



Iterative Approximations with Hybrid Techniques for Fixed Points and Equilibrium Problems

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Abstract. In this paper, we consider an iterative algorithm by using the shrinking projection method for solving the fixed point problem of the pseudo-contractive mappings and the generalized equilibrium problems. We prove some lemmas for our main result and a strong convergence theorem for the proposed algorithm.

1. Introduction

The main purpose of this article is to study algorithmic approach to the fixed point problem of pseudo-contractive mappings and the generalized equilibrium problems by using the shrinking projection method with the Meir-Keeler contraction.

The problem of finding a fixed point of a nonlinear mapping defined on a nonempty closed convex subset of a real Hilbert space is so general that it includes a number of important problems such as equilibrium problems, convex minimization problems, fixed point problems, variational inequalities, saddle point problems and other problems. Approximating the solutions of these problems by the iterative schemes has been studied by many researchers and various types of mappings have been considered (see [1]-[23], [37], [38]). In particular, the class of pseudocontractive mappings is very important due to their connection with the monotone mappings. In the literature, there are a large number references associated with the fixed point algorithms for pseudocontractive mappings (see, for instance, [24]-[33]).

In the sequel, we assume that C is a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ is an L -Lipschitz pseudocontractive mapping such that $Fix(T) \neq \emptyset$. The first interesting result for finding the fixed points of the pseudocontractive mappings was presented by Ishikawa [28] in 1974 as follows:

2010 *Mathematics Subject Classification.* 49J40; 47J20; 47H09; 65J15.

Keywords. Shrinking projection method; Meir-Keeler contraction; pseudocontractive mapping; fixed point; equilibrium problem.

Received: 19 November 2014; Accepted: 13 April 2015

Communicated by Hari M. Srivastava

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Theorem 1.1. For any $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_nTx_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_nTy_n \end{cases} \tag{1}$$

for all $n \in \mathbb{N}$, where $\{\beta_n\} \subset [0, 1]$, $\{\alpha_n\} \subset [0, 1]$ satisfy the conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (b) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

If C is a convex compact subset of H , then the sequence $\{x_n\}$ generated by (1) converges strongly to a fixed point of T .

Remark 1.2. The iteration (1) is now refereed as the Ishikawa iterative sequence. We observe that C is compact subset. This additional assumption is very rigorous. We know that strong convergence have not been achieved without compactness assumption.

Zhou [33] suggested the following algorithm which coupled Ishikawa method with the CQ-method and proved strong convergence theorems without the compactness assumption.

Theorem 1.3. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0, 1)$ satisfying the conditions:

- (a) $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N}$;
- (b) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2+1}}$.

Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_nTx_n, \\ z_n = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 \\ \quad - \alpha_n\beta_n(1 - 2\beta_n - \beta_n^2L^2)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = \text{proj}_{C_n \cap Q_n}(x_0) \end{cases} \tag{2}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by (2) converges strongly to $\text{proj}_{\text{Fix}(T)}(x_0)$.

Yao et al. [30] introduced the hybrid Mann algorithm and obtained a strong convergence theorem.

Theorem 1.4. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = \text{proj}_{C_1}(x_0)$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_nTx_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I - T)y_n\|^2 \leq 2\alpha_n\langle x_n - z, (I - T)y_n \rangle\}, \\ x_{n+1} = \text{proj}_{C_{n+1}}(x_0) \end{cases} \tag{3}$$

for all $n \in \mathbb{N}$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L+1})$. Then the sequence $\{x_n\}$ generated by (3) converges strongly to $\text{proj}_{\text{Fix}(T)}(x_0)$.

Remark 1.5. In (2) and (3), there are involved in the projection technique. Hence, how to compute the projection is an important problem. In which, the key point is how to construct C_n (or Q_n). In this respect, the following shrinking projection method is instructive. The so-called shrinking projection method

was proposed by Takahashi, Takeuchi and Kubota [34] for finding the fixed points of the nonexpansive mappings:

$$\begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \text{proj}_{C_{n+1}}(x_0) \end{cases} \quad (4)$$

for all $n \in \mathbb{N}$, where $T : C \rightarrow C$ is a nonexpansive mapping. It is clear that C_{n+1} in (4) is simpler than the one in (2) and (3). In the next section, we will draw on this shrinking projection method to construct our algorithm.

The equilibrium problems theory provides us a natural, novel and unified framework to study a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. It is known that the variational inequalities and mathematical programming problems can be viewed as special realization of the abstract equilibrium problems. Equilibrium problems have numerous applications, including but not limited to problems in economics, game theory, finance, traffic analysis, circuit network analysis and mechanics. For related works, refer to [41]-[50]. The importance of the equilibrium problem induced us to study its algorithmic approaches.

The purpose of this paper is to present the following algorithm for the fixed point problem of the pseudo-contractive mappings and the generalized equilibrium problems:

$$\begin{cases} F(z_n, y) + \langle A(x_n), y - z_n \rangle + \frac{1}{\lambda_n} \langle z_n - x_n, y - z_n \rangle \geq 0, \\ y_n = (1 - \alpha_n)z_n + \alpha_n T((1 - \beta_n)z_n + \beta_n Tz_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \text{proj}_{C_{n+1}} f(x_n) \end{cases}$$

for all $y \in C_0$ and $n \geq 0$. Also, we prove that the presented algorithm has strong convergence under some mild conditions.

2. Preliminaries

Throughout, we assume that H is a real Hilbert space with the inner $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and $C \subset H$ is a nonempty closed convex set.

Recall the following:

(1) A mapping $T : C \rightarrow C$ is said to be *pseudo-contractive* if

$$\langle Tu - Tu^\dagger, u - u^\dagger \rangle \leq \|u - u^\dagger\|^2 \quad (5)$$

for all $u, u^\dagger \in C$. It is clear that (5) is equivalent to

$$\|Tu - Tu^\dagger\|^2 \leq \|u - u^\dagger\|^2 + \|(I - T)u - (I - T)u^\dagger\|^2 \quad (6)$$

for all $u, u^\dagger \in C$.

(2) Let $\text{Fix}(T)$ denote the set of fixed points of T . A mapping $T : C \rightarrow C$ is said to be *L-Lipschitz* if

$$\|Tu - Tu^\dagger\| \leq L\|u - u^\dagger\|$$

for all $u, u^\dagger \in C$, where $L > 0$ is a constant. If $L = 1$, T is called *nonexpansive*.

(3) A mapping $A : C \rightarrow H$ is said to be *inverse strongly monotone* if there exists $\zeta > 0$ such that

$$\langle u - v, Au - Av \rangle \geq \zeta \|Au - Av\|^2$$

for all $u, v \in C$.

(4) Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction. The *generalized equilibrium problem* is to find $x^\dagger \in C$ such that

$$F(x^\dagger, y) + \langle Ax^\dagger, y - x^\dagger \rangle \geq 0 \tag{7}$$

for all $y \in C$. The solution set of (7) is denoted by $GEP(F, A)$.

(5) The *metric projection* $proj_C : H \rightarrow C$ is defined by

$$\|u - proj_C(u)\| = \inf\{\|u - u^\dagger\| : u^\dagger \in C\}.$$

The metric projection $proj$ is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is

$$\langle u - proj_C(u), u^\dagger - proj_C(u) \rangle \leq 0$$

for all $u \in H, u^\dagger \in C$.

(6) A mapping T is said to be *demiclosed* if, for any sequence $\{x_n\}$ which weakly converges to \tilde{x} , whenever the sequence $\{T(x_n)\}$ strongly converges to $x^\dagger, T(\tilde{x}) = x^\dagger$.

It is well-known that, in a real Hilbert space H , the following equality holds:

$$\|\xi u + (1 - \xi)u^\dagger\|^2 = \xi\|u\|^2 + (1 - \xi)\|u^\dagger\|^2 - \xi(1 - \xi)\|u - u^\dagger\|^2 \tag{8}$$

for all $u, u^\dagger \in H$ and $\xi \in [0, 1]$.

Assume that $F : C \times C \rightarrow \mathbb{R}$ is a bifunction which satisfies the following conditions:

- (C1) $F(u, u) = 0$ for all $u \in C$;
- (C2) F is monotone, i.e., $F(u, v) + F(v, u) \leq 0$ for all $u, v \in C$;
- (C3) for each $u, v, w \in C, \lim_{t \downarrow 0} F(tw + (1 - t)u, v) \leq F(u, v)$;
- (C4) for each $u \in C, v \mapsto F(u, v)$ is convex and lower semi-continuous.

Lemma 2.1. ([43]) *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction which satisfies conditions (C1)-(C4). Let $\tau > 0$ and $u \in C$. Then there exists $w \in C$ such that*

$$F(w, v) + \frac{1}{\tau} \langle v - w, w - u \rangle \geq 0$$

for all $v \in C$. Further, if

$$T_\tau(u) = \{w \in C : F(w, v) + \frac{1}{\tau} \langle v - w, w - u \rangle \geq 0$$

for all $v \in C\}$, then the following hold:

- (1) T_τ is single-valued and T_τ is firmly nonexpansive.
- (2) $EP(F)$ is closed and convex and $EP(F) = \text{Fix}(T_\tau)$.

Lemma 2.2. ([33]) *Let H be a real Hilbert space and C be a closed convex subset of H . Let $T : C \rightarrow C$ be a continuous pseudo-contractive mapping. Then we have*

- (1) $\text{Fix}(T)$ is a closed convex subset of C ,
- (2) $(I - T)$ is demiclosed at zero.

For convenient, in the sequel, $x_n \rightharpoonup x^\dagger$ denotes the weak convergence of x_n to x^\dagger and $x_n \rightarrow x^\dagger$ denotes the strong convergence of x_n to x^\dagger , respectively.

Let $\{C_n\}$ be the sequence of nonempty closed convex subsets of a Hilbert space H . We define $s - Li_n C_n$ and $w - Ls_n C_n$ as follows, respectively:

- (1) $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset C_n$ such that $x_n \rightarrow x$.
- (2) $x \in w - Ls_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset C_{n_i}$ such that $y_i \rightarrow y$.
- (3) If C_0 satisfies

$$C_0 = s - Li_n C_n = w - Ls_n C_n,$$

then we say that $\{C_n\}$ converges to C_0 in the sense of Mosco [35] and we write $C_0 = M - \lim_{n \rightarrow \infty} C_n$.

It is easy to show that, if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco.

Tsukada [36] proved the following theorem for the metric projection.

Lemma 2.3. ([36]) *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H . If $C_0 = M - \lim_{n \rightarrow \infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{proj_{C_n}(x)\}$ converges strongly to $proj_{C_0}(x)$, where $proj_{C_n}$ and $proj_{C_0}$ are the metric projections of H onto C_n and C_0 , respectively.*

Let (X, d) be a complete metric space. A mapping $f : X \rightarrow X$ is called the *Meir-Keeler contraction* ([39]) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \epsilon + \delta \implies d(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. It is well known that the Meir-Keeler contraction is a generalization of the contractive mapping.

Lemma 2.4. ([39]) *The Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 2.5. ([40]) *Let f be the Meir-Keeler contraction on a convex subset C of a Banach space E . Then, for any $\epsilon > 0$, there exists $r \in (0, 1)$ such that*

$$\|x - y\| \geq \epsilon \implies \|f(x) - f(y)\| \leq r\|x - y\|$$

for all $x, y \in C$.

Lemma 2.6. ([40]) *Let C be a convex subset of a Banach space E . Let T be a nonexpansive mapping on C and f be the Meir-Keeler contraction on C . Then the following hold.*

- (1) Tf is the Meir-Keeler contraction on C ;
- (2) For each $\alpha \in (0, 1)$, $(1 - \alpha)T + \alpha f$ is the Meir-Keeler contraction on C .

3. Main Results

In this section, we first introduce a hybrid iterative algorithm for finding the common element of the generalized equilibrium problem and the fixed point problem. Consequently, we show the strong convergence of our presented algorithm.

For the main result, we assume that

- (a) H is a real Hilbert space and $C \subset H$ is a nonempty closed convex set;

- (b) $f : C \rightarrow C$ is a Meir-Keeler contractive operator and $F : C \times C \rightarrow \mathbb{R}$ is a bifunction satisfying conditions (C1)-(C4);
- (c) $A : C \rightarrow H$ is a δ -inverse strongly monotone operator and $T : C \rightarrow C$ is an L -Lipschitz pseudocontractive operator, where $L > 1$;
- (d) $\{\lambda_n\} \subset [a, b] \subset (0, 2\delta)$ is a real number sequence;
- (e) $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$ satisfying

$$0 < c < \alpha_n \leq \beta_n < d < \frac{1}{\sqrt{1 + L^2} + 1}.$$

Algorithm 3.1. For $x_0 \in C_0 = C$ arbitrarily, define the sequence $\{x_n\}$ iteratively by

$$\begin{cases} F(z_n, y) + \langle A(x_n), y - z_n \rangle + \frac{1}{\lambda_n} \langle z_n - x_n, y - z_n \rangle \geq 0, \\ y_n = (1 - \alpha_n)z_n + \alpha_n T((1 - \beta_n)z_n + \beta_n Tz_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \text{proj}_{C_{n+1}} f(x_n) \end{cases} \tag{9}$$

for all $y \in C_0$ and $n \geq 0$, where proj is the metric projection.

Now, we give some lemmas for the main result in this paper as follows:

Lemma 3.2. For each $n \geq 0$, $\Omega \subset C_n$.

Proof. By induction, we prove that $\Omega \subset C_n$ for all $n \geq 0$.

- (1) $\Omega \subset C_0$ is obvious.
- (2) Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$.

Set $v_n = (1 - \beta_n)z_n + \beta_n Tz_n$ for all $n \geq 0$. Then $y_n = (1 - \alpha_n)z_n + \alpha_n T v_n$ for all $n \geq 0$. Let $x^* \in \Omega \subset C_k$. Then, by Lemma 2.1, we have

$$\begin{aligned} \|z_n - x^*\| &= \|T_{\lambda_n}(x_n - \lambda_n A(x_n)) - T_{\lambda_n}(x^* - \lambda_n A(x^*))\| \\ &\leq \|(x_n - \lambda_n A(x_n)) - (x^* - \lambda_n A(x^*))\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{10}$$

By (6), we have

$$\|Tz_n - x^*\|^2 \leq \|z_n - x^*\|^2 + \|Tz_n - z_n\|^2 \tag{11}$$

and

$$\begin{aligned} \|T v_n - x^*\|^2 &= \|T((1 - \beta_n)I + \beta_n T)z_n - x^*\|^2 \\ &\leq \|(1 - \beta_n)(z_n - x^*) + \beta_n(Tz_n - x^*)\|^2 + \|(1 - \beta_n)z_n + \beta_n Tz_n - T v_n\|^2. \end{aligned} \tag{12}$$

From (8), we have

$$\begin{aligned} &\|(1 - \beta_n)z_n + \beta_n Tz_n - T v_n\|^2 \\ &= \|(1 - \beta_n)(z_n - T v_n) + \beta_n(Tz_n - T v_n)\|^2 \\ &= (1 - \beta_n)\|z_n - T v_n\|^2 + \beta_n\|Tz_n - T v_n\|^2 - \beta_n(1 - \beta_n)\|z_n - Tz_n\|^2. \end{aligned} \tag{13}$$

Since T is L -Lipschitz and $z_n - v_n = \beta_n(z_n - Tz_n)$, by (13), we get

$$\begin{aligned} &\|(1 - \beta_n)z_n + \beta_n Tz_n - T v_n\|^2 \\ &\leq (1 - \beta_n)\|z_n - T v_n\|^2 + \beta_n^3 L^2 \|z_n - Tz_n\|^2 - \beta_n(1 - \beta_n)\|z_n - Tz_n\|^2 \\ &= (1 - \beta_n)\|z_n - T v_n\|^2 + (\beta_n^3 L^2 + \beta_n^2 - \beta_n)\|z_n - Tz_n\|^2. \end{aligned} \tag{14}$$

By (8) and (11), we have

$$\begin{aligned}
 & \| (1 - \beta_n)(z_n - x^*) + \beta_n(Tz_n - x^*) \|^2 \\
 &= \| (1 - \beta_n)(z_n - x^*) + \beta_n(Tz_n - x^*) \|^2 \\
 &= (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 - \beta_n(1 - \beta_n)\|z_n - Tz_n\|^2 \\
 &\leq (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n(\|z_n - x^*\|^2 + \|z_n - Tz_n\|^2) - \beta_n(1 - \beta_n)\|z_n - Tz_n\|^2 \\
 &= \|z_n - x^*\|^2 + \beta_n^2\|z_n - Tz_n\|^2.
 \end{aligned} \tag{15}$$

From (12), (14) and (15), we deduce

$$\|Tv_n - x^*\|^2 \leq \|x - x^*\|^2 + (1 - \beta_n)\|z_n - Tv_n\|^2 - \beta_n(1 - 2\beta_n - \beta_n^2L^2)\|z_n - Tz_n\|^2. \tag{16}$$

Since $\beta_n < d < \frac{1}{\sqrt{1+L^2+1}}$, we derive that

$$1 - 2\beta_n - \beta_n^2L^2 > 0$$

for all $n \geq 0$. This together with (16) implies that

$$\|Tv_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1 - \beta_n)\|z_n - Tv_n\|^2. \tag{17}$$

By (8), (10) and (17) and noting that $\alpha_n \leq \beta_n$ for all $n \geq 0$, we have

$$\begin{aligned}
 \|y_n - x^*\|^2 &= \|(1 - \alpha_n)z_n + \alpha_nTv_n - x^*\|^2 \\
 &= (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|Tv_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|z_n - Tv_n\|^2 \\
 &\leq \|z_n - x^*\|^2 - \alpha_n(\beta_n - \alpha_n)\|Tv_n - x^*\|^2 \\
 &\leq \|z_n - x^*\|^2 \\
 &\leq \|x_n - x^*\|^2,
 \end{aligned} \tag{18}$$

and hence $x^* \in C_{k+1}$, which implies that

$$\Omega \subset C_n$$

for all $n \geq 0$. This completes the proof. \square

Lemma 3.3. For each $n \geq 0$, C_n is closed and convex.

Proof. By the induction, we prove this lemma.

(1) It is obvious from the assumption that $C_0 = C$ is closed convex.

(2) Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For any $z \in C_k$, it follows that $\|y_k - z\| \leq \|x_k - z\|$ is equivalent to

$$\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - z \rangle \leq 0$$

and so C_{k+1} is closed and convex. Therefore, C_n is closed and convex for all $n \geq 0$. This completes the proof. \square

From Lemma 3.3, we have the following:

Lemma 3.4. The sequence $\{x_n\}$ is well-defined.

By using Lemmas 3.2–3.3, we prove the main result in this paper.

Theorem 3.5. Suppose that $\Omega := GEP(F, A) \cap \text{Fix}(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (9) converges strongly to $x^\dagger = \text{proj}_\Omega f(x^\dagger)$.

Proof. Since $\bigcap_{n=1}^{\infty} C_n$ is closed convex, we also know that $proj_{\bigcap_{n=1}^{\infty} C_n}$ is well-defined and so $proj_{\bigcap_{n=1}^{\infty} C_n} f$ is the Meir-Keeler contraction on C . By Lemma 2.4, there exists a unique fixed point $u \in \bigcap_{n=1}^{\infty} C_n$ of $proj_{\bigcap_{n=1}^{\infty} C_n} f$. Since $\{C_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of H with respect to inclusion, it follow that

$$\emptyset \neq \Omega \subset \bigcap_{n=1}^{\infty} C_n = M - \lim_{n \rightarrow \infty} C_n.$$

Setting $u_n := proj_{C_n} f(u)$ and applying Lemma 2.3, we can conclude that

$$\lim_{n \rightarrow \infty} u_n = proj_{\bigcap_{n=1}^{\infty} C_n} f(u) = u.$$

Now, we show that $\lim_{n \rightarrow \infty} \|x_n - u\| = 0$. Assume $M = \overline{\lim}_n \|x_n - u\| > 0$. Then, for all $\epsilon \in (0, M)$, we can choose $\delta_1 > 0$ such that

$$\overline{\lim}_{n \rightarrow \infty} \|x_n - u\| > \epsilon + \delta_1. \tag{19}$$

Since f is the Meir-Keeler contraction, for above ϵ , there exists another $\delta_2 > 0$ such that

$$\|x - y\| < \epsilon + \delta_2 \implies \|f(x) - f(y)\| < \epsilon \tag{20}$$

for all $x, y \in C$. In fact, we can choose a common $\delta > 0$ such that (19) and (20) hold. If $\delta_1 > \delta_2$, then

$$\overline{\lim}_{n \rightarrow \infty} \|x_n - u\| > \epsilon + \delta_1 > \epsilon + \delta_2.$$

If $\delta_1 \leq \delta_2$, then from (20), we deduce that

$$\|x - y\| < \epsilon + \delta_1 \implies \|f(x) - f(y)\| < \epsilon$$

for all $x, y \in C$. Thus we have

$$\overline{\lim}_n \|x_n - u\| > \epsilon + \delta \tag{21}$$

and

$$\|x - y\| < \epsilon + \delta \implies \|f(x) - f(y)\| < \epsilon \tag{22}$$

for all $x, y \in C$. Since $u_n \rightarrow u$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n - u\| < \delta \tag{23}$$

for all $n \geq n_0$.

We now consider two possible cases.

Case 1. There exists $n_1 \geq n_0$ such that

$$\|x_{n_1} - u\| \leq \epsilon + \delta.$$

By (22) and (23), we get

$$\begin{aligned} \|x_{n_1+1} - u\| &\leq \|x_{n_1+1} - u_{n_1+1}\| + \|u_{n_1+1} - u\| \\ &= \|proj_{C_{n_1+1}} f(x_{n_1}) - proj_{C_{n_1+1}} f(u)\| + \|u_{n_1+1} - u\| \\ &\leq \|f(x_{n_1}) - f(u)\| + \|u_{n_1+1} - u\| \\ &\leq \epsilon + \delta. \end{aligned}$$

By induction, we can obtain

$$\|x_{n_1+m} - u\| \leq \epsilon + \delta$$

for all $m \geq 1$, which implies that

$$\overline{\lim}_{n \rightarrow \infty} \|x_n - u\| \leq \epsilon + \delta,$$

which contradicts with (21). Therefore, we conclude that $\|x_n - u\| \rightarrow 0$ as $n \rightarrow \infty$.

Case 2. $\|x_n - u\| > \epsilon + \delta$ for all $n \geq n_0$.

Now, we prove that Case 2 is impossible. Suppose that Case 2 holds true. By Lemma 2.5, there exists $r \in (0, 1)$ such that

$$\|f(x_n) - f(u)\| \leq r\|x_n - u\|$$

for all $n \geq n_0$. Thus we have

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|\text{proj}_{C_{n+1}} f(x_n) - \text{proj}_{C_{n+1}} f(u)\| \\ &\leq \|f(x_n) - f(u)\| \\ &\leq r\|x_n - u\| \end{aligned}$$

for all $n \geq n_0$. It follows that

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \|x_{n+1} - u\| &= \overline{\lim}_{n \rightarrow \infty} \|x_{n+1} - u_{n+1}\| \\ &\leq r \overline{\lim}_{n \rightarrow \infty} \|x_n - u\| \\ &< \overline{\lim}_{n \rightarrow \infty} \|x_n - u\|, \end{aligned}$$

which gives a contradiction. Hence we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u\| = 0$$

and so $\{x_n\}$ is bounded. Observe that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_n - u\| + \|u - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| \\ &= \|x_n - u\| + \|u - u_{n+1}\| + \|\text{proj}_{C_{n+1}} f(x_n) - \text{proj}_{C_{n+1}} f(u)\| \\ &\leq \|x_n - u\| + \|u - u_{n+1}\| + \|f(x_n) - f(u)\|. \end{aligned}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{24}$$

Since $x_{n+1} \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

This together with (24) implies that

$$\lim_{n \rightarrow \infty} \|y_n - x_{n+1}\| = \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{25}$$

Note that

$$\begin{aligned} &\|y_n - x^*\|^2 \\ &\leq \|z_n - x^*\|^2 \\ &\leq \|(x_n - \lambda_n A(x_n)) - (x^* - \lambda_n A(x^*))\|^2 \\ &= \|x_n - x^*\|^2 + \lambda_n^2 \|A(x_n) - A(x^*)\|^2 - 2\langle A(x_n) - A(x^*), x_n - x^* \rangle \\ &= \|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\delta) \|A(x_n) - A(x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 + (b - 2\delta)b \|A(x_n) - A(x^*)\|^2. \end{aligned} \tag{26}$$

Then we have

$$\begin{aligned} (2\delta - b)b\|A(x_n) - A(x^*)\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|). \end{aligned} \tag{27}$$

By (25) and (27), we obtain

$$\lim_{n \rightarrow \infty} \|A(x_n) - A(x^*)\| = 0. \tag{28}$$

Since T_{λ_n} is firmly-nonexpansive, we have

$$\begin{aligned} &\|z_n - x^*\|^2 \\ &= \|T_{\lambda_n}(x_n - \lambda_n A(x_n)) - T_{\lambda_n}(x^* - \lambda_n A(x^*))\|^2 \\ &\leq \langle (x_n - \lambda_n A(x_n)) - (x^* - \lambda_n A(x^*)), z_n - x^* \rangle \\ &= \frac{1}{2}(\|(x_n - \lambda_n A(x_n)) - (x^* - \lambda_n A(x^*))\|^2 + \|z_n - x^*\|^2 \\ &\quad - \|(x_n - \lambda_n A(x_n)) - (x^* - \lambda_n A(x^*)) + x^* - z_n\|^2) \\ &\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - \lambda_n(A(x_n) - A(x^*))\|^2) \\ &= \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\ &\quad + 2\lambda_n \langle x_n - z_n, A(x_n) - A(x^*) \rangle - \lambda_n^2 \|A(x_n) - A(x^*)\|^2). \end{aligned} \tag{29}$$

It follows that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle x_n - z_n, A(x_n) - A(x^*) \rangle - \lambda_n^2 \|A(x_n) - A(x^*)\|^2. \tag{30}$$

From (18) and (30), we have

$$\begin{aligned} &\|y_n - x^*\|^2 \\ &\leq \|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle x_n - z_n, A(x_n) - A(x^*) \rangle - \lambda_n^2 \|A(x_n) - A(x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|A(x_n) - A(x^*)\| \end{aligned}$$

and so

$$\begin{aligned} \|x_n - z_n\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2\lambda_n \|x_n - z_n\| \|A(x_n) - A(x^*)\| \\ &\leq \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|) + 2\lambda_n \|x_n - z_n\| \|A(x_n) - A(x^*)\|. \end{aligned}$$

This together with (25) and (28) implies that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| \rightarrow 0. \tag{31}$$

Next, we prove that $u \in \text{Fix}(T) \cap \text{GEP}(F, A)$. Note that

$$\begin{aligned} \|z_n - Tz_n\| &\leq \|z_n - y_n\| + \|y_n - Tz_n\| \\ &\leq \|z_n - y_n\| + (1 - \alpha_n)\|z_n - Tz_n\| + \alpha_n \|Tv_n - Tz_n\| \\ &\leq \|z_n - y_n\| + (1 - \alpha_n)\|z_n - Tz_n\| + \alpha_n L \|v_n - z_n\| \\ &\leq \|z_n - y_n\| + (1 - \alpha_n)\|z_n - Tz_n\| + \alpha_n \beta_n L \|z_n - Tz_n\|. \end{aligned}$$

It follows that

$$\|z_n - Tz_n\| \leq \frac{1}{\alpha_n(1 - \beta_n L)} \|z_n - y_n\| \leq \frac{1}{c(1 - dL)} \|z_n - y_n\| \rightarrow 0. \tag{32}$$

Since $x_n \rightarrow u$, we have $z_n \rightarrow u$ by (31). So, from (32) and Lemma 2.2, we deduce that $u \in \text{Fix}(T)$.

Now, we show that $u \in GEP(F, A)$. For any $y \in C$, we have

$$F(z_n, y) + \langle A(x_n), y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0. \tag{33}$$

By (C2) and (33), we have

$$-F(y, z_n) + \langle A(x_n), y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0$$

and so

$$\langle A(x_n), y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq F(y, z_n). \tag{34}$$

Since A is $1/\delta$ -Lipschitzian, from (31), we have

$$\lim_{n \rightarrow \infty} \|A(z_n) - A(x_n)\| = 0. \tag{35}$$

For any $t \in (0, 1]$ and $y \in C$, let $z_t^* = ty + (1 - t)u \in C$. By (34), we have

$$\begin{aligned} \langle z_t^* - z_n, A(z_t^*) \rangle &\geq \langle z_t^* - z_n, A(z_t^*) \rangle - \langle z_t^* - z_n, A(x_n) \rangle - \langle z_t^* - z_n, \frac{z_n - x_n}{\lambda_n} \rangle + F(z_t^*, z_n) \\ &= \langle z_t^* - z_n, A(z_t^*) - A(x_n) \rangle - \langle z_t^* - z_n, \frac{z_n - x_n}{\lambda_n} \rangle + F(z_t^*, z_n) \\ &= \langle z_t^* - z_n, A(z_t^*) - A(z_n) \rangle + \langle z_t^* - z_n, A(z_n) - A(x_n) \rangle - \langle z_t^* - z_n, \frac{z_n - x_n}{\lambda_n} \rangle + F(z_t^*, z_n). \end{aligned} \tag{36}$$

By the monotonicity of A , we have

$$\langle z_t^* - z_n, A(z_t^*) \rangle \geq \langle z_t^* - z_n, A(z_n) - A(x_n) \rangle - \langle z_t^* - z_n, \frac{z_n - x_n}{\lambda_n} \rangle + F(z_t^*, z_n). \tag{37}$$

By (31),(35) and (37), we deduce

$$\langle z_t^* - u, A(z_t^*) \rangle \geq F(z_t^*, u). \tag{38}$$

From (C1), (C4) and (38), we have

$$\begin{aligned} 0 &= F(z_t^*, z_t^*) \\ &= F(z_t^*, ty + (1 - t)u) \\ &\leq tF(z_t^*, y) + (1 - t)F(z_t^*, u) \\ &\leq tF(z_t^*, y) + (1 - t)\langle z_t^* - u, A(z_t^*) \rangle \\ &\leq tF(z_t^*, y) + (1 - t)t\langle y - u, A(z_t^*) \rangle \end{aligned}$$

and hence

$$0 \leq F(z_t^*, y) + (1 - t)\langle y - u, A(z_t^*) \rangle. \tag{39}$$

Letting $t \rightarrow 0$ in (39), we have

$$0 \leq F(u, y) + \langle y - u, A(u) \rangle$$

This implies that $u \in GEP(F, A)$. Therefore, we have $u \in \Omega$. Since $x_{n+1} = \text{proj}_{C_{n+1}} f(x_n)$, we have

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y \rangle \geq 0$$

for all $y \in C_{n+1}$. Since $\Omega \subset C_{n+1}$, we get

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y \rangle \geq 0$$

for all $y \in \Omega$. Noting that $x_n \rightarrow u \in \Omega$, we deduce

$$\langle f(u) - u, u - y \rangle \geq 0$$

for all $y \in \Omega$. Thus $u = \text{proj}_{\Omega} f(u) = x^{\dagger}$. This completes the proof. \square

Remark 3.6. By Lemmas 2.1 and 2.2, Ω is a closed convex subset of C . Thus proj_{Ω} is well-defined. Since f is the Meir-Keeler contraction of C , it follows that $\text{proj}_{\Omega} f$ is the Meir-Keeler contraction of C by Lemma 2.6. According to Lemma 2.4, there exists a unique fixed point $x^{\dagger} \in C$ such that $x^{\dagger} = \text{proj}_{\Omega} f(x^{\dagger})$.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under Grant No. (31-130-35-HiCi) and the forth author was supported in part by NNSF of China (61362033).

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