Filomat 30:7 (2016), 1779–1787
DOI 10.2298/FIL1607779C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Incomplete Srivastava's Triple Hypergeometric Functions γ_B^H and Γ_B^H

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Abstract. Recently Srivastava *et al.* [26] introduced the incomplete Pochhammer symbols by means of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, and defined incomplete hypergeometric functions whose a number of interesting and fundamental properties and characteristics have been investigated. Further, Çetinkaya [6] introduced the incomplete second Appell hypergeometric functions and studied many interesting and fundamental properties and characteristics. In this paper, motivated by the above-mentioned works, we introduce two incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H by using the incomplete Pochhammer symbols and investigate certain properties, for example, their various integral representations, derivative formula, reduction formula and recurrence relation. Various (known or new) special cases and consequences of the results presented here are also considered.

1. Introduction, Definitions and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$ denote the sets of positive integers, negative integers, complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$.

The familiar *incomplete Gamma functions* $\gamma(s,x)$ and $\Gamma(s,x)$ defined by

$$\gamma(s,x) := \int_0^x t^{s-1} e^{-t} dt \qquad (\Re(s) > 0;\ x \geqq 0) \tag{1}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1} e^{-t} dt \qquad (x \geqq 0;\ \Re(s) > 0 \text{ when } x = 0) \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 33B15, 33B20, 33C05, 33C15, 33C20; Secondary 33B99, 33C99, 60B99

Keywords. Gamma functions; Incomplete gamma function; Pochhammer symbol; Incomplete Pochhammer symbol; Incomplete generalized hypergeometric functions; Appell function; Incomplete second Appell function; Srivastava's triple hypergeometric Functions; Whittaker function; Bessel and modified Bessel functions; Incomplete Srivastava's triple hypergeometric Functions

Received: 24 September 2014; Accepted: 27 November 2014

Communicated by Prof. H.M. Srivastava

The first-named author was supported, in part, by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (2010-0011005). This work was also supported by Dongguk University Research Fund. The second-named author was supported, in part, by TEQIP-II.

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respectively, satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) := \Gamma(s) \qquad (\Re(s) > 0). \tag{3}$$

These functions play an important role in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, *e.g.*, [1, 2, 4, 7, 8, 10, 11, 14, 15, 27–29, 31, 37, 38, 40]).

Recently Srivastava *et al.* [26] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$${}_p\gamma_q\left[\begin{array}{r}(\alpha_1,x),\alpha_2,\cdots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1;x)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \tag{4}$$

and

$${}_p\Gamma_q\left[\begin{array}{r}(\alpha_1,x),\alpha_2,\cdots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{[\alpha_1;x]_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{5}$$

where, in terms of the incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by (1) and (2), the *incomplete* Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ $(\lambda,\ \nu \in \mathbb{C};\ x \geqq 0)$ are defined as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\ \nu\in\mathbb{C};\ x\geqq 0) \tag{6}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\ \nu\in\mathbb{C};\ x\geqq 0) \tag{7}$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $(\lambda;x)_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu + [\lambda;x]_\nu := (\lambda)_\nu \qquad (\lambda,\ \nu\in\mathbb{C};\ x\geqq 0), \tag{8}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [28, p. 2 and p. 5]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N})\end{cases}\\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\right)\end{aligned} \tag{9}$$

and $\Gamma(\lambda)$ is the familiar Gamma function.

Remark 1. The argument $x \geqslant 0$ in the definitions (1) and (2), (4) and (5), (6) and (7), and elsewhere in this paper, is *independent* of the argument $z \in \mathbb{C}$ which occurs in the definitions (4), (5) and (10), and also in the results presented here.

As already pointed out by Srivastava *et al.* [26, Remark 7], since

$$|(\lambda;x)_n| \leqq |(\lambda)_n| \quad \text{and} \quad |[\lambda;x]_n| \leqq |(\lambda)_n| \quad (\lambda\in\mathbb{C};\ n\in\mathbb{N}_0;\ x\geqq 0)$$

the precise *sufficient* conditions under which the infinite series in definitions (4) and (5) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function ${}_pF_q$ $(p,\ q \in \mathbb{N}_0)$ (see [17, p. 72-73] and [30, p. 20]; see also [3, 5, 12] and [19]). Indeed, in their special case when $x = 0$, both ${}_p\gamma_q$ $(p,\ q \in \mathbb{N}_0)$ and ${}_p\Gamma_q$ $(p,\ q \in \mathbb{N}_0)$ would reduce immediately to the extensively-investigated generalized hypergeometric function ${}_pF_q$ $(p,\ q \in \mathbb{N}_0)$.

Furthermore, it is easy to see from the definitions (4) and (5), we have the following decomposition formula:

$$
{}_p\gamma_q\left[\begin{array}{r}(\alpha_1,x),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]+{}_p\Gamma_q\left[\begin{array}{r}(\alpha_1,x),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{array} z\right]={}_pF_q\left[\begin{array}{r}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right] \tag{10}
$$

in terms of the familiar generalized hypergeometric function ${}_pF_q$.

More recently, Çetinkaya [6] introduced and studied various properties of the following two families of the incomplete second Appell hypergeometric functions γ_2 and Γ_2:

$$
\gamma_2[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2]=\sum_{m,p=0}^{\infty}\frac{(\alpha;x)_{m+p}(\beta_1)_m(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_p}\frac{x_1^m}{m!}\frac{x_2^p}{p!} \tag{11}
$$

and

$$
\Gamma_2[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2]=\sum_{m,p=0}^{\infty}\frac{[\alpha;x]_{m+p}(\beta_1)_m(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_p}\frac{x_1^m}{m!}\frac{x_2^p}{p!}. \tag{12}
$$

Motivated essentially by the demonstrated potential for applications of these families of incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, and the incomplete second Appell hypergeometric functions γ_2 and Γ_2 in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [6, 26] and the references cited therein), here, we aim at investigating, in a rather systematic manner, two families of incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H to present various representations and formulas, for example, integral representations, derivative formula and certain integral representations involving Whittaker function, Bessel and modified Bessel functions. A reduction formula and a recurrence relation of the incomplete Srivastava's triple hypergeometric functions are also considered. For various other investigations involving generalizations of the hypergeometric function ${}_pF_q$ of p numerator and q denominator parameters, which were motivated essentially by the pioneering work of Srivastava *et al.* [26], the interested reader may be referred to several recent papers on the subject (see, for example, [13, 25, 32–36] *and* the references cited in each of these papers).

2. The Incomplete Srivastava's Triple Hypergeometric Functions

In terms of the incomplete Pochhammer symbol $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ defined by (6) and (7), we introduce two families of incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H as follows: For $\alpha,\ \beta_1,\ldots,\ \beta_n\in\mathbb{C}$ and $\gamma_1,\ldots,\gamma_n\in\mathbb{C}\setminus\mathbb{Z}_0^-$,

$$
\gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]=\sum_{m,n,p=0}^{\infty}\frac{(\alpha;x)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!} \tag{13}
$$

$$
\left(x\geqq 0;\ |x_1|<r,\ |x_2|<s,\ |x_3|<t,\ r+s+t+2\sqrt{rst}=1\ \text{ when }\ x=0\right)
$$

and

$$
\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]=\sum_{m,n,p=0}^{\infty}\frac{[\alpha;x]_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!} \tag{14}
$$

$$
\left(x\geqq 0;\ |x_1|<r,\ |x_2|<s,\ |x_3|<t,\ r+s+t+2\sqrt{rst}=1\ \text{ when }\ x=0\right).
$$

In view of (8), these incomplete families of Srivastava's triple hypergeometric functions satisfy the following decomposition formula:

$$
\begin{aligned}
&\gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]+\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\
&\qquad=H_B[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3],
\end{aligned} \tag{15}
$$

where H_B is the familiar Srivastava's triple hypergeometric functions (see, for details, [20–24, 29, 30]).

Theorem 1. *The incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H satisfy the following system of partial differential equations*:

$$\begin{cases} \Big[\theta(\theta+\gamma_1-1)-x_1(\theta+\psi+\alpha)(\theta+\phi+\beta_1)\Big]u=0 \\ \Big[\phi(\phi+\gamma_2-1)-x_2(\theta+\phi+\beta_1)(\phi+\psi+\beta_2)\Big]u=0 \\ \Big[\psi(\psi+\gamma_3-1)-x_3(\theta+\psi+\alpha)(\phi+\psi+\beta_2)\Big]u=0, \end{cases} \tag{16}$$

where $\theta = x_1\frac{\partial}{\partial x_1}$, $\phi = x_2\frac{\partial}{\partial x_2}$ and $\psi = x_3\frac{\partial}{\partial x_3}$ *and*

$$u=u(x_1,x_2,x_3):=\gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]+\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3].$$

Proof. In light of (15), it is easy to prove (16) since H_B satisfies the same system of partial differential equations as in [20]. □

Remark 2. It is interesting to note that the special cases of (13) and (14) when $x_2 = 0$ reduce to the known incomplete families of the second Appell hypergeometric functions (11) and (12). Also, the special cases of (13) and (14) when $x_2 = 0$ and $x_3 = 0$ or $x_1 = 0$ are seen to yield the known incomplete families of Gauss hypergeometric functions [26].

In view of the formula (15), it is sufficient to discuss the properties and characteristics of the incomplete Srivastava's triple hypergeometric functions Γ_B^H.

3. Integral Representations of Γ_B^H

In this section, we present certain integral representations of the incomplete Srivastava's triple hypergeometric functions by applying (2) and (7). We also obtain various integral representations involving Whittaker function, Bessel and modified Bessel functions.

Theorem 2. *The following integral representation for Γ_B^H in (14) holds true*:

$$\begin{aligned}&\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\ &\quad=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)}\int_x^\infty\int_0^\infty e^{-s-t}t^{\alpha-1}s^{\beta_1-1}\;{}_0F_1(-;\gamma_1;x_1st)\;\Psi_2(\beta_2;\gamma_2,\gamma_3;x_2s,x_3t)dtds\end{aligned} \tag{17}$$

$$\Big(x \geqq 0;\ \Re(x_2)<1, \Re(x_3)<1,\ \Re(\alpha)>0, \Re(\beta_1)>0 \text{ when } x=0\Big),$$

where Ψ_2 is one of the confluent forms of Appell series in two variables defined by (*see, e.g.*, [29, p. 26, Eq. (22)])

$$\Psi_2\,[a;c_1,c_2;\,x_1,x_2]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c_1)_m(c_2)_n}\,\frac{{x_1}^m}{m!}\frac{{x_2}^n}{n!}. \tag{18}$$

Proof. Considering the integral representations of the incomplete Pochhammer symbol $[\alpha;x]_{m+p}$ in (2) and (7), the classical Pochhammer symbol $(\beta_1)_{m+n}$ and using the definition (18) in (14), we are led to the desired result. □

Theorem 3. *The following triple integral representation for Γ_B^H in (14) holds true*:

$$\begin{aligned}\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3]&=\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)}\int_x^\infty\int_0^\infty\int_0^\infty e^{-s-t-u}t^{\alpha-1}s^{\beta_1-1}u^{\beta_2-1}\\ &\quad\times\;{}_0F_1(-;\gamma_1;x_1st)\;{}_0F_1(-;\gamma_2;x_2us)\;{}_0F_1(-;\gamma_3;x_3ut)dtdsdu\end{aligned} \tag{19}$$

$$\Big(x \geqq 0;\ \min\{\Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0 \ when \ x = 0\Big).$$

Proof. Considering the integral representation of the incomplete Pochhammer symbol $[\alpha; x]_{m+p}$ in (2) and (7),and the classical Pochhammer symbol $(\beta_1)_{m+n}$ and $(\beta_2)_{n+p}$ in (14), we are led to the desired result. □

Theorem 4. *The following Euler integral representation for* Γ_B^H in (14) *holds true*:

$$\begin{aligned}\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] &= \frac{1}{B(\beta_1,\gamma_1-\beta_1)B(\beta_2,\gamma_3-\beta_2)B(1-\gamma_1+\beta_1,\gamma_1+\gamma_2-\beta_1-1)}\\ &\times \int_0^1\int_0^1\int_0^1 u^{\beta_1-1}v^{\beta_1-\gamma_1}w^{\beta_2-1}(1-u)^{\gamma_1-\gamma_3-\beta_1+\beta_2}(1-v)^{\gamma_1+\gamma_2-\beta_1-2}(1-ux_1-wx_3)^{-\alpha}\\ &\times \frac{\Gamma(\alpha,x(1-ux_1-wx_3))}{\Gamma(\alpha)}(1-u-w+uw-uvwx_2)^{\gamma_3-\beta_2-1}du\,dv\,dw\end{aligned} \tag{20}$$

$$\Big(x \geqq 0;\ 0<\Re(\gamma_1-\beta_1)<1, \Re(\beta_1)>1,\ \Re(\gamma_1+\gamma_2-\beta_1)>1, \Re(\gamma_3)>\Re(\beta_2)>0 \ when \ x=0\Big).$$

Proof. From the definition of incomplete Srivastava's triple hypergeometric functions Γ_B^H in (14), we have

$$\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2] = \sum_{n=0}^{\infty}\frac{(\beta_1)_n(\beta_2)_n}{(\gamma_2)_n}\,\Gamma_2[(\alpha,x),\beta_1+n,\beta_2+n;\gamma_1,\gamma_3;x_1,x_3]\,\frac{x_2^n}{n!}. \tag{21}$$

Employing the integral representation of incomplete second Appell hypergeometric functions Γ_2 [6, p. 8335, Eq. (24)]:

$$\begin{aligned}\Gamma_2[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] &= \frac{1}{B(\beta_1,\gamma_1-\beta_1)B(\beta_2,\gamma_2-\beta_2)}\\ &\times\int_0^1\int_0^1 t^{\beta_1-1}s^{\beta_2-1}(1-t)^{\gamma_1-\beta_1-1}(1-s)^{\gamma_2-\beta_2-1}(1-x_1t-x_2s)^{-\alpha}\,\frac{\Gamma(\alpha,x(1-x_1t-x_2s))}{\Gamma(\alpha)}\,dt\,ds\end{aligned} \tag{22}$$

$$\Big(x \geqq 0; \Re(\gamma_j) > \Re(\beta_j) > 0 \ \ (j=1,2) \text{ when } x=0\Big).$$

in (21), and, by uniform convergence, changing the order of summation and integration, we obtain

$$\begin{aligned}\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] &= \frac{1}{B(\beta_1,\gamma_1-\beta_1)B(\beta_2,\gamma_3-\beta_2)}\\ &\times\int_0^1\int_0^1 u^{\beta_1-1}w^{\beta_2-1}(1-u)^{\gamma_1-\beta_1-1}(1-ux_1-wx_3)^{-\alpha}\frac{\Gamma(\alpha,x(1-ux_1-wx_3))}{\Gamma(\alpha)}\\ &\times{}_2F_1\left[\begin{matrix}1-\gamma_1+\beta_1,\ 1-\gamma_3+\beta_2; & \dfrac{uwx_2}{(1-u)(1-w)}\\ \gamma_2; & \end{matrix}\right]du\,dw.\end{aligned}$$

If, now, we use the following integral representation of (see [17] and [28, Chapter 1]):

$${}_2F_1\,(a,\,b;\,c;\,z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 t^{a-1}\,(1-t)^{c-a-1}\,(1-zt)^{-b}\,dt \tag{23}$$

$$(\Re(c) > \Re(a) > 0;\ |\arg(1-z)| \leq \pi-\epsilon\ (0<\epsilon<\pi)).$$

in (**??**), we are led to the desired integral representation (20) asserted by Theorem 4. □

Remark 3. The Whittaker function of two variables $M_{k,m,n}(x,y)$ (see [24, p. 100]), Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ (see, *e.g.*, [17]; see also [5, 12, 14, 15, 38, 39]) are expressible in terms of the confluent function Ψ_2 and hypergeometric functions ${}_0F_1$, respectively, as follows:

$$M_{k,m,n}(x,y) = x^{m+\frac{1}{2}}\, y^{n+\frac{1}{2}}\, e^{-\frac{1}{2}(x+y)}\, \Psi_2(m+n-k+1; 2m+1, 2n+1; x, y), \tag{24}$$

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\Big(-;\nu+1;-\frac{1}{4}z^2\Big) \quad (\nu \in \mathbb{C}\setminus\mathbb{Z}^-) \tag{25}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\Big(-;\nu+1;\frac{1}{4}z^2\Big) \quad (\nu \in \mathbb{C}\setminus\mathbb{Z}^-). \tag{26}$$

Now, applying the relationships (24) to (17), (25) and (26) to (17), and (25) and (26) to (19), respectively, we can deduce certain interesting integral representations for the incomplete Srivastava's triple hypergeometric function in (14) asserted by Corollaries 1, 2, 3 and 4 below. Each of their proofs will be omitted.

Corollary 1. *The following integral representation for* Γ_B^H in (14) *holds true*:

$$\begin{aligned}&\Gamma_B^H[(\lambda,x),\mu,\sigma+\rho-k+1;\nu,2\sigma+1,2\rho+1;x_1,x_2,x_3]\\ &\quad= \frac{x_2^{-\sigma-\frac{1}{2}}x_3^{-\rho-\frac{1}{2}}}{\Gamma(\lambda)\Gamma(\mu)}\int_x^\infty\int_0^\infty e^{-(1-\frac{1}{2}x_2)s-(1-\frac{1}{2}x_3)t}\,t^{\lambda-\rho-\frac{3}{2}}s^{\mu-\sigma-\frac{3}{2}}\,{}_0F_1(-;\nu;x_1st)\,M_{k,\sigma,\rho}(x_2s,x_3t)\,dtds,\end{aligned} \tag{27}$$

provided that the involved integral is convergent.

Corollary 2. *Each of the following double integral representations holds true*:

$$\begin{aligned}&\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1+1,\gamma_2,\gamma_3;-x_1,x_2,x_3]\\ &\quad= \frac{\Gamma(\gamma_1+1)\,x_1^{-\frac{\gamma_1}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)}\int_x^\infty\int_0^\infty e^{-s-t}t^{\alpha-\frac{\gamma_1}{2}-1}s^{\beta_1-\frac{\gamma_1}{2}-1}\,J_{\gamma_1}(2\sqrt{x_1st})\,\Psi_2(\beta_2;\gamma_2,\gamma_3;x_2s,x_3t)\,dt\,ds\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1+1,\gamma_2,\gamma_3;x_1,x_2,x_3]\\ &\quad= \frac{\Gamma(\gamma_1+1)\,x_1^{-\frac{\gamma_1}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)}\int_x^\infty\int_0^\infty e^{-s-t}t^{\alpha-\frac{\gamma_1}{2}-1}s^{\beta_1-\frac{\gamma_1}{2}-1}\,I_{\gamma_1}(2\sqrt{x_1st}))\,\Psi_2(\beta_2;\gamma_2,\gamma_3;x_2s,x_3t)\,dt\,ds,\end{aligned} \tag{29}$$

provided that the involved integrals are convergent.

Corollary 3. *The following integral representations for* Γ_B^H in (14) *hold true*:

$$\begin{aligned}\Gamma_B^H[(\lambda,x),\mu,\sigma+\rho-k+1;\nu+1,2\sigma+1,2\rho+1;-x_1,x_2,x_3] &= \frac{\Gamma(\nu+1)x_1^{-\frac{1}{2}\nu}x_2^{-\sigma-\frac{1}{2}}x_3^{-\rho-\frac{1}{2}}}{\Gamma(\lambda)\Gamma(\mu)}\\ \times\int_x^\infty\int_0^\infty e^{-(1-\frac{1}{2}x_2)s-(1-\frac{1}{2}x_3)t}\,t^{\lambda-\rho-\frac{1}{2}\nu-\frac{3}{2}}s^{\mu-\sigma-\frac{1}{2}\nu-\frac{3}{2}}&\,J_\nu(2\sqrt{x_1st})\,M_{k,\sigma,\rho}(x_2s,x_3t)\,dtds\end{aligned} \tag{30}$$

and

$$\Gamma_B^H[(\lambda,x),\mu,\sigma+\rho-k+1;\nu+1,2\sigma+1,2\rho+1;x_1,x_2,x_3] = \frac{\Gamma(\nu+1)x_1^{-\frac{1}{2}\nu}x_2^{-\sigma-\frac{1}{2}}x_3^{-\rho-\frac{1}{2}}}{\Gamma(\lambda)\Gamma(\mu)} \times \int_x^\infty \int_0^\infty e^{-(1-\frac{1}{2}x_2)s-(1-\frac{1}{2}x_3)t} t^{\lambda-\rho-\frac{1}{2}\nu-\frac{3}{2}} s^{\mu-\sigma-\frac{1}{2}\nu-\frac{3}{2}}\, I_\nu(2\sqrt{x_1 st})\, M_{k,\sigma,\rho}(x_2 s, x_3 t)\, dtds, \tag{31}$$

provided that the involved integrals are convergent.

Corollary 4. *The following triple integral representations for* Γ_B^H in (14) *hold true*:

$$\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1+1,\gamma_2+1,\gamma_3+1;-x_1,-x_2,-x_3] = \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\Gamma(\gamma_3+1)x_1^{\frac{-\gamma_1}{2}}x_2^{\frac{-\gamma_2}{2}}x_3^{\frac{-\gamma_3}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_x^\infty \int_0^\infty \int_0^\infty e^{-s-t-u} t^{\alpha-\frac{\gamma_1}{2}-\frac{\gamma_3}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-\frac{\gamma_2}{2}-1} u^{\beta_2-\frac{\gamma_2}{2}-\frac{\gamma_3}{2}-1}\, J_{\gamma_1}(2\sqrt{x_1 st})\, J_{\gamma_2}(2\sqrt{x_2 us})\, J_{\gamma_3}(2\sqrt{x_3 ut})\, dtdsdu \tag{32}$$

and

$$\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1+1,\gamma_2+1,\gamma_3+1;x_1,x_2,x_3] = \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\Gamma(\gamma_3+1)x_1^{\frac{-\gamma_1}{2}}x_2^{\frac{-\gamma_2}{2}}x_3^{\frac{-\gamma_3}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_x^\infty \int_0^\infty \int_0^\infty e^{-s-t-u} t^{\alpha-\frac{\gamma_1}{2}-\frac{\gamma_3}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-\frac{\gamma_2}{2}-1} u^{\beta_2-\frac{\gamma_2}{2}-\frac{\gamma_3}{2}-1}\, I_{\gamma_1}(2\sqrt{x_1 st})\, I_{\gamma_2}(2\sqrt{x_2 us})\, I_{\gamma_3}(2\sqrt{x_3 ut})\, dtdsdu, \tag{33}$$

provided that the involved integrals are convergent.

4. Derivative Formula

Differentiating, partially, both sides of (14) with respect to x_1, x_2 and x_3, m, n and p times, respectively, we obtain a derivative formula for the incomplete Srivastava's triple hypergeometric function Γ_B^H given in the following theorem.

Theorem 5. *The following derivative formula for* Γ_B^H *holds true*:

$$\frac{\partial^{m+n+p}}{\partial x_1^m \partial x_2^n \partial x_3^p}\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] = \frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} \times \Gamma_B^H[(\alpha+m+p,x),\beta_1+m+n,\beta_2+n+p;\gamma_1+m,\gamma_2+n,\gamma_3+p;x_1,x_2,x_3]. \tag{34}$$

5. Reduction Formula of Γ_B^H

Here we give a reduction formula of the incomplete Srivastava triple hypergeometric function Γ_B^H.

Theorem 6. *The following reduction formula for* Γ_B^H *holds true*:

$$\Gamma_B^H[(\alpha,x),\beta_1,\gamma_2;\gamma_1,\gamma_2,\gamma_2;x_2x_3,x_2,x_3] = (1-x_2)^{-\beta_1}(1-x_3)^{-\alpha}\,{}_4F_3\left[\begin{array}{r}(\alpha,x(1-x_3)),\beta_1,(\gamma_1+\gamma_2)/2,(\gamma_1+\gamma_2-1)/2;\\ \gamma_1,\gamma_2,\gamma_1+\gamma_2-1;\end{array}\ \frac{4x_2x_3}{(1-x_2)(1-x_3)}\right]. \tag{35}$$

Proof. Setting $\beta_2 = \gamma_2, \gamma_3 = \gamma_2, \ x_1 = x_2x_3$ and using $\Psi_2(\gamma;\gamma,\gamma;x,y) = e^{x+y}\ {}_0F_1(-;\gamma;xy)$ in the integral representation (17), we have

$$\Gamma_B^H[(\alpha,x),\beta_1,\gamma_2;\gamma_1,\gamma_2,\gamma_2;x_2x_3,x_2,x_3] = \tag{36}$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta_1)}\int_x^\infty\int_0^\infty e^{-s(1-x_2)-t(1-x_3)}t^{\alpha-1}s^{\beta_1-1}\ {}_0F_1(-;\gamma_1;x_2x_3st)\ {}_0F_1(-;\gamma_2;x_2x_3st)\,dt\,ds. \tag{37}$$

Setting $t(1-x_3) = u, \ s(1-x_2) = v$ and using well-known result of Erdélyi *et al.* (see [8, p. 185])

$${}_0F_1(-;a;x)\ {}_0F_1(-;b;x) = \ {}_2F_3\left[\begin{array}{r} (a+b)/2, (a+b-1)/2; \\ a, b, a+b-1; \end{array} 4x\right] \tag{38}$$

in (36), we are led to the desired reduction formula (35). □

6. Recurrence Relation

Here we present a recurrence relation for incomplete Srivastava triple hypergeometric function Γ_B^H.

Theorem 7. *The following recurrence relation for* Γ_B^H *holds true*:

$$\begin{aligned}\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] &= \Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1-1,\gamma_2,\gamma_3;x_1,x_2,x_3] \\ &+ \frac{\alpha\beta_1x_1}{\gamma(1-\gamma)}\Gamma_B^H[(\alpha+1,x),\beta_1+1,\beta_2;\gamma_1+1,\gamma_2,\gamma_3;x_1,x_2,x_3]\end{aligned} \tag{39}$$

Proof. Using the well-known contiguous relation for the function ${}_0F_1$ (see [18, p. 12])

$${}_0F_1(-;\gamma-1;x) - \ {}_0F_1(-;\gamma;x) - \frac{x}{\gamma(\gamma-1)}\ {}_0F_1(-;\gamma+1;x) = 0$$

in the integral representation (17), we are led to the desired result. □

7. Concluding Remarks and Observations

In our present investigation, with the help of the incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$, we have introduced the incomplete Srivastava triple hypergeometric function Γ_B^H, whose special cases when $x_2 = 0$ reduces to the incomplete Appell functions of two variables (see [6]) and when $x_2 = 0$, $x_3 = 0$ or $x_1 = 0$ reduces to the incomplete Gauss hypergeometric function (see [26]), respectively, and investigated their diverse properties such mainly as integral representations, derivative formula, reduction formula and recurrence relation. The special cases of the results presented here when $x = 0$ would reduce to the corresponding well-known results for the Srivastava's triple hypergeometric function (see, for details, [20–24, 29, 30]).

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