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L^1-Solutions for Implicit Fractional Order Differential Equations with Nonlocal Conditions

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Abstract. In this paper we study the existence of integrable solutions of the nonlocal problem for fractional order implicit differential equations with nonlocal condition. Our results are based on Schauder's fixed point theorem and the Banach contraction principle fixed point theorem.

1. Introduction

The topic of fractional calculus (integration and differentiation of fractional-order), which concerns singular integral and integro-differential operators, is enjoying interest among mathematicians, physicists and engineers. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 13, 16, 17, 19]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [14], Lakshmikantham *et al.* [15], and the papers by Agarwal *et al* [1, 2], Belarbi *et al.* [6], Benchohra *et al.* [7], and the references therein.

To our knowledge, the literature on integral solutions for fractional differential equations is very limited. El-Sayed and Hashem [12] studied the existence of integral and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered L^p-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

Motivated by the above papers, in this paper we deal with the existence of solutions of the nonlocal problem, for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \ a.e, \ t \in J =: (0, T], \tag{1}$$

$$\sum_{k=1}^{m} a_k y(t_k) = y_0, \tag{2}$$

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where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, $a_k \in \mathbb{R}$, ${}^cD^\alpha$ is the Caputo fractional derivative, and $0 < t_1 < t_2 < ..., t_m < T$, $k = 1, 2, ..., m$.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 3.3) and the second one on the Banach contraction principle (Theorem 3.4). An example is given in Section 4 to demonstrate the application of our main results. Let us mention that most of the existing results for fractional order differential equations are devoted to continuous or Carathéodory solutions. Thus, the main results of the present paper constitute a contribution to this emerging field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $L^1(J)$ denotes the class of Lebesgue integrable functions on the interval $J = [0, T]$, with the norm $\|u\|_{L_1} = \int_J |u(t)|dt$.

Definition 2.1. *.([14, 18]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where $\Gamma(.)$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2. *. ([14, 18]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}_+)$, is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α. If $\alpha \in (0, T]$, then

$$(D_{a+}^\alpha h)(t) = \frac{d}{dt} I_{a+}^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} h(s)ds.$$

Definition 2.3. *. ([14]). The Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}_+)$ is given by*

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$. If $\alpha \in (0, T]$, then

$$({}^cD_{a+}^\alpha h)(t) = I_{a+}^{1-\alpha} \frac{d}{dt} h(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} h(s)ds.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

Proposition 2.4. *[14] Let α, $\beta > 0$. Then we have*

(i) *$I^\alpha: L^1(J,\mathbb{R}_+) \to L^1(J,\mathbb{R}_+)$, and if $f \in L^1(J,\mathbb{R}_+)$, then*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t).$$

(ii) *If $f \in L^p(J,\mathbb{R}_+)$, $1 \le p \le +\infty$, then $\|I^\alpha f\|_{L_p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|f\|_{L_p}$.*

(iii) $\lim_{\alpha\to n} I^\alpha f(t) = I^n f(t)$, $n = 1, 2, \ldots$ *uniformly.*

The following theorems will be needed.

Theorem 2.5. *(Schauder fixed point theorem [10]) Let E a Banach space and Q be a convex subset of E and $T : Q \longrightarrow Q$ is compact, and continuous map. Then T has at least one fixed point in Q.*

Theorem 2.6. *(Kolmogorov compactness criterion [10]) Let $\Omega \subseteq L^p([0,T],\mathbb{R})$, $1 \le p \le \infty$. If*

(i) *Ω is bounded in $L^p([0,T],\mathbb{R})$, and*

(ii) *$u_h \longrightarrow u$ as $h \longrightarrow 0$ uniformly with respect to $u \in \Omega$,*
then Ω is relatively compact in $L^p([0,T],\mathbb{R})$,

where

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds.$$

3. Existence of Solutions

Let us start by defining what we mean by an integrable solution of the nonlocal problem (1) − (2).

Definition 3.1. . *A function $y \in L^1([0,T],\mathbb{R})$ is said to be a solution of problem* (1) − (2) *if y satisfies (1) and (2).*

In what follows, we assume that $\sum_{k=1}^m a_k \neq 0$. Set

$$a = \frac{1}{\sum_{k=1}^m a_k}.$$

For the existence of solutions for the nonlocal problem (1) − (2), we need the following auxiliary lemma.

Lemma 3.2. *The nonlocal problem* (1) − (2) *is equivalent to the integral equation*

$$y(t) = ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \tag{3}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, x(t)\right). \tag{4}$$

Proof. Let ${}^cD^\alpha y(t) = x(t))$ in equation (1), then

$$x(t) = f(t, y(t), x(t)) \tag{5}$$

and

$$\begin{aligned} y(t) &= y(0) + I^\alpha x(t)) \\ &= y(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \end{aligned} \tag{6}$$

Let $t = t_k$ in (6), we obtain

$$y(t_k) = y(0) + \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

and

$$\sum_{k=1}^m a_k y(t_k) = \sum_{k=1}^m a_k y(0) + \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \tag{7}$$

Substitute from (2) into (7), we get

$$y_0 = \sum_{k=1}^m a_k y(0) + \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

and

$$y(0) = a\left(y_0 - \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right). \tag{8}$$

Substitute from (8) into (6) and (5), we obtain (3) and (4).
For complete the proof, we prove that equation (3) satisfies the nonlocal problem (1) – (2). Differentiating (3), we get

$${}^cD^\alpha y(t) = x(t) = f(t, y(t), {}^cD^\alpha y(t)).$$

Let $t = t_k$ in (3), we obtain

$$\begin{aligned} y(t_k) &= ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds) + \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ &= ay_0 + \left(1 - a\sum_{k=1}^m a_k\right) \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \end{aligned}$$

Then

$$\sum_{k=1}^m a_k y(t_k) = \sum_{k=1}^m a_k a y_0 + \sum_{k=1}^m a_k \left(1 - a\sum_{k=1}^m a_k\right) \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds = y_0.$$

This complete the proof of the equivalent between the nonlocal problem (1)-(2) and the integral equation (3).
Leu us introduce the following assumptions:

(H1) $f : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in [0,T]$, for any $(u_1,u_2) \in \mathbb{R}^2$ and continuous in $(u_1,u_2) \in \mathbb{R}^2$, for almost all $t \in [0,T]$.

(H2) There exist a positive function $a \in L^1[0,T]$ and constants, $b_i > 0; i = 1,2$ such that:

$$|f(t,u_1,u_2)| \le |a(t)| + b_1|u_1| + b_2|u_2|, \forall (t,u_1,u_2) \in [0,T] \times \mathbb{R}^2.$$

Our first result is based on Schauder fixed point theorem.

Theorem 3.3. *Assume that the assumptions* $(H1)-(H2)$ *are satisfied. If*

$$\frac{2b_1T^{\alpha}}{\Gamma(\alpha+1)}+b_2<1, \tag{9}$$

then the problem $(1)-(2)$ *has at least one solution* $y\in L^1([0,T],\mathbb{R})$.

Proof. Transform the nonlocal problem $(1)-(2)$ into a fixed point problem. Consider the operator

$$H: L^1([0,T],\mathbb{R})\longrightarrow L^1([0,T],\mathbb{R})$$

defined by:

$$(Hx)(t)=f\left(t, ay_0-a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right). \tag{10}$$

Let

$$r=\frac{Tab_1y_0+\|a\|_{L_1}}{1-\left(\frac{2b_1T^{\alpha}}{\Gamma(\alpha+1)}+b_2\right)},$$

and consider the set

$$B_r=\{x\in L^1([0,T],\mathbb{R}):\|x\|_{L_1}\leq r\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $HB_r\subset B_r$, indeed, for each $x\in B_r$, from (9) and (10) we get

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T|Hx(t)|dt\\
&= \int_0^T\left|f\left(t, ay_0-a\sum_{k=1}^{m}a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right)\right|dt\\
&\leq \int_0^T\left[|a(t)|+b_1|ay_0-a\sum_{k=1}^{m}a_kI^{\alpha}x(t)|_{t=t_k}+I^{\alpha}x(t)|+b_2|x(t)|\right]dt\\
&\leq Tab_1y_0+\|a\|_{L_1}+\frac{b_1a\sum_{k=1}^{m}a_kt_k^{\alpha}}{\Gamma(\alpha+1)}\|x\|_{L_1}+\frac{b_1T^{\alpha}}{\Gamma(\alpha+1)}\|x\|_{L_1}+b_2\|x\|_{L_1}\\
&\leq Tab_1y_0+\|a\|_{L_1}+\left(\frac{2b_1T^{\alpha}}{\Gamma(\alpha+1)}+b_2\right)\|x\|_{L_1}\\
&\leq r.
\end{aligned}$$

Then $HB_r\subset B_r$. Assumption **(H1)** implies that H is continuous. Now, we will show that H is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in $L^1([0,T],\mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h\longrightarrow(Hx)$ in $L^1([0,T],\mathbb{R})$ for each $x\in B_r$.

Let $x \in B_r$, then we have

$$\begin{aligned}
& \|(Hx)_h - (Hx)\|_{L^1} \\
= & \int_0^T |(Hx)_h(t) - (Hx)(t)| dt \\
= & \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s) ds - (Hx)(t) \right| dt \\
\leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hx)(s) - (Hx)(t)| ds \right) dt \\
\leq & \int_0^T \frac{1}{h} \int_t^{t+h} \left| f\left(t, ay_0 - a \sum_{k=1}^m a_k \int_0^{s_k} \frac{(s_k - \tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau) d\tau + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau) d\tau, x(s) \right) \right. \\
& \left. - f\left(t, ay_0 - a \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds, x(t) \right) \right| ds dt.
\end{aligned}$$

Since $x \in B_r \subset L^1([0,T], \mathbb{R})$ and assumption **(H2)** that implies $f \in L^1([0,T], \mathbb{R})$, it follows that
$\frac{1}{h} \int_t^{t+h} \Big| f\left(t, ay_0 - a \sum_{k=1}^m a_k \int_0^{s_k} \frac{(s_k-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau) d\tau + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} x(\tau) d\tau, x(s) \right)$
$- f\left(t, ay_0 - a \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds, x(t) \right) \Big| ds \to 0 \text{ as } h \to 0$. Hence

$$(Hx)_h \to (Hx) \text{ uniformly as } h \to 0.$$

Then by Kolmogorov compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem the nonlocal problem (1) − (2) has at least one solution in B_r.

The following result is based on the Banach contraction principle.

Theorem 3.4. *Assume that **(H1)** and the following condition hold.*

(H3) *There exist constants $k_1, k_2 > 0$ such that*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq k_1 |x_1 - x_2| + k_2 |y_1 - y_2|, \ t \in [0,T], \ x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

If

$$\frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} + k_2 < 1, \tag{11}$$

then the problem (1) − (2) *has a unique solution* $y \in L^1([0,T], \mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (10) has a fixed point. Let $x, y \in L^1([0,T], \mathbb{R})$, and $t \in [0,T]$. Then we have,

$$\begin{aligned}
& |(Hx)(t) - (Hy)(t)| \\
= & \Big| f(t, ay_0 - a \sum_{k=1}^m a_k I^\alpha x(t)|_{t=t_k} + I^\alpha x(t), x(t)) \\
& - f(t, ay_0 - a \sum_{k=1}^m a_k I^\alpha y(t)|_{t=t_k} + I^\alpha y(t), y(t)) \Big| \\
\leq & k_1 a \sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - y(s)| ds \\
& + k_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - y(s)| ds + k_2 |x - y|.
\end{aligned}$$

Thus

$$\begin{aligned}\|(Hx)-(Hy)\|_{L_1} &\leq \frac{k_1 t_k^{\alpha} a \sum_{k=1}^{m} a_k}{\Gamma(\alpha+1)} \int_0^T |x(t)-y(t)|dt + \frac{k_1 T^{\alpha}}{\Gamma(\alpha+1)} \int_0^T |x(t)-y(t)|dt \\ &\quad + k_2 \int_0^T |x(t)-y(t)|dt \\ &\leq \frac{2k_1 T^{\alpha}}{\Gamma(\alpha+1)} \|x-y\|_{L_1} + k_2 \|x-y\|_{L_1} \\ &\leq \left(\frac{2k_1 T^{\alpha}}{\Gamma(\alpha+1)} + k_2\right) \|x-y\|_{L_1}.\end{aligned}$$

Consequently by (11) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the nonlocal problem (1) – (2).

4. Example

Let us consider the following fractional nonlocal problem,

$$^{c}D^{\alpha}y(t) = \frac{1}{(e^t+5)(1+|y(t)|+|^{c}D^{\alpha}y(t)|)},\ t \in J := [0,1],\ \alpha \in (0,1], \tag{12}$$

$$\sum_{k=1}^{m} a_k y(t_k) = 1, \tag{13}$$

where $a_k \in \mathbb{R},\ 0 < t_1 < t_2 < ... < 1$.
Set

$$f(t,y,z) = \frac{1}{(e^t+5)(1+y+z)},\ (t,y,z) \in J \times [0,+\infty) \times [0,+\infty).$$

Let $y, z \in [0,+\infty)$ and $t \in J$. Then we have

$$\begin{aligned}|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \left|\frac{1}{e^t+5}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\ &\leq \frac{|y_1-y_2|+|z_1-z_2|}{(e^t+5)(1+y_1+z_1)(1+y_2+z_2)} \\ &\leq \frac{1}{(e^t+5)}(|y_1-y_2|+|z_1-z_2|) \\ &\leq \frac{1}{6}|y_1-y_2| + \frac{1}{6}|z_1-z_2|.\end{aligned}$$

Hence condition **(H3)** holds with $k_1 = k_2 = \frac{1}{6}$. We shall check that condition (11) is satisfied. Indeed

$$\frac{2k_1}{\Gamma(\alpha+1)} + k_2 = \frac{1}{3\Gamma(\alpha+1)} + \frac{1}{6} < 1. \tag{14}$$

Then by Theorem 3.2, the nonlocal problem (12) – (13) has a unique integrable solution on $[0,1]$.

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