Filomat 30:7 (2016), 1757–1765
DOI 10.2298/FIL1607757K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Identities and Recurrence Relations on the Two Variables Bernoulli, Euler and Genocchi Polynomials

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Abstract. Mahmudov in ([16], [17], [18]) introduced and investigated some q-extensions of the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α and the q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ of order α. In this article, we give some identities for the q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and the recurrence relation between these polynomials. We give a different form of the analogue of the Srivastava-Pintér addition theorem.

1. Introduction, Definitions and Notations

In the usual notations, let $B_n(x)$, $E_n(x)$ and $G_n(x)$ denote, respectively, the classical Bernoulli, Euler and Genocchi polynomials of degree n in x, defined by the generating functions;

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt},\ |t|<2\pi,$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt},\ |t|<\pi,$$

and

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt},\ |t|<\pi.$$

Also let

$$B_n := B_n(0),\ E_n := E_n(0)\ \text{ and }\ G_n := G_n(0)$$

2010 *Mathematics Subject Classification*. Primary 05A10; Secondary 11B73, 11B68.
Keywords. Bernoulli numbers and polynomials, Euler polynomials and numbers, Genocchi polynomials and numbers, the Stirling numbers of second kind, q-exponential functions.
Received: 13 September 2014; Accepted: 12 December 2014
Communicated by Hari M. Srivastava
Research supported by the Scientific Research Project Administration of Akdeniz University.The authors are grateful for the valuable comments and suggestions of referees.
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where B_n, E_n and G_n are, respectively, the Bernoulli, the Euler and the Genocchi numbers of order n.

Carlitz was the first to extended the classical Bernoulli polynomials and numbers, Euler polynomials and numbers. Carlitz gave some recurrence relations between q-Bernoulli polynomials and q-Euler polynomials ([3], [4]). Choi et. al. [8] defined q-Eta polynomial and proved some relations between q-Eta functions and q-Eta numbers and q-Stirling numbers of the second kind. Choi et. al. [7] defined and investigated the Apostol-Bernoulli polynomials $B_k^{(n)}(x,\lambda;q)$ of order $n \in \mathbb{N}$ and Apostol-Euler polynomials $E_k^{(n)}(x,\lambda;q)$ of order $n \in \mathbb{N}$. He proved some relations the Apostol-Bernoulli $B_k^{(n)}(x,\lambda;q)$ of order n and the multiple Hurwitz-Lerch zeta function $\Phi_n(z,s,a)$. Luo [14], Luo et. al. ([15], [16]) defined the q-Bernoulli polynomials $B_k^{(\alpha)}(x,\lambda;q)$ of order α in q^x and the q-Euler polynomials $E_k^{(\alpha)}(x,\lambda;q)$ of order α in q^x. They proved some explicit relationships between the q-Bernoulli and q-Euler polynomials. Srivastava et. al. [23, In Chapter 9] gave basic knowledge for q-Analysis. Cenkci *et. al.* [5] defined and investigated (i,q)-Bernoulli numbers and (i,q)-Euler numbers.

Firstly, Mahmudov ([17], [18]) and Mahmudov et. al. [19] defined and studied the properties of the generalized q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α, q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ of order α. Mahmudov extended the Addition theorems of Srivastava-Pintér for two variables q-Euler polynomials $\mathcal{E}_{n,q}(x,y)$. Kim *et. al.* [11] gave two identities and two recurrence relations for q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ and q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$. Kurt [13] gave new identities and relations between the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ and q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$.

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers.

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, q \neq 1,$$

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q ,$$

respectively, where $[0]_q! = 1$ and $n \in \mathbb{N}$, $a \in \mathbb{C}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}.$$

The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k.$$

The q-binomial formula is known as

$$(n:q) = (1-a)_q^n = \prod_{j=0}^{n-1} \left(1-q^j a\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} (-1)^k a^k.$$

The q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)\,q^k z)}, 0 < \left|q\right| < 1, \ |z| < \frac{1}{\left|1-q\right|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1+(1-q)\,q^k z\right), 0 < \left|q\right| < 1, z \in \mathbb{C}.$$

From these forms, we easily see that $e_q(z)E_q(-z) = 1$. Moreover $D_q e_q(z) = e_q(z)$, $D_q E_q(z) = E_q(qz)$ where D_q is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, 0 < \left|q\right| < 1, 0 \neq z \in \mathbb{C}.$$

The above q-standard notation can be found in ([1], [7], [8], [11]-[18], [23]). Mahmudov defined and studied properties of the following generalized q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x, y)$ of order α, q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x, y)$ of order α and q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x, y)$ of order α as follows ([18], [19]).

Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$ and $0 < \left|q\right| < 1$. The q-Bernoulli numbers $\mathcal{B}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x, y)$ in x, y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha}, \ |t| < 2\pi, \tag{1}$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha} e_q(tx)E_q(ty), \ |t| < 2\pi. \tag{2}$$

The q-Euler numbers $\mathcal{E}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x, y)$ in x, y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha}, \ |t| < \pi, \tag{3}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha} e_q(tx)E_q(ty), \ |t| < \pi. \tag{4}$$

The q-Genocchi numbers $\mathcal{G}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x, y)$ in x, y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha}, \ |t| < \pi, \tag{5}$$

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha} e_q(tx)E_q(ty), \ |t| < \pi. \tag{6}$$

The familiar q-Stirling numbers $\mathcal{S}_{2,q}(n, k)$ of the second kind are defined by

$$\frac{\left(e_q(t) - 1\right)^k}{[k]_q!} = \sum_{n=k}^{\infty} \mathcal{S}_{2,q}(n, k) \frac{t^n}{[n]_q!}. \tag{7}$$

It is obvious that

$$\mathcal{B}_{n,q}^{(1)}(x, y) = \mathcal{B}_{n,q}(x, y), \mathcal{E}_{n,q}^{(1)}(x, y) = \mathcal{E}_{n,q}(x, y), \mathcal{G}_{n,q}^{(1)}(x, y) = \mathcal{G}_{n,q}(x, y),$$

$$\lim_{q \to 1^-} \mathcal{B}_{n,q}^{(\alpha)}(x, y) = B_n^{(\alpha)}(x + y), \lim_{q \to 1^-} \mathcal{E}_{n,q}^{(\alpha)}(x, y) = E_n^{(\alpha)}(x + y), \lim_{q \to 1^-} \mathcal{G}_{n,q}^{(\alpha)}(x, y) = G_n^{(\alpha)}(x + y).$$

From (2), (4) and (6), it is easy to check that

$$\mathcal{B}_{n,q}^{(\alpha)}(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{B}_{n-k,q}(x, 0) \mathcal{B}_{k,q}^{(\alpha-1)}(0, y),$$

$$\mathcal{E}_{n,q}^{(\alpha)}(x,y)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathcal{E}_{n-k,q}(x,0)\mathcal{E}_{k,q}^{(\alpha-1)}(0,y),$$

$$\mathcal{G}_{n,q}^{(\alpha)}(x,y)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathcal{G}_{n-k,q}(x,0)\mathcal{G}_{k,q}^{(\alpha-1)}(0,y),$$

$$\mathcal{B}_{n,q}^{(\alpha-m)}(x,y)=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathcal{B}_{n-k,q}^{(\alpha)}\mathcal{B}_{k,q}^{(-m)}(x,y).$$

In this work, we give some identities for the q-Bernoulli polynomials. Also, we give some relations between the q-Stirling numbers $\mathcal{S}_{2,q}(n,k)$ of the second kind, q-Bernoulli polynomials and q-Euler polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.

2. Main Theorems

Theorem 2.1. *There is the following relation between for the Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Stirling numbers $\mathcal{S}_{2,q}(n,k)$ of the second kind*

$$\mathcal{B}_{n,q}^{(\alpha-k)}(x,y)=\frac{[n]_q!\,[m]_q!}{[m+n]_q!}\sum_{k=0}^{n+m}\left[\begin{array}{c} n+m \\ k \end{array}\right]_q \mathcal{S}_{2,q}(k,m)\mathcal{B}_{n+m-k,q}^{(\alpha)}(x,y). \tag{8}$$

Proof. The equation (7) may be rearranged as

$$\left(\frac{e_q(t)-1}{t}\right)^k\frac{1}{[k]_q!}=\sum_{n=0}^{\infty}\mathcal{S}_{2,q}(n+k,k)\frac{t^n}{[n+k]_q!}. \tag{9}$$

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha-k)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{t}{e_q(t)-1}\right)^{(-k)}\left(\frac{t}{e_q(t)-1}\right)^{(\alpha)}e_q(tx)E_q(ty).$$

If we carry out the necessary operations,

$$=\frac{1}{t^k}\,[k]_q!\sum_{n=0}^{\infty}\mathcal{S}_{2,q}(n,m)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}\left\{\frac{[m]_q!\,[n-m]_q!}{[n]_q!}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathcal{S}_{2,q}(k,m)\mathcal{B}_{n-k,q}^{(\alpha)}(x,y)\right\}\frac{t^{n-m}}{[n-m]_q!}.$$

If we make mathematical operation, comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (8). □

Theorem 2.2. *The following relation is true:*

$$\sum_{m=0}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q \mathcal{S}_{2,q}(m,k)(x+y)_q^{(n-m)}=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}\mathcal{B}_{n-k,q}^{(-k)}(x,y). \tag{10}$$

Proof. From (7), we write as

$$\sum_{n=k}^{\infty}\mathcal{S}_{2,q}(n,k)\frac{t^n}{[n]_q!}=\left(\frac{e_q(t)-1}{t}\right)^{(k)}e_q(tx)E_q(ty)\frac{t^k}{[k]_q!}\frac{1}{e_q(tx)E_q(ty)},$$

$$\sum_{n=0}^{\infty} \mathcal{S}_{2,q}(n,k)\frac{t^n}{[n]_q!}e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(-k)}(x,y)\frac{t^n}{[n]_q!}\frac{t^k}{[k]_q!},$$

$$\sum_{m=0}^{\infty} \mathcal{S}_{2,q}(m,k)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty}(x+y)_q^n\frac{t^n}{[n]_q!} = \frac{1}{[k]_q!}\sum_{n=0}^{\infty}\frac{[n+k]_q!}{[n]_q!}\mathcal{B}_{n,q}^{(-k)}(x,y)\frac{t^{n+k}}{[n+k]_q!}.$$

Using the Cauchy product, comparing the coefficients of $\frac{t^n}{[n]_q!}$, we obtain the result. □

Theorem 2.3. *There is the following relation between the q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ and the Stirling numbers $\mathcal{S}_{2,q}(n,k)$ of the second kind:*

$$\mathcal{E}_{n,q}^{(\alpha)}(x,y) = \sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\mathcal{S}_{2,q}(m,j)\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q (x+y)_q^{n-m} \tag{11}$$

where $q \in \mathbb{C}$, α, j, $n \in \mathbb{N}$ and $0 < |q| < 1$.

Proof. We write the following identity,

$$\begin{aligned}\left(\frac{2}{e_q(t)+1}\right)^{(\alpha)} &= \left(1+\frac{e_q(t)-1}{2}\right)^{(-\alpha)} \\ &= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)}.\end{aligned}$$

From this last identity and two variable q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$, we can write as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \left(\frac{2}{e_q(t)+1}\right)^{(\alpha)} e_q(tx)E_q(ty) \\ &= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\frac{\left(e_q(t)-1\right)^j}{[j]_q!}e_q(tx)E_q(ty),\end{aligned}$$

$$= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\sum_{m=k}^{\infty}\mathcal{S}_{2,q}(m,k)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty}(x+y)_q^n\frac{t^n}{[n]_q!}.$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{E}_{n+j,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} \\ =& \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\mathcal{S}_{2,q}(m,j)(x+y)_q^{n-m}\right\}\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we have the result. □

3. Explicit Relations Between the q-Bernoulli Polynomials, q-Euler Polynomials and q-Genocchi Polynomials

In this section, we prove the interesting relations on the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α, q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ of order α.

Theorem 3.1. *The following relation is true:*

$$\mathcal{G}_{n,q}^{(\alpha)} = \frac{m^{k-n}}{2\,[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \mathcal{G}_{n+1-k,q}m^k\left\{\mathcal{G}_{k,q}^{(\alpha)}\left(\frac{1}{m},0\right)+\mathcal{G}_{k,q}^{(\alpha)}\right\}. \tag{12}$$

Proof. From (5);

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t)+1}\right)^{\alpha}\frac{e_q(\frac{t}{m})+1}{\frac{2t}{m}}\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1},$$

$$= \frac{m}{2t}\left\{\left(\frac{2t}{e_q(t)+1}\right)^{\alpha}e_q(\frac{t}{m})\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}+\left(\frac{2t}{e_q(t)+1}\right)^{\alpha}\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}\right\},$$

$$= \frac{m}{2t}\left\{\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}\left(\frac{1}{m},0\right)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}\right\}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}\frac{t^n}{m^n\,[n]_q!},$$

$$= \frac{m}{2t}\sum_{n=-1}^{\infty}\left\{\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \mathcal{G}_{n+1-k,q}m^{k-n-1}\left(\mathcal{G}_{k,q}^{(\alpha)}(\frac{1}{m},0)+\mathcal{G}_{k,q}^{(\alpha)}\right)\frac{t^n}{[n]_q!}\right\}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we have the result (12). □

Theorem 3.2. *The q-Bernoulli polynomials $\mathcal{B}_{n,q}(x,y)$ satisfy the following relation*

$$\begin{aligned}&\mathcal{B}_{n,q}^{(\alpha)}(x,y)\\ =\;& \frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{B}_{j,q}^{(\alpha)}(x,0)m^{j-k}-\mathcal{B}_{k,q}^{(\alpha)}(x,0)\right)\mathcal{B}_{n+1-k,q}(0,my)m^{k-n}.\end{aligned} \tag{13}$$

Proof. From (2)

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{(\alpha)}e_q(tx)E_q(ty),$$

$$= \left(\frac{t}{e_q(t)-1}\right)^{(\alpha)}e_q(tx)\frac{e_q(\frac{t}{m})-1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q(\frac{t}{m})-1}E_q(\frac{t}{m}my),$$

$$= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n\,[n]_q!}-\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\right\}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(0,my)\frac{t^n}{m^n\,[n]_q!},$$

$$\begin{aligned}=\;& m\sum_{n=-1}^{\infty}\left\{\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q\left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{B}_{j,q}^{(\alpha)}(x,0)m^{j-k}-\mathcal{B}_{k,q}^{(\alpha)}(x,0)\right)\right.\\ &\left.\times\,\mathcal{B}_{n+1-k,q}(0,my)m^{k-n-1}\right\}\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (13). □

Theorem 3.3. *The q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ satisfy the following relation*

$$\begin{aligned}&\mathcal{G}_{n,q}^{(\alpha)}(x,y)\\ =\ &\frac{1}{2}\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\left(\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\mathcal{G}_{j,q}^{(\alpha)}(0,y)m^{j-k}+\mathcal{G}_{k,q}^{(\alpha)}(0,y)\right)\\ &\times\mathcal{G}_{n+1-k,q}(mx,0)m^{k-n}.\end{aligned} \tag{14}$$

Proof. From (6)

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{2t}{e_q(t)+1}\right)^{(\alpha)}e_q(tx)E_q(ty),$$

$$=\left(\frac{2t}{e_q(t)+1}\right)^{(\alpha)}E_q(ty)\frac{e_q(\frac{t}{m})+1}{\frac{2t}{m}}\frac{\frac{2t}{m}}{e_q(\frac{t}{m})+1}e_q(\frac{t}{m}mx),$$

$$=\frac{m}{2t}\left\{\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n\,[n]_q!}+\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\right\}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(mx,0)\frac{t^n}{m^n\,[n]_q!},$$

$$\begin{aligned}=\ &\frac{m}{2}\sum_{n=-1}^{\infty}\left\{\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\right.\\ &\left.\times\left(\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\mathcal{G}_{j,q}^{(\alpha)}(0,y)m^{j-k}+\mathcal{G}_{k,q}^{(\alpha)}(0,y)\right)\mathcal{G}_{n+1-k,q}(mx,0)m^{k-n-1}\right\}\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, gives the result. □

Theorem 3.4. *There are the following relations between q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Berboulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α:*

$$\mathcal{E}_{n,q}^{(\alpha)}(x,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\left(\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\mathcal{E}_{j,q}^{(\alpha)}(x,0)-\mathcal{E}_{k,q}^{(\alpha)}(x,0)\right)\mathcal{B}_{n+1-k,q}(0,y), \tag{15}$$

$$\mathcal{E}_{n,q}^{(\alpha)}(x,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\left(\mathcal{E}_{k,q}^{(\alpha)}(1,y)-\mathcal{E}_{k,q}^{(\alpha)}(0,y)\right)\mathcal{B}_{n+1-k,q}(x,y), \tag{16}$$

$$\mathcal{E}_{n,q}^{(\alpha)}(x,y)=\frac{1}{[n+1]_q}\sum_{m=0}^{n+1}\left[\begin{array}{c}n+1\\m\end{array}\right]_q\left(\sum_{k=0}^{m}\left[\begin{array}{c}m\\k\end{array}\right]_q\mathcal{E}_{k,q}^{(\alpha)}(1+y)_q^{m-k}-\mathcal{E}_{m,q}^{(\alpha)}(y,0)\right)\mathcal{B}_{n+1-k,q}(x,0). \tag{17}$$

Proof. Proof of (16): From (4),

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{2}{e_q(t)+1}\right)^{(\alpha)}e_q(tx)E_q(ty),$$

$$=\left(\frac{2}{e_q(t)+1}\right)^{(\alpha)}E_q(ty)\frac{e_q(t)-1}{t}\frac{t}{e_q(t)-1}e_q(tx),$$

$$= \frac{1}{t}\left\{\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}(1,y)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\right\}\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(x,0)\frac{t^n}{[n]_q!},$$

$$= \sum_{n=0}^{\infty}\left\{\frac{1}{[n]_q}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\left(\mathcal{E}_{k,q}^{(\alpha)}(1,y)-\mathcal{E}_{k,q}^{(\alpha)}(0,y)\right)\mathcal{B}_{n-k,q}(x,0)\right\}\frac{t^{n-1}}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we have (16).

The proof of equation (15) and (17) are similar. We omit it. □

Theorem 3.5. *There are the following relations between q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ of order α:*

$$\begin{aligned}&\mathcal{B}_{n,q}^{(\alpha)}(x,y)\\ =\;&\frac{1}{2\,[n+1]_q}\sum_{j=0}^{n+1}\left[\begin{array}{c} n+1 \\ j \end{array}\right]_q\left(\sum_{k=0}^{j}\left[\begin{array}{c} j \\ k \end{array}\right]_q\mathcal{B}_{k,q}^{(\alpha)}\times(\frac{1}{m}+y)_q^{j-k}+\mathcal{B}_{j,q}^{(\alpha)}(0,y)\right)\mathcal{G}_{n+1-k,q}(mx,0)m^{j-n}.\end{aligned} \tag{18}$$

Proof. Proof of (18), from (2):

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\left(\frac{t}{e_q(t)-1}\right)^{(\alpha)}e_q(tx)E_q(ty),$$

$$=\left(\frac{t}{e_q(t)-1}\right)^{(\alpha)}E_q(ty)\frac{e_q(\frac{t}{m})+1}{\frac{t}{m}}\frac{\frac{t}{m}}{e_q(\frac{t}{m})+1}e_q(\frac{t}{m}mx),$$

$$=\frac{m}{2t}\left\{\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\left(\frac{1}{m}+y\right)_q^n\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\right\}\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(mx,0)\frac{t^n}{m^n\,[n]_q!},$$

$$=\frac{m}{2t}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q\left\{\sum_{k=0}^{j}\left[\begin{array}{c} j \\ k \end{array}\right]_q\mathcal{B}_{k,q}^{(\alpha)}\times\left(\frac{1}{m}+y\right)_q^{j-k}+\mathcal{B}_{j,q}^{(\alpha)}(0,y)\right\}\mathcal{G}_{n-j,q}(mx,0)m^{j-n}\frac{t^n}{[n]_q!},$$

$$=\frac{1}{2}\sum_{n=-1}^{\infty}\left\{\sum_{j=0}^{n+1}\left[\begin{array}{c} n+1 \\ j \end{array}\right]_q\left(\sum_{k=0}^{j}\left[\begin{array}{c} j \\ k \end{array}\right]_q\mathcal{B}_{k,q}^{(\alpha)}\times\left(\frac{1}{m}+y\right)_q^{j-k}+\mathcal{B}_{j,q}^{(\alpha)}(0,y)\right)\mathcal{G}_{n+1-j,q}(mx,0)m^{j-n}\right\}\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we have (18). □

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