Statistical Approximation Properties of Stancu Type q-Baskakov-Kantorovich Operators

Vishnu Narayan Mishra[a,b], Preeti Sharma[a], Adem Kiliçman[c], Dilip Jain[a,d]

[a]*Department of Applied Mathematics & Humanities, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat -395 007 (Gujarat), India*
[b]*L. 1627 Awadh Puri Colony Beniganj, Phase -III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad-224 001, (Uttar Pradesh), India*
[c]*Department of Mathematics, Universiti Putra Malaysia (UPM), 43400 Serdang, Selangor, Malaysia*
[d]*Ward No. 12, Sumerganjmandi, Indergarh, (Bundi), Rajasthan - 323614, India*

Abstract. In the present paper, we consider Stancu type generalization of Baskakov-Kantorovich operators based on the q-integers and obtain statistical and weighted statistical approximation properties of these operators. Rates of statistical convergence by means of the modulus of continuity and the Lipschitz type function are also established for said operators. Finally, we construct a bivariate generalization of the operator and also obtain the statistical approximation properties.

1. Introduction

In the last decade, some new generalizations of well known positive linear operators based on q-integers were introduced and studied by several authors. Our aim is to investigate statistical approximation properties of a Stancu type q-Baskakov-Kantorovich operators. Firstly, Baskakov-Kantorovich operators based on q-integers was introduced by Gupta and Radu in [14] and they established some approximation results.

Later, I. Büyükyazici and Atakut [5] introduced a new Stancu type generalization of q-Baskakov operators which is defined as

$$\mathcal{L}_n^{(\alpha,\beta)}(f;q,x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k f\left(\frac{1}{q^{k-1}}\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right), \tag{1}$$

where $0 \le \alpha \le \beta$, $q \in (0,1)$, $f \in C[0,\infty)$.

Let $\{\phi_n\}$ $(n = 1, 2, 3, \ldots)$, $\phi_n : \mathbb{R} \to \mathbb{R}$ be sequence which satisfies following conditions:

(i) ϕ_n $(n = 1, 2, 3, \ldots)$ k-times continuously q-differentiable in any closed interval $[0, A]$, where $A > 0$,

(ii) $\phi_n(0) = 1$, $(n = 1, 2, \ldots)$,

(iii) for all $x \in [0, A]$ and $(k = 1, 2, \ldots; n = 1, 2, \ldots)$, $(-1)^k D_q^k(\phi_n(x)) \ge 0$,

(iv) there exist a positive integer $m(n)$, such that

$$D_q^k(\phi_n(x)) = -[n]_q D_q^{k-1} \phi_{m(n)}(x), \ (k = 1, 2, \ldots; n = 1, 2, \ldots),$$

(v) $\displaystyle\lim_{n \to \infty} \frac{[n]_q}{[m(n)]_q} = 1$.

We first start by recalling some basic definitions and notations of q-calculus. We consider q as a real number satisfying $0 < q < 1$.

For each non negative integer n, we define the q-integer $[n]_q$ as

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \ne 1, \\ n, & q = 1. \end{cases}$$

The q-factorial is defined as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q \ldots [1]_q, & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases}$$

We observe that

$$(1+x)_q^n = (-x; q)_n = \begin{cases} (1+x)(1+qx)(1+q^2x) \ldots (1+q^{n-1}x), & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases}$$

Also, for any real number α, we have

$$(1+x)_q^\alpha = \frac{(1+x)_q^\infty}{(1+q^\alpha x)_q^\infty}.$$

In special case, when α is a whole number, this definition coincides with the above definition. The q-binomial coefficients are given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \ 0 \le k \le n.$$

The q-derivative $D_q f$ of a function f is given by

$$D_q(f(x)) = \frac{f(x) - f(qx)}{(1-q)x}, \ x \ne 0.$$

The q-Jackson integral is defined as

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^\infty f(aq^n)q^n, \ a > 0.$$

Over a general interval $[a, b]$, $0 < a < b$, one defines

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Throughout the paper, we use e_i the test functions defined by $e_i(t) := t^i$, where $i = 0, 1, 2$. First we need the following auxiliary result.

Let $\{\phi_n\}$ be a sequence of real functions on $\mathbb{R}_+ = [0, \infty)$ which are k-times continuously q-differentiable on $\mathbb{R}_+$ satisfying following conditions:

(p_1) $\phi_n(0) = 1$, $(n = 1, 2, \ldots)$,

(p_2) for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, $(-1)^k D_q^k(\phi_n(x)) \geq 0$, $x \in \mathbb{R}_+$,

(p_3) there exist a positive integer $m(n)$, such that

$$D_q^k(\phi_n(x)) = -[n]_q D_q^{k-1}\phi_{m(n)}(x), \ (k = 1, 2, \ldots; \ n = 1, 2, \ldots),$$

(p_4) $\lim_{n\to\infty} \frac{[n]_q}{[m(n)]_q} = 1$.

Under the condition (p_1) - (p_4), Ç. Atakut and İ. Büyükyazici [3] defined a new generalization of Stancu type q-Baskakov-Kantorovich operators as follows

$$\mathcal{L}_n^{*(\alpha,\beta)}(f; q, x) = ([n]_q + \beta) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q+\beta}} f(q^{-k+1}t)d_q t, \tag{2}$$

where $x \in \mathbb{R}_+$, $n \in \mathbb{N}$, $0 \leq \alpha \leq \beta$.

To obtain the approximation results we need the following Lemmas in what follows.

Lemma 1.1. *[5] $\mathcal{L}_n^{(\alpha,\beta)}$ be defined by (1). Then the following identities hold*

$$\mathcal{L}_n^{(\alpha,\beta)}(e_0; q, x) = 1,$$

$$\mathcal{L}_n^{(\alpha,\beta)}(e_1; q, x) = \frac{[n]_q}{[n]_q + \beta}x + \frac{\alpha}{[n]_q + \beta},$$

$$\mathcal{L}_n^{(\alpha,\beta)}(e_2; q, x) = \frac{[n]_q[m(n)]_q}{q([n]_q + \beta)^2}x^2 + \frac{[n]_q(2\alpha + 1)}{([n]_q + \beta)^2}x + \frac{\alpha^2}{([n]_q + \beta)^2}.$$

Lemma 1.2. *[3] The following relations are satisfied:*

$$\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q+\beta}} d_q t = \frac{1}{[n]_q + \beta},$$

$$\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q+\beta}} t\,d_q t = \frac{[2]_q[k]_q + q^k(1 + 2\alpha)}{[2]_q([n]_q + \beta)^2},$$

$$\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q+\beta}} t^2\,d_q t = \frac{[3]_q[k]_q^2 + q^k[k]_q\left((1 + 3\alpha)[2]_q + 1\right) + (1 + 3\alpha + 3\alpha^2)q^{2k}}{[3]_q([n]_q + \beta)^3}.$$

Lemma 1.3. *[3] Let $e_i = t^i$, where $i = 0, 1, 2$. For all $x \in \mathbb{R}_+$, $n \in \mathbb{N}$, α, $\beta \geq 0$ and $0 < q < 1$, we have*

$$\mathcal{L}_n^{*(\alpha,\beta)}(e_0; q, x) = 1,$$

$$\mathcal{L}_n^{*(\alpha,\beta)}(e_1; q, x) = \frac{[n]_q}{[n]_q + \beta}x + \frac{q(1 + 2\alpha)}{[2]_q([n]_q + \beta)},$$

$$\mathcal{L}_n^{*(\alpha,\beta)}(e_2; q, x) = \frac{[n]_q[m(n)]_q}{q([n]_q + \beta)^2}x^2 + \frac{[n]_q\left[[3]_q + q\left((1 + 3\alpha)[2]_q + 1\right)\right]}{[3]_q([n]_q + \beta)^2}x + \frac{q^2(1 + 3\alpha + 3\alpha^2)}{[3]_q([n]_q + \beta)^2}.$$

Remark 1.4. *From Lemma (1.3), we have*

$$\alpha_n(x) = \mathcal{L}_n^{*(\alpha,\beta)}(t - x; q, x) = \left(\frac{[n]_q}{[n]_q + \beta} - 1\right)x + \frac{q_n(1 + 2\alpha)}{[2]_q([n]_q + \beta)},$$

$$\delta_n(x) = \mathcal{L}_n^{*(\alpha,\beta)}((t - x)^2; q, x) = \left(\frac{[n]_q[m(n)]_q}{q([n]_q + \beta)^2} + 1 - \frac{2[n]_q}{[n]_q + \beta}\right)x^2$$

$$+ \left(\frac{[n]_q\left[[3]_q + q\left((1 + 3\alpha)[2]_q + 1\right)\right]}{([n]_q + \beta)^2[3]_q} - \frac{2q(1 + 2\alpha)}{[2]_q([n]_q + \beta)}\right)x + \left(\frac{q^2(1 + 3\alpha + 3\alpha^2)}{[3]_q([n]_q + \beta)^2}\right).$$

Remark 1.5. *If we put q=1, we get the moment of Stancu type Baskakov-Kantorovich operators as*

$$\mathcal{L}_n^{*(\alpha,\beta)}(e_1; 1, x) = \frac{n}{(n + \beta)}x + \frac{(1 + 2\alpha)}{2(n + \beta)},$$

$$\mathcal{L}_n^{*(\alpha,\beta)}(e_2; 1, x) = \frac{n\,m(n)}{(n + \beta)^2}x^2 + \frac{2n(\alpha + 1)}{(n + \beta)^2}x + \frac{1 + 3\alpha + 3\alpha^2}{3(n + \beta)^2},$$

$$\mathcal{L}_n^{*(\alpha,\beta)}(t - x; 1, x) = \left(\frac{n}{(n + \beta)} - 1\right)x + \frac{(1 + 2\alpha)}{2(n + \beta)},$$

$$\mathcal{L}_n^{*(\alpha,\beta)}((t - x)^2; 1, x) = \left(\frac{n\,m(n)}{(n + \beta)^2} + 1 - \frac{2n}{(n + \beta)}\right)x^2 + \left(\frac{2n(1 + \alpha)}{(n + \beta)^2} - \frac{(1 + 2\alpha)}{(n + \beta)}\right)x + \left(\frac{(1 + 3\alpha + 3\alpha^2)}{3(n + \beta)^2}\right).$$

2. Korovkin Type Statistical Approximation Properties

The idea of statistical convergence was introduced independently by Steinhaus [31], Fast [11] and Schoenberg [32]. The study of the statistical convergence for sequences of linear positive operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan [12]. Recently the idea of statistical convergence has been used in proving some approximation theorems. It was shown that the statistical versions are stronger than the classical ones. Authors have used many types of classical operators and test functions to study the Korovkin type approximation theorems which further motivate to continue the study. In particular, Korovkin type approximation theorems [15] was proved by using statistical convergence by various authors, e.g. [4, 10, 13, 16–18, 33]. In the recent years, Stancu type generalization of the certain operators and trigonometric approximation of signals by different types of summability operators have been studied by several other researchers, we refer some of the important papers in this direction as ([19]-[30]) etc.
Korovkin type approximation theory has also many useful connections, other than classical approximation theory, in other branches of mathematics (see Altomare and Campiti in [1]).

Now, we recall the concept of statistical convergence for sequences of real numbers which was introduced by Fast [11] and further studied by many others.

Let $K \subseteq \mathbb{N}$ and $K_n = \{j \le n : j \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set K_n.

A sequence $x = (x_j)_{j\ge1}$ of real numbers is said to be *statistically convergent* to L provided that for every $\epsilon > 0$ the set $\{j \in \mathbb{N} : |x_j - L| \ge \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \le n : |x_j - L| \ge \epsilon\}| = 0.$$

It is denoted by $st - \lim_n x_n = L$.
We consider a sequence $q = (q_n)$, $q_n \in (0, 1)$, such that

$$\lim_{n\to\infty} q_n = 1. \tag{3}$$

The condition (3) guarantees that $[n]_{q_n} \to \infty$ as $n \to \infty$.

Now, let us recall the following theorem given by Gadjiev and Orhan [12].

Theorem 2.1. *If the sequence of linear positive operators $A_n : C_M[a,b] \to C[a,b]$ satisfies the conditions*

$$st - \lim_n \|A_n(e_\nu; \cdot) - e_\nu\|_{C[a,b]} = 0, \quad e_\nu(t) = t^\nu \text{ for } \nu = 0,1,2, \tag{4}$$

then, for any function $f \in C_M[a,b]$, we have

$$st - \lim_n \|A_n(f; \cdot) - f\|_{C[a,b]} = 0,$$

where $C_M[a,b]$ denotes the space of all functions f which are continuous in $[a,b]$ and bounded on the all positive axis.

In [6] Doğru and Kanat defined the Kantorovich-type modification of Lupaş operators as follows:

$$\tilde{R}_n(f;q;x) = [n+1] \sum_{k=0}^{n} \left(\int_{\frac{[k]}{[n+1]}}^{\frac{[k+1]}{[n+1]}} f(t) d_q t \right) \binom{n}{k} \frac{q^{-k} q^{k(k-1)/2} x^k (1-x)^{(n-k)}}{(1-x+qx)\cdots(1-x+q^{n-1}x)}. \tag{5}$$

Doğru and Kanat [6] proved the following statistical Korovkin-type approximation theorem for operators (5).

Theorem 2.2. *Let $q := (q_n)$, $0 < q < 1$, be a sequence satisfying the following conditions:*

$$st - \lim_n q_n = 1, \ st - \lim_n q_n^n = a \ (a < 1) \text{ and } st - \lim_n \frac{1}{[n]_q} = 0, \tag{6}$$

then if f is any monotone increasing function defined on $[0,1]$, for the positive linear operator $\tilde{R}_n(f;q;x)$, then

$$st - \lim_n \|\tilde{R}_n(f;q;\cdot) - f\|_{C[0,1]} = 0$$

holds.

In [7] Doğru gave some examples so that (q_n) is statistically convergent to 1 but it may not convergent to 1 in the ordinary case.

Theorem 2.3. *Let $\mathcal{L}_n^{*(\alpha,\beta)}$ be the sequence of the operators (2) and the sequence $q = (q_n)$ satisfies (6). Then for any function $f \in C[0,A] \subset C[0,\infty)$, $A > 0$, we have*

$$st - \lim_n \|\mathcal{L}_n^{*(\alpha,\beta)}(f;q,\cdot) - f\| = 0, \tag{7}$$

where $C[0,A]$ denotes the space of all real bounded functions f which are continuous in $[0,A]$.

Proof. Let $e_i = t^i$, where $i = 0,1,2$. Using $\mathcal{L}_n^{*(\alpha,\beta)}(1;q_n,x) = 1$, it is clear that

$$st - \lim_n \|\mathcal{L}_n^{*(\alpha,\beta)}(1;q_n,x) - 1\| = 0.$$

Now by Lemma (1.3)(ii), we have

$$\lim_{n\to\infty} \|L_n^{*(\alpha,\beta)}(t;q_n,x) - x\| = \left\| \frac{[n]_q}{[n]_q + \beta} x + \frac{q(1+2\alpha)}{[2]_q([n]_q + \beta)} - x \right\| \le \frac{\beta}{[n]_q + \beta} x + \frac{q(1+2\alpha)}{[2]_q([n]_q + \beta)}.$$

For given $\epsilon > 0$, we define the following sets:

$$S = \{k : \|\mathcal{L}_n^{*(\alpha,\beta)}(t;q_k,x) - x\| \ge \epsilon\},$$

and

$$S' = \left\{ k : \frac{\beta}{[k]_q + \beta} x + \frac{q(1 + 2\alpha)}{[2]_q([k]_q + \beta)} \geq \epsilon \right\}. \tag{8}$$

It is obvious that $S \subset S'$, it can be written as

$$\delta\left(\{k \leq n : \|L_n^{*(\alpha,\beta)}(t; q_k, x) - x\| \geq \epsilon\}\right) \leq \delta\left(\{k \leq n : \frac{\beta}{[k]_q + \beta} x + \frac{q(1 + 2\alpha)}{[2]_q([k]_q + \beta)} \geq \epsilon\}\right).$$

By using (6), we get

$$st - \lim_n \left(\frac{\beta}{[n]_q + \beta} x + \frac{q(1 + 2\alpha)}{[2]_q([n]_q + \beta)} \right) = 0.$$

So, we have

$$\delta\left(\{k \leq n : \frac{\beta}{[k]_q + \beta} x + \frac{q(1 + 2\alpha)}{[2]_q([k]_q + \beta)} \geq \epsilon\}\right) = 0,$$

then

$$st - \lim_n \|\mathcal{L}_n^{*(\alpha,\beta)}(t; q_n, x) - x\| = 0.$$

Similarly, by Lemma (1.3)(iii), we have

$$\|L_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\| = \left\| \frac{[n]_q[m(n)]_q}{q([n]_q + \beta)^2} x^2 + \frac{[n]_q\big[[3]_q + q\big((1 + 3\alpha)[2]_q + 1\big)\big]}{[3]_q([n]_q + \beta)^2} x + \frac{q^2(1 + 3\alpha + 3\alpha^2)}{[3]_q([n]_q + \beta)^2} - x^2 \right\|$$

$$\leq \left| \frac{[n]_q[m(n)]_q}{q([n]_q + \beta)^2} - 1 \right| A^2 + \left| \frac{[n]_q[[3]_q + q((1 + 3\alpha)[2]_q + 1)]}{[3]_q([n]_q + \beta)^2} \right| A + \left| \frac{q^2(1 + 3\alpha + 3\alpha^2)}{[3]_q([n]_q + \beta)^2} \right|$$

$$\leq \mu^2 \left(\left(\frac{1}{q} - 1 \right) + \left(\frac{[n]_q(2 + 3\alpha)}{([n]_q + \beta)^2} + \frac{[n]_q(q - (1 + 3\alpha))}{[3]_q([n]_q + \beta)^2} \right) + \frac{q^2(1 + 3\alpha + 3\alpha^2)}{[3]_q([n]_q + \beta)^2} \right),$$

where $\mu^2 = \max\{A^2, A, 1\} = A^2$.
Now, if we choose

$$\alpha_n = \left(\frac{1}{q} - 1 \right),$$

$$\beta_n = \frac{[n]_q(2 + 3\alpha)}{([n]_q + \beta)^2} + \frac{[n]_q(q - (1 + 3\alpha))}{[3]_q([n]_q + \beta)^2},$$

$$\gamma_n = \frac{q^2(1 + 3\alpha + 3\alpha^2)}{[3]_q([n]_q + \beta)^2},$$

then by (6), we can write

$$st - \lim_{n \to \infty} \alpha_n = 0 = st - \lim_{n \to \infty} \beta_n = st - \lim_{n \to \infty} \gamma_n. \tag{9}$$

Now for given $\epsilon > 0$, we define the following four sets

$$U = \{k : \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_k, x) - x^2\| \geq \epsilon\},$$

$$U_1 = \{k : \alpha_k \geq \frac{\epsilon}{3\mu^2}\},$$

$$U_2 = \{k : \beta_k \geq \frac{\epsilon}{3\mu^2}\},$$

$$U_3 = \{k : \gamma_k \geq \frac{\epsilon}{3\mu^2}\}.$$

It is obvious that $U \subseteq U_1 \cup U_2 \cup U_3$. Then, we obtain

$$\delta\Big(\{k \leq n : \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\| \geq \epsilon\}\Big)$$

$$\leq \delta\Big(\{k \leq n : \alpha_k \geq \frac{\epsilon}{3\mu^2}\}\Big) + \delta\Big(\{k \leq n : \beta_k \geq \frac{\epsilon}{3\mu^2}\}\Big) + \delta\Big(\{k \leq n : \gamma_k \geq \frac{\epsilon}{3\mu^2}\}\Big).$$

Using (9), we get

$$st - \lim_{n\to\infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\| = 0.$$

Since,

$$\|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f\| \leq \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\| + \|\mathcal{L}_n^{*(\alpha,\beta)}(t; q_n, x) - x\| + \|\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) - 1\|,$$

we get

$$st - \lim_{n\to\infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f\| \leq st - \lim_{n\to\infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\|$$

$$+ st - \lim_{n\to\infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(t; q_n, x) - x\|$$

$$+ st - \lim_{n\to\infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) - 1\|,$$

which implies that

$$st - \lim_{n\to\infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f\| = 0.$$

This completes the proof of theorem.

3. Weighted Statistical Approximation

Let $B_{x^2}[0, \infty)$ be set of all function f defined on $[0, \infty)$ and satisfying the condition $|f(x)| \leq M_f \, \rho(x)$, M_f being a constant depending on f and $\rho(x) = (1 + x^2) \geq 1$ is called weighted function, it is continuous on the positive real axis and $\lim_{x\to\infty} \rho(x) = \infty$. By $C_{x^2}[0, \infty)$, we denote the subspace of all continuous function belonging to $B_{x^2}[0, \infty)$. Also, $C_{x^2}^*[0, \infty)$ is subspace of all function $f \in C_{x^2}[0, \infty)$ for which $\lim_{x\to\infty} \frac{f(x)}{1 + x^2}$ is finite. The norm on $C_{x^2}^*[0, \infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1 + x^2}$.

Theorem 3.1. *Let $q = (q_n)$ be a sequence satisfying (6) for $0 < q_n \leq 1$. Then, for all non decreasing functions $f \in C_{x^2}^*[0, \infty)$, we have*

$$st - \lim_n \|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, \cdot) - f\|_{x^2} = 0. \tag{10}$$

Proof. By Lemma (1.3)(iii), we have $\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) \leq Cx^2$, where C is a positive constant, $\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x)$ is a sequence of positive linear operator acting from $C_{x^2}^*[0, \infty)$ to $B_{x^2}[0, \infty)$.

Using $\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) = 1$, it is clear that

$$st - \lim_n \|\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) - 1\|_{x^2} = 0.$$

Now, by Lemma (1.3)(ii), we have

$$\|\mathcal{L}_n^{*(\alpha,\beta)}(t;q_n,x) - x\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|\mathcal{L}_n^{*(\alpha,\beta)}(t;q_n,x) - x|}{1+x^2} \leq \frac{\beta}{[n]_q + \beta} + \frac{q(1+2\alpha)}{[2]_q([n]_q+\beta)}.$$

Using (6), we get

$$st - \lim_n \left(\frac{\beta}{[n]_q+\beta} + \frac{q(1+2\alpha)}{[2]_q([n]_q+\beta)} \right) = 0,$$

then

$$st - \lim_n \|\mathcal{L}_n^{*(\alpha,\beta)}(t;q_n,x) - x\|_{x^2} = 0.$$

Finally, by Lemma(1.3)(iii), we have

$$\|\mathcal{L}_n^{*(\alpha,\beta)}(t^2;q_n,x) - x^2\|_{x^2} \leq \left(\frac{[n]_q[m(n)]_q}{([n]_q+\beta)^2} - 1 \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}$$

$$+ \left(\frac{[n]_q\big[[3]_q + q\big((1+3\alpha)[2]_q + 1\big)\big]}{[3]_q([n]_q+\beta)^2} \right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \left(\frac{q^2(1+3\alpha+3\alpha^2)}{[3]_q([n]_q+\beta)^2} \right)$$

$$\leq \left(\frac{[n]_q[m(n)]_q}{([n]_q+\beta)^2} - 1 \right) + \left(\frac{[n]_q\big[[3]_q + q\big((1+3\alpha)[2]_q + 1\big)\big]}{[3]_q([n]_q+\beta)^2} \right) + \left(\frac{q^2(1+3\alpha+3\alpha^2)}{[3]_q([n]_q+\beta)^2} \right)$$

$$= \left(\frac{1}{q_n} - 1 \right) + \left(\frac{[n]_q(2+3\alpha)}{([n]_q+\beta)^2} + \frac{[n]_q\big(q - (1+3\alpha)\big)}{[3]_q([n]_q+\beta)^2} \right) + \left(\frac{q^2(1+3\alpha+3\alpha^2)}{[3]_q([n]_q+\beta)^2} \right).$$

If we choose

$$\alpha_n = \left(\frac{1}{q_n} - 1 \right),$$

$$\beta_n = \frac{[n]_q(2+3\alpha)}{([n]_q+\beta)^2} + \frac{[n]_q(q - (1+3\alpha))}{[3]_q([n]_q+\beta)^2},$$

$$\gamma_n = \frac{q^2(1+3\alpha+3\alpha^2)}{[3]_q([n]_q+\beta)^2},$$

then by (6), we can write

$$st - \lim_{n\to\infty} \alpha_n = 0 = st - \lim_{n\to\infty} \beta_n = st - \lim_{n\to\infty} \gamma_n. \tag{11}$$

Now for given $\epsilon > 0$, we define the following four sets:

$$S = \{k : \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2;q_k,x) - x^2\|_{x^2} \geq \epsilon\},$$

$$S_1 = \{k : \alpha_k \geq \frac{\epsilon}{3}\},$$

$$S_2 = \{k : \beta_k \geq \frac{\epsilon}{3}\},$$

$$S_3 = \{k : \gamma_k \geq \frac{\epsilon}{3}\}.$$

It is obvious that $S \subseteq S_1 \cup S_2 \cup S_3$. Then, we obtain

$$\delta\Big(\{k \leq n : \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\| \geq \epsilon\}\Big)$$
$$\leq \delta\Big(\{k \leq n : \alpha_k \geq \frac{\epsilon}{3}\}\Big) + \delta\Big(\{k \leq n : \beta_k \geq \frac{\epsilon}{3}\}\Big) + \delta\Big(\{k \leq n : \gamma_k \geq \frac{\epsilon}{3}\}\Big).$$

Using (11), we get

$$st - \lim_{n \to \infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\|_{x^2} = 0.$$

Since

$$\|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f\|_{x^2}$$
$$\leq \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\|_{x^2} + \|\mathcal{L}_n^{*(\alpha,\beta)}(t; q_n, x) - x\|_{x^2} + \|\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) - 1\|_{x^2},$$

we get

$$st - \lim_{n \to \infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f\|_{x^2} \leq st - \lim_{n \to \infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(t^2; q_n, x) - x^2\|_{x^2}$$
$$+ st - \lim_{n \to \infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(t; q_n, x) - x\|_{x^2}$$
$$+ st - \lim_{n \to \infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) - 1\|_{x^2},$$

which implies that

$$st - \lim_{n \to \infty} \|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f\|_{x^2} = 0.$$

This completes the proof of the theorem.

4. Rates of Statistical Convergence

In this section, using the modulus of continuity, we study rates of statistical convergence of operator (2) and Lipschitz functions are introduced.

Lemma 4.1. *Let $0 < q < 1$ and $a \in [0, bq]$, $b > 0$. The inequality*

$$\int_a^b |t - x| d_q t \leq \left(\int_a^b |t - x|^2 d_q t\right)^{1/2} \left(\int_a^b d_q t\right)^{1/2} \tag{12}$$

is satisfied.

Let $C_B[0, \infty)$, the space of all bounded and continuous functions on $[0, \infty)$ and $x \geq 0$. Then, for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f; \delta)$ is defined to be

$$\omega(f; \delta) = \sup_{|t-x| \leq \delta} |f(t) - f(x)|, \ t \in [0, \infty).$$

It is known that $\lim_{\delta \to 0} \omega(f; \delta) = 0$ for $f \in C_B[0, \infty)$ and also, for any $\delta > 0$ and each $t, \ x \geq 0$, we have

$$|f(t) - f(x)| \leq \omega(f; \delta)\left(1 + \frac{|t - x|}{\delta}\right). \tag{13}$$

Theorem 4.2. *Let (q_n) be a sequence satisfying (6). For every non-decreasing $f \in C_B[0, \infty)$, $x \geq 0$ and $n \in \mathbb{N}$, we have*

$$|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| \leq 2\omega\left(f; \sqrt{\delta_n(x)}\right),$$

where

$$
\begin{aligned}
\delta_n(x) &= \left(\frac{[n]_{q_n}[m(n)]_{q_n}}{q_n([n]_{q_n} + \beta)^2} + 1 - \frac{2[n]_{q_n}}{[n]_{q_n} + \beta}\right)x^2 \\
&+ \left(\frac{[n]_{q_n}\left[[3]_{q_n} + q_n((1 + 3\alpha)[2]_{q_n} + 1)\right]}{([n]_{q_n} + \beta)^2[3]_{q_n}} - \frac{2q_n(1 + 2\alpha)}{[2]_{q_n}([n]_{q_n} + \beta)}\right)x \\
&+ \left(\frac{q_n^2(1 + 3\alpha + 3\alpha^2)}{[3]_{q_n}([n]_{q_n} + \beta)^2}\right).
\end{aligned}
\tag{14}
$$

Proof. Let non-decreasing $f \in C_B[0, \infty)$ and $x \geq 0$. Using linearity and positivity of the operators $\mathcal{L}_n^{*(\alpha,\beta)}$ and then applying (13), we get for $\delta > 0$ and $n \in \mathbb{N}$ that

$$
\begin{aligned}
|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| &\leq \mathcal{L}_n^{*(\alpha,\beta)}\left(|f(t) - f(x)|; q_n, x\right) \\
&\leq \omega(f, \delta)\left\{\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) + \frac{1}{\delta}\mathcal{L}_n^{*(\alpha,\beta)}(|t - x|; q_n, x)\right\}.
\end{aligned}
$$

Taking $\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x) = 1$ and then applying Lemma (4.1) with $a = q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)$ and $b = \frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}$, we can write

$$
\begin{aligned}
|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| &\leq \omega(f; \delta)\left\{1 + \frac{1}{\delta}([n]_q + \beta)\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k\right. \\
&\left. \times \left(\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} |q^{-k+1}t - x|^2 d_q t\right)^{1/2}\left(\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} d_q t\right)^{1/2}\right\}.
\end{aligned}
$$

$$
\begin{aligned}
&\leq \omega(f; \delta)\left\{1 + \frac{1}{\delta}\left(([n]_q + \beta)\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} |q^{-k+1}t - x|^2 d_q t\right)^{1/2}\right. \\
&\left. \times \left(([n]_q + \beta)\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} d_q t\right)^{1/2}\right\} \\
&\leq \omega(f; \delta)\left\{1 + \frac{1}{\delta}\left(\mathcal{L}_n^{*(\alpha,\beta)}((t - x)^2; q_n, x)\right)^{1/2}\left(\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x)\right)^{1/2}\right\}.
\end{aligned}
$$

Taking $q = (q_n)$, a sequence satisfying (6), and using $\delta_n(x) = \mathcal{L}_n^{*(\alpha,\beta)}\left((t - x)^2; q_n, x\right)$ and then choosing $\delta = \delta_n(x)$ as in (14), the theorem is proved.

Observe that by the conditions in (6), $st - \lim_n \delta_n = 0$. By (13), we have

$$st - \lim_n \omega(f; \delta_n) = 0.$$

This gives us the pointwise rate of statistical convergence of the operators $\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x)$ to $f(x)$.

Now, we will study the rate of convergence of the operator $\mathcal{L}_n^{*(\alpha,\beta)}$ with the help of functions of the Lipschitz class $Lip_M(a)$, where $M > 0$ and $0 < a \le 1$. Recall that a function $f \in C_B[0, \infty)$ belongs to $Lip_M(a)$ if the inequality

$$|f(t) - f(x)| \le M|t - x|^a; \ \forall \, t, x \in [0, \infty) \tag{15}$$

holds.

Now, we have the following theorem.

Theorem 4.3. *Let the sequence $q = (q_n)$ satisfy the condition given in (6), and let $f \in Lip_M(a)$, $x \ge 0$ with $0 \le a \le 1$ and $M > 0$. Then*

$$|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| \le M\delta_n^{a/2}(x), \tag{16}$$

where $\delta_n(x)$ is given as in (14).

Proof. Since $\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x)$ are linear positive operators and $f \in Lip_M(a)$, on $x \ge 0$ with $0 < a < 1$, we can write

$$|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| \le \mathcal{L}_n^{*(\alpha,\beta)}(|f(t) - f(x)|; q_n, x)$$
$$\le M\mathcal{L}_n^{*(\alpha,\beta)}(|t - x|^a; q_n, x).$$

Now, we take $p = \frac{2}{a}$, $q = \frac{2}{2-a}$, applying Lemma 4.1 and Hölder's inequality, we obtain

$$|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| \le M\Bigg\{([n]_q + \beta) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k$$
$$\times \left(\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} |q^{-k+1}t - x|^2 d_q t \right)^{a/2} \left(\int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} d_q t \right)^{(2-a)/2} \Bigg\}$$

$$\le M\Bigg\{\left(([n]_q + \beta) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} |q^{-k+1}t - x|^2 d_q t \right)^{a/2}$$

$$\times \left(([n]_q + \beta) \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{D_q^k(\phi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q + q^{k-1}\alpha}{[n]_q + \beta}\right)}^{\frac{[k+1]_q + q^k\alpha}{[n]_q + \beta}} d_q t \right)^{(2-a)/2} \Bigg\}$$

$$\le M\Bigg\{1 + \frac{1}{\delta}\Big(\mathcal{L}_n^{*(\alpha,\beta)}((t - x)^2; q_n, x)\Big)^{1/2}\Big(\mathcal{L}_n^{*(\alpha,\beta)}(1; q_n, x)\Big)^{1/2}\Bigg\}.$$

Taking $\delta_n(x) = \left(\mathcal{L}_n^{*(\alpha,\beta)}((t - x)^2; q_n, x)\right)$, as in (14), we get

$$|\mathcal{L}_n^{*(\alpha,\beta)}(f; q_n, x) - f(x)| \le M\delta_n^{a/2}(x).$$

Thus, the proof is complete.

5. The Construction of the Bivariate Operators of Kantorovich Type

The aim of this part is to construct the bivariate extension of the operator (2), introduce the statistical convergence of the operators to the function f and show the rate of statistical convergence of these operators.

$f : C\big([0, \infty) \times [0, \infty)\big) \to C\big([0, \infty) \times [0, \infty)\big)$ and $0 < q_{n_1}, q_{n_2} \le 1$, let us define the bivariate case of operator (2) as follows:

$$\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f;q_{n_1},q_{n_2},x,y) = ([n_1]_{q_{n_1}} + \beta)([n_2]_{q_{n_2}} + \beta)$$

$$\times \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} q_{n_1}^{k_1(k_1-1)/2} \frac{D_{q_{n_1}}^{k_1}(\phi_{n_1}(x))}{[k_1]_{q_{n_1}}!}(-x)^{k_1} q_{n_2}^{k_2(k_2-1)/2} \frac{D_{q_{n_2}}^{k_2}(\phi_{n_2}(x))}{[k_2]_{q_{n_2}}!}(-x)^{k_2}$$

$$\times \int_{q_{n_1}\left(\frac{[k_1]q_{n_1}+q_{n_1}^{k_1-1}\alpha}{[n_1]q_{n_1}+\beta}\right)}^{\frac{[k_1+1]q_{n_1}+q_{n_1}^{k_1}\alpha}{[n_1]q_{n_1}+\beta}} \int_{q_{n_2}\left(\frac{[k_2]q_{n_2}+q_{n_2}^{k_2-1}\alpha}{[n_2]q_{n_2}+\beta}\right)}^{\frac{[k_2+1]q_{n_2}+q_{n_2}^{k_2}\alpha}{[n_2]q_{n_2}+\beta}} f\left(q_{n_1}^{-k_1+1}t,\ q_{n_2}^{-k_2+1}s\right)d_{q_{n_1}}t\ d_{q_{n_2}}s. \tag{17}$$

In [9], Erkuş and Duman proved the statistical Korovkin type approximation theorem for the bivariate linear positive operators to the functions in space H_{ω_2}.

Recently, Ersan and Doğru [8] obtained the statistical Korovkin type theorem and lemma for the bivariate linear positive operators defined in the space H_{ω_2} as follows

Theorem 5.1. *[8] Let $\mathcal{D}_{n_1,n_2}$ be the sequence of linear positive operator acting from $H_{\omega_2}(\mathbb{R}_+^2)$ into $C_B(\mathbb{R}_+)$, where $\mathbb{R}_+ = [0,\infty)$. Then, for any $f \in H_{\omega_2}$,*

$$st - \lim_{n_1,n_2} \|\mathcal{D}_{n_1,n_2}(f) - f\| = 0.$$

Lemma 5.2. *The bivariate operators defined in [8] satisfy the following :*

(i) $\mathcal{D}_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y) = q_{n_1}q_{n_2}$,

(ii) $\mathcal{D}_{n_1,n_2}(f_1;q_{n_1},q_{n_2},x,y) = q_{n_1}q_{n_2}\dfrac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}\dfrac{x}{1+x}$,

(iii) $\mathcal{D}_{n_1,n_2}(f_2;q_{n_1},q_{n_2},x,y) = q_{n_1}q_{n_2}\dfrac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}\dfrac{y}{1+y}$,

(iv) $\mathcal{D}_{n_1,n_2}(f_3;q_{n_1},q_{n_2},x,y) = q_{n_1}^3 q_{n_2}\dfrac{[n_1]_{q_{n_1}}[n_1-1]_{q_{n_1}}}{[n_1+1]^2_{q_{n_1}}}\dfrac{x^2}{(1+x)(1+q_{n_1}x)} + q_{n_1}q_{n_2}\dfrac{[n_1]_{q_{n_1}}}{[n_1+1]^2_{q_{n_1}}}\dfrac{x}{(1+x)}$

$\qquad + q_{n_1}q_{n_2}^3\dfrac{[n_2]_{q_{n_2}}[n_2-1]_{q_{n_2}}}{[n_2+1]^2_{q_{n_2}}}\dfrac{y^2}{(1+y)(1+q_{n_2}y)} + q_{n_1}q_{n_2}\dfrac{[n_2]_{q_{n_2}}}{[n_2+1]^2_{q_{n_2}}}\dfrac{y}{1+y}.$

In order to obtain the statistical convergence of bivariate operator (17), we need the following lemma.

Lemma 5.3. *The bivariate operators defined in (17) satisfy the following equalities:*

$$\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(e_0;q_{n_1},q_{n_2},x,y) = 1,$$

$$\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(e_1;q_{n_1},q_{n_2},x,y) = \frac{[n_1]_{q_{n_1}}}{[n_1]_{q_{n_1}}+\beta}x + \frac{q_{n_1}(1+2\alpha)}{[2]_{q_{n_1}}([n_1]_{q_{n_1}}+\beta)},$$

$$\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(e_2;q_{n_1},q_{n_2},x,y) = \frac{[n_2]_{q_{n_2}}}{[n_2]_{q_{n_2}}+\beta}y + \frac{q_{n_2}(1+2\alpha)}{[2]_{q_{n_2}}([n_2]_{q_{n_2}}+\beta)},$$

$$\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(e_3;q_{n_1},q_{n_2},x,y) = \frac{[n]_{q_{n_1}}[m(n)]_{q_{n_1}}}{q_{n_1}([n]_{q_{n_1}}+\beta)^2}x^2 + \frac{[n]_{q_{n_1}}\left[[3]_{q_{n_1}}+q_{n_1}((1+3\alpha)[2]_{q_{n_1}}+1)\right]}{[3]_{q_{n_1}}([n]_{q_{n_1}}+\beta)^2}x + \frac{q_{n_1}^2(1+3\alpha+3\alpha^2)}{[3]_{q_{n_1}}([n]_{q_{n_1}}+\beta)^2}$$

$$+ \frac{[n]_{q_{n_2}}[m(n)]_{q_{n_2}}}{([n]_{q_{n_2}}+\beta)^2}y^2 + \frac{[n]_{q_{n_2}}\left[[3]_{q_{n_2}}+q_{n_2}((1+3\alpha)[2]_{q_{n_2}}+1)\right]}{[3]_{q_{n_2}}([n]_{q_{n_2}}+\beta)^2}y + \frac{q_{n_2}^2(1+3\alpha+3\alpha^2)}{[3]_{q_{n_2}}([n]_{q_{n_2}}+\beta)^2}.$$

Proof. The proof can be obtained similar to the proof of bivariate operator in [8]. So, we shall omit this proof.

Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be the sequence that converges statistically to 1 but does not converge in ordinary sense, so for $0 < q_{n_1}, q_{n_2} \le 1$, it can be written as

$$st - \lim_{n_1} q_{n_1} = st - \lim_{n_2} q_{n_2} = 1. \tag{18}$$

Now, under the condition in (18), let us show the statistical convergence of bivariate operator (17) with the help of the proof of Theorem 2.3.

Theorem 5.4. *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be sequence satisfying (18) for $0 < q_{n_1}, q_{n_2} \le 1$, and let $\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}$ be sequence of linear positive operator from $C(K)$ into $C(K)$ given by (17). Then, for any function $f \in C(K_1 \times K_1) \subset C(K \times K)$ and $x \in K_1 \times K_1 \subset K \times K$, where $K = [0, \infty) \times [0, \infty)$, $K_1 = [0, A] \times [0, A]$, we have*

$$st - \lim_{n_1, n_2} \| \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f) - f \|_{C(K_1 \times K_1)} = 0. \tag{19}$$

Proof. Using Lemma (5.3), the proof can be obtained similar to the proof of Theorem 2.3. So, we shall omit this proof.

6. Rates of Convergence of Bivariate Operators

Let $K = [0, \infty) \times [0, \infty)$. Then the sup norm on $C_B(K)$ is given by

$$\|f\| = \sup_{(x,y) \in K} |f(x,y)|, \quad f \in C_B(K).$$

We consider the modulus of continuity $\omega(f; \delta_1, \delta_2)$, where $\delta_1, \delta_2 > 0$, for bivariate case given by

$$\omega(f; \delta_1, \delta_2) = \{\sup |f(x', y') - f(x, y)| : (x', y'), (x, y) \in K \text{ and } |x' - x| \le \delta_1, |y' - y| \le \delta_2\}. \tag{20}$$

It is clear that a necessary and sufficient condition for a function $f \in C_B(K)$ is

$$\lim_{\delta_1, \delta_2 \to 0} \omega(f; \delta_1, \delta_2) = 0$$

and $\omega(f; \delta_1, \delta_2)$ satisfy the following condition:

$$|f(x', y') - f(x, y)| \le \omega(f; \delta_1, \delta_2) \left(1 + \frac{|x' - x|}{\delta_1}\right)\left(1 + \frac{|y' - y|}{\delta_2}\right) \tag{21}$$

for each $f \in C_B(K)$. Then observe that any function in $C_B(K)$ is continuous and bounded on K. Details of the modulus of continuity for bivariate case can be found in [2].

Now, the rate of statistical convergence of bivariate operator (17) by means of modulus of continuity in $f \in C_B(K)$ will be given in the following theorem.

Theorem 6.1. *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be sequence satisfying (18). So, we have*

$$|\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \le 4\omega\left(f; \sqrt{\delta_{n_1}(x)}, \sqrt{\delta_{n_2}(y)}\right), \tag{22}$$

where

$$\delta_{n_1}(x) = \left(\frac{[n_1]_{q_{n_1}}[m(n_1)]_{q_{n_1}}}{q_{n_1}([n_1]_{q_{n_1}} + \beta)^2} + 1 - \frac{2[n_1]_{q_{n_1}}}{([n_1]_{q_{n_1}} + \beta)}\right)x^2$$

$$+ \left(\frac{[n_1]_{q_{n_1}}\left[[3]_{q_{n_1}} + q_{n_1}((1 + 3\alpha)[2]_{q_{n_1}} + 1)\right]}{([n_1]_{q_{n_1}} + \beta)^2 [3]_{q_{n_1}}} - \frac{2q_{n_1}(1 + 2\alpha)}{[2]_{q_{n_1}}([n_1]_{q_{n_1}} + \beta)}\right)x$$

$$+ \left(\frac{q_{n_1}^2(1 + 3\alpha + 3\alpha^2)}{[3]_{q_{n_1}}([n_1]_{q_{n_1}} + \beta)^2}\right), \tag{23}$$

$$\delta_{n_2}(y) = \left(\frac{[n_2]_{q_{n_2}} [m(n_2)]_{q_{n_2}}}{q_{n_2}([n_2]_{q_{n_2}} + \beta)^2} + 1 - \frac{2[n_2]_{q_{n_2}}}{([n_2]_{q_{n_2}} + \beta)} \right) y^2$$

$$+ \left(\frac{[n_2]_{q_{n_2}} \left[[3]_{q_{n_2}} + q_{n_2}((1 + 3\alpha)[2]_{q_{n_2}} + 1) \right]}{([n_2]_{q_{n_2}} + \beta)^2 [3]_{q_{n_2}}} - \frac{2q_{n_2}(1 + 2\alpha)}{[2]_{q_{n_2}} ([n_2]_{q_{n_2}} + \beta)} \right) y$$

$$+ \left(\frac{q_{n_2}^2 (1 + 3\alpha + 3\alpha^2)}{[3]_{q_{n_2}} ([n_2]_{q_{n_2}} + \beta)^2} \right). \tag{24}$$

Proof. By using the condition in (21), for $\delta_{n_1}, \delta_{n_2} > 0$ and $n \in \mathbb{N}$, we get

$$|\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)|$$

$$\leq \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|f(x', y') - f(x, y)|; q_{n_1}, q_{n_2}, x, y)$$

$$\leq \omega(f; \delta_{n_1}(x), \delta_{n_2}(y)) \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f_0; q_{n_1}, q_{n_2}, x, y) + \frac{1}{\delta_{n_1}} \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|x' - x|; q_{n_1}, q_{n_2}, x, y) \right)$$

$$\times \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f_0; q_{n_1}, q_{n_2}, x, y) + \frac{1}{\delta_{n_2}} \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|y' - y|; q_{n_1}, q_{n_2}, x, y) \right)$$

If the Cauchy-Schwarz inequality is applied, we have

$$\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|x' - x|; q_{n_1}, q_{n_2}, x, y) \leq \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}((x' - x)^2; q_{n_1}, q_{n_2}, x, y) \right)^{1/2} \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f_0; q_{n_1}, q_{n_2}, x, y) \right)^{1/2}.$$

So, if it is substituted in the above equation, the proof is completed.

At last, the following theorem represents the rate of statistical convergence of bivariate operator (17) by means of Lipschitz $Lip_M(\alpha_1, \alpha_2)$ functions for the bivariate case, where $f \in C_B[0, \infty)$ and $M > 0$ and $0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$, then let us define $Lip_M(\alpha_1, \alpha_2)$ as

$$|f(x', y') - f(x, y)| \leq M|x' - x|^{\alpha_1} |y' - y|^{\alpha_2}; \quad \forall x, x', y, y' \in [0, \infty).$$

We have the following theorem.

Theorem 6.2. *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be sequence satisfying the condition given in (18), and let $Lip_M(\alpha_1, \alpha_2)$, $x \geq 0$ and $0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$. Then*

$$|\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq M \delta_{n_1}^{\alpha_1/2}(x) \delta_{n_2}^{\alpha_2/2}(y), \tag{25}$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in (23), (24).

Proof. Since $\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f; q_{n_1}, q_{n_2}, x, y)$ are linear positive operators and $f \in Lip_M(\alpha_1, \alpha_2)$, $x \geq 0$ and $0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$, we can write

$$|\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f; q_{n_1}, q_{n_2}, x, y) - f(x, y)| \leq \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|f(x', y') - f(x, y)|; q_{n_1}, q_{n_2}, x, y)$$

$$\leq M \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|x' - x|^{\alpha_1} |y' - y|^{\alpha_2}; q_{n_1}, q_{n_2}, x, y)$$

$$= M \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|x' - x|^{\alpha_1}; q_{n_1}, q_{n_2}, x, y) \mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(|y' - y|^{\alpha_2}; q_{n_1}, q_{n_2}, x, y).$$

If we take $p_1 = \frac{2}{\alpha_1}$, $p_2 = \frac{2}{\alpha_2}$, $q_1 = \frac{2}{2-\alpha_1}$, $q_2 = \frac{2}{2-\alpha_2}$, applying Hölder's inequality, we obtain

$$
\begin{aligned}
|\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| &\leq \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}\big((x'-x)^{\alpha_1};q_{n_1},q_{n_2},x,y\big)\right)^{\alpha_1/2} \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f_0;q_{n_1},q_{n_2},x,y)\right)^{(2-\alpha_1)/2} \\
&\quad \times \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}\big((y'-y)^{\alpha_2};q_{n_1},q_{n_2},x,y\big)\right)^{\alpha_2/2} \left(\mathcal{L}_{n_1,n_2}^{*(\alpha,\beta)}(f_0;q_{n_1},q_{n_2},x,y)\right)^{(2-\alpha_2)/2} \\
&= M\delta_{n_1}^{\alpha_1/2}(x)\delta_{n_2}^{\alpha_2/2}(y).
\end{aligned}
$$

Which is the required result.

Conclusion
Our proposed family of integral operators $\mathcal{L}_n^{*(\alpha,\beta)}$ are generalization of summation-integral type operators. The results established here are more general rather than the results of any other previous proved lemmas and theorems. The strong convergence in weighted spaces is highlighted and Bivariate generalization also established for said operators. Some special cases are also considered. Problems considered in this paper may open further research opportunities in these fields. The researchers and professionals working or intend to work in the areas of analysis and its applications will find this research article to be quite useful.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] F. Altomare, M. Campiti, Korovkin type approximation theory and its applications. de Gruyter Stud. Math. 17, Berlin, 1994.
[2] G.A. Anastassiou, S.G. Gal, Approximation Theory: Moduli of Continuity and Global Smoothness Preservation, Birkhäuser, Boston (2000).
[3] Ç. Atakut, İ. Büyükyazici, Approximation Properties for Stancu Type q-Baskakov-Kantorovich Operators, Math. Sci. Lett. 3, No. 1, 53–58 (2014).
[4] N. L. Braha, H. M. Srivastava and S. A. Mohiuddine, A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean, Appl. Math. Comput. 228 (2014), 162–169.
[5] İ. Büyükyazici, Ç. Atakut, On Stancu type generalization of q-Baskakov operators, Mathematical and Computer Modeling, 52 (2010) 752–759.
[6] O. Doğru, K. Kanat, On statistical approximation properties of the Kantorovich type Lupaş operators. Math. Comput. Model. 55 (2012) no. 3–4, 1610–1621.
[7] O. Doğru, On statistical approximation properties of Stancu type bivariate generalization of q-Balazs-Szabados operators, Proc. of the Int. Conf. on Numer. Anal. and Approx. Th., Cluj-Napoca, Romania, July 5-8,(2006) 179–194.
[8] S. Ersan, Approximation properties of bivariate generalization of Bleimann, Butzer and Hahn operators based on the q-integers. In, Proc. of the 12th WSEAS Int. Conference on Applied Mathematics, Cairo, Egypt, pp. 122-127 (2007).
[9] E. Erkuş, O. Duman, A-Statistical extension of the Korovkin type approximation theorem, Proc. Indian Acad. Sci. Math. Sci. 115 (4) (2003) 499–507.
[10] E. Erkuş, O. Duman and H. M. Srivastava, Statistical approximation of certain positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials, *Appl. Math. Comput.* 182 (2006), 213–222.

[11] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[12] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002), 129–138.

[13] S.G. Gal, On the approximation by polynomials in $C_{[0,1]}^q$, Journal of Approximation Theory, Vol. 42, Issue 1, September 1984, Pages 27–29.

[14] V. Gupta, C. Radu, Statistical approximation properties of q-Baskakov-Kantorovich operators, Cent. Eur. J. Math., 7 (2009) 809–818.

[15] P.P. Korovkin, Linear Operators and Approximation Theory,[Translated from the Russian edition (1959)], Russian Monographs and Texts on Advanced Mathematics and Physics, Vol. III, Gordon and Breach Publishers, New York; Hindustan Publishing Corporation, Delhi, 1960.

[16] H. Karsli, I.U. Tiryaki, H.E. Altin, Some approximation properties of a certain nonlinear Bernstein operators, Filomat 28:6 (2014), 12951305.

[17] N.I. Mahmudov and P. Sabancigil, A q-analogue of the Meyer-König and Zeller Operators, Bull. Malays. Math. Sci. Soc.(2), 35 (1) (2012), 39–51.

[18] M. Mursaleen, Faisal Khan, Asif Khan, Statistical approximation for new positive linear operators of Lagrange type, Appl. Math. Comput. 232 (2014) 548–558.

[19] M. Mursaleen, Asif Khan, H.M. Srivastava, K.S. Nisar, Operators constructed by means of q-Lagrange polynomials and A-statistical approximation. Appl. Math. Comput. 219, 6911–6918 (2013).

[20] M. Mursaleen, Asif Khan, Generalized q-Bernstein-Schurer Operators and Some Approximation Theorems, J. Funct. Spaces Appl., Volume 2013, Article ID 719834, 7 pages, http://dx.doi.org/10.1155/2013/719834.

[21] M. Mursaleen, F. Khan, Asif Khan, A. Kılıçman, Some approximation results for generalized Kantorovich-type operators, Jour. Ineq. Appl., 2013, 2013:585, 10.1186/1029-242X-2013–585.

[22] M. Mursaleen, S.A. Mohiuddine, Convergence Methods for Double Sequences and Applications, Springer, New York, 2014.

[23] V.N. Mishra, K. Khatri, L.N. Mishra, Statistical approximation by Kantorovich-type discrete q-Beta operators, Advances in Difference Equations 2013, 2013:345.

[24] V.N. Mishra, H.H. Khan, K. Khatri, L.N. Mishra, Hypergeometric Representation for Baskakov-Durrmeyer-Stancu Type Operators, Bulletin of Mathematical Analysis and Applications, ISSN: 1821-1291, Vol. 5 Issue 3 (2013), Pages 18–26.

[25] V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, Journal of Inequalities and Applications 2013, 2013:586. doi:10.1186/1029-242X-2013–586.

[26] V.N. Mishra, K. Khatri, L.N. Mishra, Using Linear Operators to Approximate Signals of $Lip(\alpha, p), (p \geq 1)$-Class, Filomat 27:2 (2013), 353–363.

[27] V.N. Mishra, P. Sharma, A short note on approximation properties of q-Baskakov-Szász-Stancu Operators, Southeast Asian Bulletin of Mathematics, Vol. 38 (6) (2014) 857–871.

[28] L.N. Mishra, V.N. Mishra, K. Khatri, Deepmala, On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)-$ class by matrix $(C^1.N_p)$ Operator of conjugate series of its Fourier series, Appl. Math. Comput., Vol. 237 (2014), 252-263. doi: 10.1016/j.amc.2014.03.085.

[29] M. A. Özarslan, O. Duman and H. M. Srivastava, Statistical approximation results for Kantorovich-type operators involving some special polynomials, Math. Comput. Modelling 48 (2008), 388–401.

[30] B. Ahmad, J.J. Nietoa, A. Alsaedi, H. Al-Hutami, Boundary value problems of Nonlinear Fractional q-Di erence (Integral) Equations with two Fractional orders and four-point nonlocal integral boundary conditions, Filomat 28:8 (2014), 17191736.

[31] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.

[32] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.

[33] H.M. Srivastava, M. Mursaleen, Asif Khan, Generalized equi-statistical convergence of positive linear operators and associated approximation theorems, Math. Comput. Model. 55 (2012) 2040–2051.