The domain $\mathcal{D}_{\min}$ of the operator $\mathcal{L}_{\min}$ consists of precisely those functions $f \in \mathcal{D}_{\max}$ satisfying the following boundary conditions*

$$f(a) = f^{[\Delta]}(a) = 0,\ [f, \varphi](\infty) = [f, \psi](\infty) = 0. \tag{2.7}$$

Proof. As noted above, the domain $\mathcal{D}_{\min}$ of $\mathcal{L}_{\min}$ coincides with the set of all functions $f \in \mathcal{D}_{\max}$, satisfying (2.3). By virtue of Lemma 2.4, (2.3) is equivalent to

$$f(a) = f^{[\Delta]}(a) = 0,\ [f, \varphi](\infty)[\bar{g}, \psi](\infty) - [f, \psi](\infty)[\bar{g}, \varphi](\infty) = 0. \tag{2.8}$$

Further $[\bar{g}, \psi](\infty)$ and $[\bar{g}, \varphi](\infty)$ $(y \in \mathcal{D}_{\max})$ can be arbitrary, therefore equality (2.8) for all $y \in \mathcal{D}_{\max}$ is possible if and only if the conditions (2.7) hold. The theorem is proved. □

We denote by Θ_1 and Θ_2 the linear mappings of $\mathcal{D}_{\max}$ into $\mathbb{C}^2$ defined by

$$\Theta_1 f = \begin{pmatrix} -f(a) \\ [f, \varphi](\infty) \end{pmatrix}, \Theta_2 f = \begin{pmatrix} f^{[\Delta]}(a) \\ [f, \psi](\infty) \end{pmatrix}. \tag{2.9}$$

Then we have

Theorem 2.6. *The triplet* $\left(\mathbb{C}^2, \Theta_1, \Theta_2\right)$ *defined according to (2.9) is a space of boundary values of the operator* $\mathcal{L}_{\min}$.

Proof. The first condition of the definition of a space of boundary values holds in view of (1.2) and Lemma 2.4:

$$(\Theta_1 f, \Theta_2 g)_{\mathbb{C}^2} - (\Theta_2 f, \Theta_1 g)_{\mathbb{C}^2} = -f(a)\,\bar{g}^{[\Delta]}(a) + f^{[\Delta]}(a)\,\bar{g}(a))$$

$$+ [f, \varphi](\infty)[\bar{g}, \psi](\infty) - [f, \psi](\infty)[\bar{g}, \varphi](\infty)$$

$$= [f, g](\infty) - [f, g](a) = (\mathcal{L}_{\max} f, g) - (f, \mathcal{L}_{\max} g), \ \forall f, g \in \mathcal{D}_{\max}.$$

The second condition will be proved as the following lemma.

Lemma 2.7. *For any complex numbers* α, β, γ *and* δ, *there is a function* $x \in \mathcal{D}_{\max}$ *satisfying the boundary conditions*

$$x(a) = \alpha, \ x^{[\Delta]}(a) = \beta, \ [x, \varphi](\infty) = \gamma, \ [x, \psi](\infty) = \delta. \tag{2.10}$$

Proof. Let z be an arbitrary vector in $\mathcal{L}_r^2(\mathbb{T}_+)$ satisfying

$$(z, \varphi) = \gamma + \beta, \ (z, \psi) = \delta - \alpha. \tag{2.11}$$

There is such an z even among the linear combination of φ and ψ. Indeed, if we set $z = c_1\varphi + c_2\psi$, then conditions (2.11) are a system of equations for constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent functions φ and ψ and is therefore nonzero.

Denote by $x(t)$ the solution of $\tau(x) = z(t)$ $(t \in \mathbb{T}_+)$ satisfying the initial conditions $x(a) = \alpha$, $x^{[\Delta]}(a) = \beta$. We claim that x is the desired function. We first observe that $x(t)$ is expressed by

$$x(t) = \alpha\varphi(t) + \beta\psi(t) + \int_a^t \{\varphi(t)\psi(\xi) - \varphi(\xi)\psi(t)\}\, r(\xi) z(\xi) \nabla\xi.$$

Observing that $\varphi, \psi \in \mathcal{L}_r^2(\mathbb{T}_+)$, we have $x \in \mathcal{L}_r^2(\mathbb{T}_+)$ and, moreover, $x \in \mathcal{D}_{\max}$. Further, applying Green's formula (1.2) to x and φ, we obtain

$$(z, \varphi) = (\tau(x), \varphi) = [x, \varphi](\infty) - [x, \varphi](a) + (x, \tau(\varphi)).$$

But $\tau(\varphi) = 0$, and thus $(x, \tau(\varphi)) = 0$. Moreover, since $x(a) = \alpha$, $x^{[\Delta]}(a) = \beta$, we have

$$[x, \varphi](a) = x(a)\,\varphi^{[\Delta]}(a) - x^{[\Delta]}(a)\,\varphi(a) = -\beta.$$

Therefore,

$$(z, \varphi) = [x, \varphi](\infty) + \beta. \tag{2.12}$$

Then, from (2.11) and (2.12), we obtain $[x, \varphi](\infty) = \gamma$.

Analogously,

$$(z, \psi) = (\tau(x), \psi) = [x, \psi](\infty) - [x, \psi](a) + (x, \tau(\psi)) = [x, \psi](\infty) - \alpha. \tag{2.13}$$

Then, from (2.11) and (2.13), we obtain $[x, \psi](\infty) = \delta$. Lemma 2.7 is proved and consequently, so is Theorem 2.6. □

Using Theorem 2.6 and [9, Theorem 1.6], we can state the following theorem.

Theorem 2.8. *For any contraction* K *in* $\mathbb{C}^2$ *the restriction of the operator* $\mathcal{L}$ *to the set of functions* $f \in \mathcal{D}_{\max}$ *satisfying the boundary condition*

$$(K - I)\,\Theta_1 f + i\,(K + I)\,\Theta_2 f = 0 \tag{2.14}$$

or

$$(K - I)\,\Theta_1 f - i\,(K + I)\,\Theta_2 f = 0 \tag{2.15}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $\mathcal{L}_{\min}$. Conversely, every maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$ is the restriction of $\mathcal{L}_{\max}$ to the set of vectors $f \in \mathcal{D}_{\max}$ satisfying (2.14) ((2.15)), and the contraction K is uniquely determined by the extensions. These conditions define a selfadjoint extension if and only if K is unitary. In the latter case (2.14) and (2.15) are, equivalent to the condition $(\cos A)\,\Theta_1 f - (\sin A)\,\Theta_2 f = 0$, where A is a Hermitian matrix in $\mathbb{C}^2$. The general form of the dissipative and accumulative extensions of the operator $\mathcal{L}_{\min}$ is given by the conditions

$$K(\Theta_1 f + i\Theta_2 f) = \Theta_1 f - i\Theta_2 f, \Theta_1 f + i\Theta_2 f \in \mathcal{D}(K) \tag{2.16}$$

$$K(\Theta_1 f - i\Theta_2 f) = \Theta_1 f + i\Theta_2 f, \Theta_1 f - i\Theta_2 f \in \mathcal{D}(K) \tag{2.17}$$

respectively, where K is a linear operator in $\mathbb{C}^2$ with $\left\|Kf\right\| \le \left\|f\right\|, f \in \mathcal{D}(K)$. The general form of symmetric extensions is given by the formulae (2.16) and (2.17), where K is an isometric operator.

In particular, the boundary conditions ($f \in \mathcal{D}_{\max}$)

$$f^{[\Delta]}(a) - h_1 f(a) = 0, \tag{2.18}$$

$$[f,\varphi](\infty) - h_2 [f,\psi](\infty) = 0 \tag{2.19}$$

with $\Im h_1 \ge 0$ or $h_1 = \infty$, and $\Im h_2 \ge 0$ or $h_2 = \infty$ ($\Im h_1 \le 0$ or $h_1 = \infty$, and $\Im h_2 \le 0$ or $h_2 = \infty$) *describe all maximal dissipative (maximal accumulative) extensions of $\mathcal{L}_{\min}$ with separated boundary conditions. The selfadjoint extensions of $\mathcal{L}_{\min}$ are obtained precisely when $\Im h_1 = 0$ or $h_1 = \infty$, and $\Im h_2 = 0$ or $h_2 = \infty$. Here for $h_1 = \infty$ ($h_2 = \infty$), condition (2.18) ((2.19)) should be replaced by $f(a) = 0$ ($[f,\psi](\infty) = 0$).*

3. Selfadjoint and Nonselfadjoint Extensions of the symmetric Operator with Two Singular End Points

We consider the second-order dynamic expression

$$(\tau x)(t) = \frac{1}{r(t)}(-(p(t)x^{\Delta})^{\nabla} + q(t)x),\ t \in \mathbb{T} := \mathbf{T} \cap (-\infty,\infty), \tag{3.1}$$

where $\mathbf{T}$ denotes a time scale unbounded above and below. We assume that p, q and r are real-valued, and p^{-1}, q and r are locally ∇ integrable functions on $\mathbb{T}$, and $r > 0$ on $\mathbb{T}$.

Denote by $\mathcal{D}_{\max}$ the linear set of all vectors $x \in \mathcal{L}_r^2(\mathbb{T})$ such that x is locally Δ absolutely continuous function on $\mathbb{T}$ and px^{Δ} is locally ∇ absolutely continuous functions on $\mathbb{T}$ ([6]), and $\tau x \in \mathcal{L}_r^2(\mathbb{T})$. We define the *maximal operator* $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{L}_{\max} x = \tau(x)$.

For two arbitrary vectors $x, y \in \mathcal{D}_{\max}$ we have Green's formula

$$(\mathcal{L}_{\max} x, y) - (x, \mathcal{L}_{\max} y) = [x,y](\infty) - [x,y](-\infty). \tag{3.2}$$

In $\mathcal{L}_r^2(\mathbb{T})$, we consider the dense linear set $\mathcal{D}_0$ consisting of smooth, compactly supported functions on $\mathbb{T}$. Denote by $\mathcal{L}_0$ the restrictions of the operator $\mathcal{L}_{\max}$ to $\mathcal{D}_0$. It follows from (3.2) that $\mathcal{L}_0$ is symmetric. Consequently, it admits closure which is denoted by $\mathcal{L}_{\min}$. The domain of *minimal operator* $\mathcal{L}_{\min}$ consists of precisely those vectors $x \in \mathcal{D}_{\max}$ satisfying the conditions

$$[x,y](\infty) - [x,y](-\infty) = 0,\ \forall y \in \mathcal{D}_{\max}. \tag{3.3}$$

The operator $\mathcal{L}_{\min}$ is a symmetric operator with deficiency indices $(0,0)$, $(1,1)$ or $(2,2)$ and $\mathcal{L}_{\max} = \mathcal{L}_{\min}^*$ (see [7-8, 11, 14, 18-20]). For deficiency indices $(0,0)$ the operator $\mathcal{L}_{\min}$ is selfadjoint, that is, $\mathcal{L}_{\min}^* = \mathcal{L}_{\min} = \mathcal{L}_{\max}$.

Let symmetric operator $\mathcal{L}_{\min}$ has deficiency indices (m,m) $(1 \le m \le 2)$. We denote by $\mathcal{L}_{\min}^-$ and $\mathcal{L}_{\min}^-$ the minimal symmetric operators generated by the expression τ on the intervals $(-\infty,c] \cap \mathbf{T}$ and $[c,\infty) \cap \mathbf{T}$ ($c \in \mathbf{T}$), respectively. We know ([7, 8, 14, 18]) that the defect number def $\mathcal{L}_{\min}$ of the operator $\mathcal{L}_{\min}$ is determined by the formula $def\mathcal{L}_{\min} = def\mathcal{L}_{\min}^+ + def\mathcal{L}_{\min}^- - 2$. From this we have $def\mathcal{L}_{\min}^+ + def\mathcal{L}_{\min}^- = 2 + m$. If we put $m_+ := def\mathcal{L}_{\min}^+$, $m_- := def\mathcal{L}_{\min}^-$, then we obtain $1 \le m_+ \le 2$, $1 \le m_- \le 2$.

We say that $\mathcal{L}_{\min}$ has deficiency indices (m,m) at $-\infty$ (respectively, at ∞), where $1 \le m \le 2$ if the operator $\mathcal{L}^-_{\min}$ (respectively $\mathcal{L}^+_{\min}$) has deficiency indices (m,m). If $\mathcal{L}_{\min}$ has deficiency indices $(1,1)$ at $-\infty$ (at ∞) then $[x,y](-\infty) = 0$ ($[x,y](\infty) = 0$) for all $x, y \in \mathcal{D}_{\max}$.

We denote by ϕ and χ the solutions of equation $\tau x = 0$ when $t \in \mathbb{T}$ satisfying the conditions

$$\phi(c) = 1,\ \phi^{[\Delta]}(c) = 0,\ \chi(c) = 0,\ \chi^{[\Delta]}(c) = 1\ (c \in \mathbb{T}). \tag{3.4}$$

Lemma 2.4 clearly remains in force also for the $x, y \in \mathcal{D}_{\max}$ ($t \in \mathbb{T}\cup\{-\infty,\infty\}$). Therefore, the next theorem can be proved in the same way as in the case of the two singular end points.

Theorem 3.1. *If $\mathcal{L}_{\min}$ has deficiency indices $(2,2)$, then its domain $\mathcal{D}_{\min}$ consists of precisely those $x \in \mathcal{D}_{\max}$ satisfying the boundary conditions*

$$\left[x,\phi\right](-\infty) = [x,\chi](-\infty) = \left[x,\phi\right](\infty) = \left[\phi,\chi\right](\infty) = 0.$$

But if $\mathcal{L}_{\min}$ has deficiency indices $(2,2)$ at $-\infty$ (at ∞) and $(1,1)$ at ∞ (at $-\infty$), then $\mathcal{D}_{\min}$ consists of all the functions $x \in \mathcal{D}_{\max}$ satisfying the boundary conditions

$$\left[x,\phi\right](-\infty) = [x,\chi](-\infty) = 0\ (\left[x,\phi\right](\infty) = [x,\chi](\infty) = 0).$$

(a) We assume the symmetric operator $\mathcal{L}_{\min}$ has deficiency indices $(2,2)$. We consider the following linear maps of $\mathcal{D}_{\max}$ into $\mathbb{C}^2$

$$\Phi_1 x = \begin{pmatrix} [x,\chi](-\infty) \\ \left[x,\phi\right](\infty) \end{pmatrix},\ \Phi_2 x = \begin{pmatrix} \left[x,\phi\right](-\infty) \\ [x,\chi](\infty) \end{pmatrix}. \tag{3.5}$$

Then we have

Theorem 3.2. *The triplet $\left(\mathbb{C}^2,\Phi_1,\Phi_2\right)$ defined according to (3.5) is a space of boundary values of the operator $\mathcal{L}_{\min}$.*

Proof. The first condition of the definition of a space of boundary values holds in view of (3.2) and Lemma 2.4 (for $t \in \mathbb{T}\cup\{-\infty,\infty\}$):

$$(\Phi_1 x,\Phi_2 y)_{\mathbb{C}^2} - (\Phi_2 x,\Phi_1 y)_{\mathbb{C}^2} = [x,\chi](-\infty)\left[\overline{y},\phi\right](-\infty)$$

$$-\left[x,\phi\right](-\infty)[\overline{y},\chi](-\infty) + \left[x,\phi\right](\infty)[\overline{y},\chi](\infty) - [x,\chi](\infty)\left[\overline{y},\phi\right](\infty)$$

$$= [x,y](\infty) - [x,y](-\infty) = (\mathcal{L}_{\max}x,y) - (x,\mathcal{L}_{\max}y)\ \ (\forall\, x,y \in \mathcal{D}_{\max}).$$

The second condition will be proved as the following lemma.

Lemma 3.3. *For any complex numbers α,β,γ and δ, there is a function $x \in \mathcal{D}_{\max}$ satisfying*

$$\left[x,\phi\right](-\infty) = \alpha, [x,\chi](-\infty) = \beta, \left[x,\phi\right](\infty) = \gamma, [x,\chi](\infty) = \delta. \tag{3.6}$$

Proof. The operators $\mathcal{L}^+_{\min}$ and $\mathcal{L}^-_{\min}$ have deficiency indices $(2,2)$. By Lemma 2.7, there is a function $x_+ \in \mathcal{D}^+_{\max}$ (where $\mathcal{D}^+_{\max}$ denotes the domain of a corresponding maximal operator $\mathcal{L}^+_{\max} = (\mathcal{L}^+_{\min})^*$) satisfying the conditions

$$x_+(c) = \gamma_0,\ x_+^{[\Delta]}(c) = \gamma_1,\ \forall\gamma_0,\gamma_1 \in \mathbb{C}, \tag{3.7}$$

$$\left[x_+,\phi\right](\infty) = \gamma,\ \ [x_+,\chi](\infty) = \delta. \tag{3.8}$$

By Lemma 2.7, there is a function $x_- \in \mathcal{D}^-_{\max}$ (where $\mathcal{D}^-_{\max}$ denotes the domain of a corresponding maximal operator $\mathcal{L}^-_{\max} = (\mathcal{L}^-_{\min})^*$) satisfying the conditions

$$x_-(c) = \gamma_0,\ x_-^{[\Delta]}(c) = \gamma_1, \tag{3.9}$$

$$\left[x_-,\phi\right](-\infty)=\alpha,\ \ [x_-,\chi](-\infty)=\beta. \tag{3.10}$$

Now we let

$$x(t)=\begin{cases} x_-(t), & -\infty<t\le c,\ t\in\mathbf{T},\\ x_+(t), & c\le t<\infty,\ t\in\mathbf{T}.\end{cases}$$

The conditions (3.8) and (3.9) then ensure that the functions x and $x^{[\Delta]}$ are continuous at the point $t=c$. Hence, $x\in\mathcal{D}_{\max}$ and the conditions (3.6) are satisfied. Lemma 3.3 and Theorem 3.2 is proved. □

Using Theorem 3.2 and [9, Theorem 1.6] we can state the following theorem.

Theorem 3.4. *For any contraction S in $\mathbb{C}^2$ the restriction of the operator $\mathcal{L}_{\max}$ to the set of vectors $g\in\mathcal{D}_{\max}$ satisfying the boundary condition*

$$(S-I)\,\Phi_1 g+i\,(S+I)\,\Phi_2 g=0 \tag{3.11}$$

or

$$(S-I)\,\Phi_1 g-i\,(S+I)\,\Phi_2 g=0 \tag{3.12}$$

is, respectively, a maximal dissipative or a maximal accumulative extension of the operator $\mathcal{L}_{\min}$. Conversely, every maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$ is the restriction of $\mathcal{L}_{\max}$ to the set of vectors $g\in\mathcal{D}_{\max}$ satisfying (3.11) ((3.12)), and the contraction S is uniquely determined by the extension. These conditions define a selfadjoint extension if and only if S is unitary. In the latter case, (3.11) and (3.12) are equivalent to the condition $(\cos A)\,\Phi_1 g-(\sin A)\,\Phi_2 g=0$, *where A is a Hermitian matrix in $\mathbb{C}^2$. The general form of dissipative and accumulative extensions of the operator $\mathcal{L}_{\min}$ is given by the conditions*

$$S\,(\Phi_1 g+i\Phi_2 g)=\Phi_1 g-i\Phi_2 g,\ \Phi_1 g+i\Phi_2 g\in\mathcal{D}(S) \tag{3.13}$$

$$S\,(\Phi_1 g-i\Phi_2 g)=\Phi_1 g+i\Phi_2 g,\ \Phi_1 g-i\Phi_2 g\in\mathcal{D}(S) \tag{3.14}$$

respectively, where S is a linear operator in $\mathbb{C}^2$ with $\left\|Sf\right\|\le\left\|f\right\|$, $f\in\mathcal{D}(S)$. The general form of symmetric extensions is given by the formulae (3.13) and (3.14), where S is an isometric operator.

In particular, the boundary conditions ($g\in\mathcal{D}_{\max}$)

$$[g,\chi](-\infty)-\theta_1\left[g,\phi\right](-\infty)=0 \tag{3.15}$$

$$\left[g,\phi\right](\infty)-\theta_2\,[g,\chi](\infty)=0 \tag{3.16}$$

with $\Im\theta_1\ge 0$ or $\theta_1=\infty$, and $\Im\theta_2\ge0$ or $\theta_2=\infty$ ($\Im\theta_1\le0$ or $\theta_1=\infty$, and $\Im\theta_2\le0$ or $\theta_2=\infty$) *describe all maximal dissipative (maximal accumulative) extensions of $\mathcal{L}_{\min}$ with separated boundary conditions. The selfadjoint extensions of $\mathcal{L}_{\min}$ are obtained precisely when $\Im\theta_1=0$ or $\theta_1=\infty$, and $\Im\theta_2=0$ or $\theta_2=\infty$. Here for $\theta_1=\infty$ ($\theta_2=\infty$), condition (3.15) ((3.16)) should be replaced by* $\left[g,\phi\right](-\infty)=0$ ($[g,\chi](\infty)=0$).

(b) We now let $\mathcal{L}^-_{\min}$ has deficiency indices $(1,1)$ and $\mathcal{L}^+_{\min}$ has deficiency indices $(2,2)$. The operator $\mathcal{L}_{\min}$ has deficiency indices $(1,1)$. Then $[x,y](-\infty)=0$ for all $x,y\in\mathcal{D}_{\max}$ and

$$(\mathcal{L}_{\max}x,y)-(x,\mathcal{L}_{\max}y)=[x,y](\infty),\ \forall x,y\in\mathcal{D}_{\max}. \tag{3.17}$$

We consider the following linear maps of $\mathcal{D}_{\max}$ into $\mathbb{C}$

$$\Psi_1 x=\left[x,\phi\right](\infty),\ \Psi_2 x=[x,\chi](\infty). \tag{3.18}$$

Then we have

Theorem 3.5. *The triplet* $(\mathbb{C},\Psi_1,\Psi_2)$ *defined according to (3.18) is a space of boundary values of the operator $\mathcal{L}_{\min}$.*

Proof. The first condition of the definition of a space of boundary values holds in view of (3.17) and Lemma 2.4:

$$(\Psi_1 x, \Psi_2 y)_{\mathbb{C}} - (\Psi_2 x, \Psi_1 y)_{\mathbb{C}} = \left[x, \phi\right](\infty)\left[\overline{y}, \chi\right](\infty) - [x, \chi](\infty)\left[\overline{y}, \phi\right](\infty)$$

$$= [x, y](\infty) = (\mathcal{L}_{\max} x, y) - (x, \mathcal{L}_{\max} y) \ \ (\forall x, y \in \mathcal{D}_{\max}).$$

The second condition will be proved as the following lemma.

Lemma 3.6. *For any complex numbers α, β, there is a function $x \in \mathcal{D}_{\max}$ satisfying $\left[x, \phi\right](\infty) = \alpha$, $[x, \chi](\infty) = \beta$.*

Proof. The operator $\mathcal{L}^{+}_{\min}$ has deficiency indices (2.2). By Lemma 2.5, there is a function $x_+ \in \mathcal{D}^{+}_{\max}$ satisfying the conditions

$$x_+(c) = 0, \ x_+^{[\Delta]}(c) = 0, \ \left[x_+, \phi\right](\infty) = \alpha, \ \ [x_+, \chi](\infty) = \beta. \tag{3.19}$$

Now we let

$$x(t) = \begin{cases} 0, \ -\infty < t \le c, \ t \in \mathbf{T}, \\ x_+(t), \ c \le t < \infty, \ t \in \mathbf{T}. \end{cases}$$

Then we have $x \in \mathcal{D}_{\max}$ and $\left[x, \phi\right](\infty) = \alpha$, $[x, \chi](\infty) = \beta$. The Lemma 3.6 and Theorem 3.5 is proved. □

Using Theorem 3.5 and [9, Theorem 1.6] we can state the following theorem.

Theorem 3.7. *Every maximal dissipative (accumulative) extensions $\mathcal{L}_\omega$ of $\mathcal{L}_{\min}$ is determined by the equality $\mathcal{L}_\omega x = \mathcal{L}_{\max} x$ of the vectors x in $\mathcal{D}_{\max}$ satisfying the boundary conditions*

$$\left[x, \phi\right](\infty) - \omega\,[x, \chi](\infty) = 0, \tag{3.20}$$

where $\Im\omega \ge 0$ or $\omega = \infty$ ($\Im\omega \le 0$ or $\omega = \infty$). Conversely, for an arbitrary number ω with $\Im\omega \ge 0$ or $\omega = \infty$ ($\Im\omega \le 0$ or $\omega = \infty$), condition (3.20) determines a maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$. The selfadjoint extension of $\mathcal{L}_{\min}$ are obtained precisely when $\Im\omega = 0$ or $\omega = \infty$.

(c) We now let $\mathcal{L}^{-}_{\min}$ has deficiency indices $(2, 2)$ and $\mathcal{L}^{+}_{\min}$ has deficiency indices $(1, 1)$. The operator $\mathcal{L}_{\min}$ has deficiency indices $(1, 1)$. Then $[x, y](\infty) = 0$ for all $x, y \in \mathcal{D}_{\max}$ and

$$(\mathcal{L}_{\max} x, y) - (x, \mathcal{L}_{\max} y) = -[x, y](-\infty), \ \forall x, y \in \mathcal{D}_{\max}. \tag{3.21}$$

We consider the following linear maps of $\mathcal{D}_{\max}$ into $\mathbb{C}$

$$\Upsilon_1 x = [x, \chi](-\infty), \ \Upsilon_2 x = \left[x, \phi\right](-\infty). \tag{3.22}$$

Then, the next two theorem can be proved in the same way as in the case (b).

Theorem 3.8. *The triplet $(\mathbb{C}, \Upsilon_1, \Upsilon_2)$ defined according to (3.22) is a space of boundary values of the operator $\mathcal{L}_{\min}$.*

Theorem 3.9. *Every maximal dissipative (accumulative) extension $\mathcal{L}_\vartheta$ of $\mathcal{L}_{\min}$ is determined by the equality $\mathcal{L}_\vartheta x = \mathcal{L}_{\max} x$ on the vectors x in $\mathcal{D}_{\max}$ satisfying the boundary condition*

$$[x, \chi](-\infty) - \vartheta\left[x, \phi\right](-\infty) = 0, \tag{3.23}$$

where $\Im\vartheta \ge 0$ or $\vartheta = \infty$ ($\Im\vartheta \le 0$ or $\vartheta = \infty$). Conversely, for an arbitrary number ϑ with $\Im\vartheta \ge 0$ or $\vartheta = \infty$ ($\Im\vartheta \le 0$ or $\vartheta = \infty$), condition (3.23) determines a maximal dissipative (accumulative) extension of $\mathcal{L}_{\min}$. The selfadjoint extensions of $\mathcal{L}_{\min}$ are obtained precisely when $\Im\vartheta = 0$ or $\vartheta = \infty$.

References

[1] R.P. Agarval, M. Bohner, P.J.Y. Wong, Sturm-Liouville eigenvalue problems on time scales, Appl. Math. Comput., 99 (1999), 153-165.

[2] B.P. Allahverdiev, Dissipative Sturm-Liouville operators with nonseparated boundary conditions, Monatsh. Math., 140 (1) (2003), 1-17.
[3] B.P. Allahverdiev, Self-adjoint and non-self-adjoint extensions of symmetric q-Sturm-Liouville operators, (submitted).
[4] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhauser, Boston, 2001.
[5] V.M. Bruk, On a class of boundary-value problems with a spectral parameter in the boundary conditions, Mat. Sb., 100 (1976), 210-216; English transl. Mat USSR Sb., 28 (1976), 186-192.
[6] A. Cabada, D.R. Vivero, Criterions for absolute continuity on time scales, J. Differ. Eq. Appl., 11 (2005), 1013-1028.
[7] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.
[8] N. Dunford, J.T. Schwartz, Linear operators, Part II, Interscience, New York, 1964.
[9] V.I. Gorbachuk, M.L. Gorbachuk, Boundary Value Problems for Operator Differential Equations, Naukova Dumka, Kiev, 1984; English transl. Dordrecht: Kluwer, 1991.
[10] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, Result Math., 18 (1990), 18-56.
[11] A. Huseynov, Limit point and limit circle cases for dynamic equations on time scales, Hacet. J. Math. Stat., 39 (2010), 379-392.
[12] A.N. Kochubei, Extensions of symmetric operators and symmetric binary relations, Mat. Zametki, 17 (1975), 41-48; English transl. Math. Notes, 17 (1975), 25-28.
[13] Q. Kong, Sturm-Liouville problems on time scales with seperated boundary conditions, Result Math., 52 (2008), 111-121.
[14] M.A. Naimark, Linear Differential Operators, 2nd edn , Moscow, Nauka, 1969; English transl. of 1st edn, Parts 1, 2: Ungar, New York, 1967, 1968.
[15] J. von Neumann, Allgemeine Eigenwerttheorie Hermitischer Functional Operatoren, Math. Ann., 102 (1929), 49-131.
[16] R.S. Phillips, Dissipative operators and hyperbolic systems of partial differential equations, Trans. Amer. Math. Soc., 90 (1959), 193-254.
[17] F.S. Rofe-Beketov, Self-adjoint extensions of differential operators in space of vector-valued functions, Dokl. Akad. Nauk SSSR, 184 (1969), 1034-1037; English transl. Soviet Math. Dokl., 10 (1969), 188-192.
[18] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Lecture Notes in Mathematics, vol. 1258, Springer-Verlag, Berlin, Heidelberg, 1987.
[19] A. Zettl, Sturm-Liouville Theory, Mathematical Surveys and Monographs, 121, American Mathematical Society, Providence, RI, 2005.
[20] C. Zhang, Y. Shi, Classification and criteria of the limit cases for singular second-order linear equations on time scales, Bound. Value Probl., 103 (2012), 1-17.