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Strong Disk Property for Domains in Open Riemann Surfaces

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Abstract. We study the relation between the holomorphic approximation property and the strong disk property for an open set of an open Riemann surface or a Stein space of pure dimension 1.

1. Introduction

Let R be a Stein space and D an open set of R. If D is Stein and Runge in R, then D satisfies the strong disk property in R, that is, if $\varphi : \overline{\Delta} \to R$ is a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset D$, then we have that $\varphi(\overline{\Delta}) \subset D$, where $\Delta := \{t \in \mathbb{C} \mid |t| < 1\} \subset \mathbb{C}$.

The converse of this fact is not true in general even if $R = \mathbb{C}^n$ and D is connected. By Nishino [12, 13], by Duval [6], or by Wold [17], there exists a connected, simply connected, and rationally convex open set D_2 of $\mathbb{C}^2$ such that D_2 is not Runge in $\mathbb{C}^2$. Then $D := D_2 \times \mathbb{C}^{n-2}$ satisfies the strong disk property in $\mathbb{C}^n$ but is not Runge in $\mathbb{C}^n$, where $n \geq 2$ (see Abe [1, Theorem 7]). On the other hand, an open set D of $\mathbb{C}$ is Runge in $\mathbb{C}$ if and only if D satisfies the strong disk property in $\mathbb{C}$ (see Abe [1, Proposition 3]).

In this paper, we study the relation between the Rungeness, or the holomorphic approximation property, and the strong disk property for an open set of an open Riemann surface or, more generally, a Stein space of pure dimension 1.

Let R be a planar open Riemann surface and D an open set of R. Then, we prove that every connected component of D is Runge in R if and only if D satisfies the strong disk property in R (see Theorem 3.3). We generalize this fact to the case when R has singular points (see Theorem 4.1).

Moreover, we give an example of the pair (R, D) of an open Riemann surface R, which is not planar, and a connected open set D of R such that D is not Runge in R whereas D satisfies the strong disk property in R (see Example 3.4). We also give an example of the pair (R, D) of an irreducible Stein space R of pure dimension 1, whose normalization is planar, and a connected open set D of R such that D is not Runge in R whereas every irreducible component of D is Runge in R (see Example 4.3).

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2. Preliminaries

Throughout this paper, complex spaces are always supposed to be reduced and second countable.

Let R be a complex space. Let E be a *locally analytic set* of R, that is, an analytic set of an open set D of R. Then E is said to be *Runge* in R or (R, E) is said to be a *Runge pair* if the image of the restriction $\mathscr{O}(R) \to \mathscr{O}(E)$ is dense in $\mathscr{O}(E)$, that is, for every $f \in \mathscr{O}(E)$, for every compact set K of D, and for every $\varepsilon > 0$, there exists $h \in \mathscr{O}(R)$ such that $\|f - h\|_K < \varepsilon$.

An open set D of a complex space R is said to be $\mathscr{O}(R)$*-convex* if for every compact set K of D the set $\hat{K}_R \cap D$ is also compact, where

$$\hat{K}_R := \{x \in R \mid |h(x)| \leq \|h\|_K \text{ for every } h \in \mathscr{O}(R)\}$$

is the *holomorphically convex hull* of K in R.

Lemma 2.1. *Let R be a complex space. Let E be a locally analytic set of R such that E is Runge in R. Then for every compact set K of E we have that $\hat{K}_R \cap E = \hat{K}_E$.*

Proof. Apply the argument in lines 16–20 of Stein [16, p. 356]. □

Lemma 2.2. *Let R be a Stein space. Let K be a compact set of R and Q a finite set of R. Then we have that $\hat{L}_R = \hat{K}_R \cup Q$, where $L := K \cup Q$.*

Proof. Applying Abe-Tabata [3, Lemma 2.5], the proof proceeds by induction on $\#Q$. □

By Stein [16, Satz 1.1], we have the following characterizations of an $\mathscr{O}(R)$-convex open set D of a Stein space R.

Lemma 2.3. *Let R be a Stein space and D an open set of R. Then the following four conditions are equivalent.*

(1) *D is $\mathscr{O}(R)$-convex.*
(2) *For every compact set K of D we have that $\hat{K}_R \subset D$.*
(3) *For every compact set K of D we have that $\hat{K}_R = \hat{K}_D$.*
(4) *D is Stein and Runge in R.*

Lemma 2.4. *Let R be a Stein space. Let $\pi : \tilde{R} \to R$ be the normalization of R. Let D be a Stein open set of R. Let $\tilde{E}$ be the connected component of $\tilde{D} := \pi^{-1}(D)$. If $E := \pi(\tilde{E})$ is Runge in R, then $\tilde{E}$ is Runge in $\tilde{R}$.*

Proof. Let L be an arbitrary compact set of $\tilde{E}$. Then $K := \pi(L)$ is a compact set of E and we have that $\hat{L}_{\tilde{R}} \subset \pi^{-1}(\hat{K}_R)$. Since E is Stein, the set $\hat{K}_E$ is compact. Since π is proper, the set $\pi^{-1}(\hat{K}_E)$ is also compact. Since $\hat{K}_R \cap E = \hat{K}_E$ by Lemma 2.1, we have that $\hat{L}_{\tilde{R}} \cap \tilde{E} \subset \pi^{-1}(\hat{K}_R) \cap \pi^{-1}(E) = \pi^{-1}(\hat{K}_E) \subset \tilde{D}$. Therefore the set $\hat{L}_{\tilde{R}} \cap \tilde{E}$ is compact. It follows that $\tilde{E}$ is Runge in $\tilde{R}$ by Lemma 2.3. □

A non-compact connected complex manifold of dimension 1 is said to be an *open Riemann surface*. By Behnke-Stein [4], every open Riemann surface is Stein. We have the following characterization of a Runge open set of an open Riemann surface, which is also due to Behnke-Stein [4].

Lemma 2.5. *Let R be an open Riemann surface and D an open set of R. Then the following two conditions are equivalent.*

(1) *D is Runge in R.*
(2) *No connected component of $R \setminus D$ is compact.*

Let $\Delta := \{t \in \mathbb{C} \mid |t| < 1\} \subset \mathbb{C}$. An open set D of a complex space R is said to satisfy the *strong disk property* in R if D satisfies the condition that if $\varphi : \overline{\Delta} \to R$ is a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset D$, then $\varphi(\overline{\Delta}) \subset D$ (see Abe [1] and Abe et al. [2]).

Proposition 2.6. *Let R be a Stein space and D a Stein open set of R. If every connected component E of D is Runge in R, then D satisfies the strong disk property in R.*

Proof. Let $\varphi : \overline{\Delta} \to R$ be a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset D$. Let E be the connected component of D which contains the connected set $\varphi(\partial\Delta)$. Since by assumption E is Runge in R, the open set E satisfies the strong disk property in R (see Abe [1, Proposition 1]). Therefore, we have that $\varphi(\partial\Delta) \subset E \subset D$. It follows that D satisfies the strong disk property in R. □

Remark 2.7. Let D be an open set of a complex space R. If D is Runge in R, then every connected component of D is also Runge in R. As is noted in Narasimhan [10, p. 159], the converse of this fact is not true in general even if R is a connected open set of $\mathbb{C}$. As an example, let $R := \mathbb{C} \setminus \{0\}$, $E_1 := \{t \in \mathbb{C} \mid 0 < |t| < 1\}$, $E_2 := \{t \in \mathbb{C} \mid |t| > 1\}$, and $D := E_1 \cup E_2$. Then D is not Runge in R whereas both E_1 and E_2 are Runge in R.

We also need the following simple topological lemma.

Lemma 2.8. *Let X be a locally connected topological space and D an open set of X. Let $\mathscr{L}$ be the set of connected components of $X \setminus D$. Let $\mathscr{C}$ be the set of connected components of ∂D. Let $I(L) := \{C \in \mathscr{C} \mid L \cap C \neq \emptyset\}$ for every $L \in \mathscr{L}$. Then we have the following three statements.*

i) $\bigcup_{C\in I(L)} C = \partial L$ *for every* $L \in \mathscr{L}$.
ii) $\bigcup_{L\in\mathscr{L}} I(L) = \mathscr{C}$.
iii) *If* $L, L' \in \mathscr{L}$, $L \neq L'$, *then we have that* $I(L) \cap I(L') = \emptyset$.

Proof. i) Take an arbitrary $L \in \mathscr{L}$. Let $C \in I(L)$. Since C is a connected set of $X \setminus D$, L is a connected component of $X \setminus D$, and $L \cap C \neq \emptyset$, we have that $C \subset L$. Since $\mathring{L} \cap C \subset (X \setminus D)^\circ \cap \partial D = \emptyset$, we have that $C \subset \partial L$. It follows that $\bigcup_{C\in I(L)} C \subset \partial L$. Let $p \in \partial L$. Assume that $p \notin \overline{D}$. Then there exists a connected neighborhood U of p in X such that $U \cap D = \emptyset$. Since U is a connected set of $X \setminus D$, L is a connected component of $X \setminus D$, and $p \in U \cap L$, we have that $U \subset L$. It is a contradiction because $p \notin \mathring{L}$. It follows that $p \in \overline{D}$ and we have that $p \in (X \setminus D) \cap \overline{D} = \partial D$. Therefore there exists $C \in \mathscr{C}$ such that $p \in C$. Since $p \in L \cap C$, we have that $C \in I(L)$. Therefore we have that $\partial L \subset \bigcup_{C\in I(L)} C$. Thus we proved that $\bigcup_{C\in I(L)} C = \partial L$. ii) Let $C \in \mathscr{C}$. Since C is a connected set of $X \setminus D$, there exists $L \in \mathscr{L}$ such that $C \subset L$. Then we have that $C \in I(L)$. It follows that $\bigcup_{L\in\mathscr{L}} I(L) = \mathscr{C}$. iii) Let $L, L' \in \mathscr{L}$. Assume that there exists $C \in I(L) \cap I(L')$. By i) we have that $C \subset \partial L \cap \partial L' \subset L \cap L'$, which implies that $L = L'$. Therefore if $L \neq L'$, then $I(L) \cap I(L') = \emptyset$. □

Corollary 2.9. *Let X be a connected and locally connected topological space and D an open set of X. If ∂D has only finitely many connected components, then $X \setminus D$ also has only finitely many connected components.*

Proof. We use the notation in Lemma 2.8. We have that $\sum_{L\in\mathscr{L}} \#I(L) = \#\mathscr{C}$ by Lemma 2.8 ii) and iii). Take an arbitrary $L \in \mathscr{L}$. Since X is connected, we have that $\partial L \neq \emptyset$ and therefore $\#I(L) \geq 1$ by Lemma 2.8 i). It follows that $\#\mathscr{L} \leq \#\mathscr{C} < +\infty$. □

3. Planar Open Riemann Surfaces

Lemma 3.1. *Let S be an open Riemann surface. Let R be an open set of S and D a connected open set of R. Then for every compact connected component L of $R \setminus D$ we have that $\hat{L}_S = L$.*

Proof. Let P be a connected component of R which contains D. Since L is a connected component of $P \setminus D$, we have that $P \setminus L$ is connected (see Bourbaki [5, p. 169]). Since $S \setminus P$ is open and closed in $S \setminus (P \setminus L)$, we have that $(P \setminus L) \cup (S \setminus P) = S \setminus L$ is connected (see Bourbaki [5, p. 169]). Since $S \setminus L$ is not relatively compact in S, it follows that $\hat{L}_S = L$ (see Guenot-Narasimhan [8, p. 300]). □

Lemma 3.2. *Let D, E, and R be open sets of $\mathbb{C}$ such that $D \subset E \subset R$. Assume that D is connected and E satisfies the strong disk property in R. Then for every open and compact connected component L of $R \setminus D$ we have that $L \subset E$.*

Proof. Let L be an arbitrary open and compact connected component of $R \setminus D$. There exists an open set O of R such that $L = O \cap (R \setminus D)$. We have that $D \cup L = D \cup O$, which is an open set of $\mathbb{C}$. By Lemma 3.1, we have that $\hat{L}_{\mathbb{C}} = L$. Therefore there exists a Runge open set P of $\mathbb{C}$ such that $L \subset P \Subset D \cup L$. Let W be the connected component of P which contains L. Since W is simply connected, there exists a biholomorphic map $\lambda : \Delta \to W$ by the Riemann mapping theorem. Take a number $\rho \in (0, 1)$ such that $L \subset \lambda(\{t \in \mathbb{C} \mid |t| < \rho\})$. Let $\varphi : \overline{\Delta} \to R$, $t \mapsto \lambda(\rho t)$. Then we have that $L \subset \varphi(\Delta)$ and $\varphi(\partial\Delta) \subset W \setminus L \subset (D \cup L) \setminus L = D \subset E$. Since E satisfies the strong disk property in R, we have that $\varphi(\overline{\Delta}) \subset E$. It follows that $L \subset E$. □

We say that an open Riemann surface R is *planar* if R is biholomorphic to a connected open set of $\mathbb{C}$. We have the following theorem.

Theorem 3.3. *Let R be a planar open Riemann surface and D an open set of R. Then the following two conditions are equivalent.*

(1) *D satisfies the strong disk property in R.*
(2) *Every connected component E of D is Runge in R.*

Proof. (1) → (2). Let E be an arbitrary connected component of D. Then E satisfies the strong disk property in R. Take a sequence $\{E_\nu\}_{\nu=1}^{\infty}$ of connected open sets with $\mathscr{C}^\infty$ smooth boundaries such that $\bigcup_{\nu=1}^{\infty} E_\nu = E$ and $E_\nu \Subset E_{\nu+1}$ for every $\nu \in \mathbb{N}$. Let K be an arbitrary compact set of E. Then there exists $m \in \mathbb{N}$ such that $K \subset E_m$. Since ∂E_m is locally connected and compact, the set ∂E_m has only finitely many connected components. Let $\mathscr{L}$ be the set of connected components of $R \setminus E_m$. Let $\mathscr{L}' := \{L \in \mathscr{L} \mid L \text{ is compact}\}$ and $\mathscr{L}'' := \mathscr{L} \setminus \mathscr{L}'$. By Corollary 2.9, we have that $\#\mathscr{L} < +\infty$ and therefore every $L \in \mathscr{L}$ is open in $R \setminus E_m$. By Lemma 3.2, we have that $G := E_m \cup (\bigcup_{L \in \mathscr{L}'} L) \subset E$. Since $R \setminus G = \bigcup_{L \in \mathscr{L}''} L$, the set G is open and Runge in R. It follows that $\hat{K}_R \subset G \subset E$. Thus we proved that E is Runge in R.
(2) → (1). The assertion is a direct consequence of Proposition 2.6. □

We have the following example of the pair (R, D) of an open Riemann surface R and a connected open set D of R such that D is not Runge in R whereas D satisfies the strong disk property in R.

Example 3.4. Let $T := \mathbb{C}/\Gamma$ be a complex torus of dimension 1, where $\Gamma := \langle \alpha, \beta \rangle_{\mathbb{Z}}$ is a lattice in $\mathbb{C}$. Let $\pi : \mathbb{C} \to T$ be the projection. Let

$$\tilde{W}_0 := \left\{ x\alpha + y\beta \mid x,\, y \in \mathbb{R},\, x^2 + y^2 < \rho^2 \right\},$$

where $0 < \rho < 1/2$. Let $R := T \setminus \{\pi(0)\}$, $W := \pi(\tilde{W}_0)$, and $D := W \setminus \{\pi(0)\}$. Then D is a connected open set of R and satisfies the strong disk property in R. Nevertheless D is not Runge in R.

Proof. It is clear that D is a connected open set of R. Let $\varphi : \overline{\Delta} \to R$ be a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset D$. Since $\pi : \mathbb{C} \to T$ is a topological covering and $\overline{\Delta}$ is simply connected, there exists a continuous map $\tilde{\varphi} : \overline{\Delta} \to \mathbb{C}$ such that $\pi \circ \tilde{\varphi} = \varphi$. Since π is locally biholomorphic, the map $\tilde{\varphi}|_\Delta : \Delta \to \mathbb{C}$ is holomorphic. Since $\pi(\tilde{\varphi}(\partial\Delta)) = \varphi(\partial\Delta) \subset D \subset W$, we have that $\tilde{\varphi}(\partial\Delta) \subset \tilde{W} := \pi^{-1}(W)$. Since the family $\left\{ c + \tilde{W}_0 \mid c \in \Gamma \right\}$ is the set of connected components of $\tilde{W}$, there exists $c \in \Gamma$ such that $\tilde{\varphi}(\partial\Delta) \subset c + \tilde{W}_0$. Since $c + \tilde{W}_0$ is simply connected and therefore Runge in $\mathbb{C}$, we have that $\tilde{\varphi}(\overline{\Delta}) \subset c + \tilde{W}_0$ by Proposition 2.6. Therefore $\varphi(\overline{\Delta}) = \pi(\tilde{\varphi}(\overline{\Delta})) \subset \pi(c + \tilde{W}_0) = W$. Since $\varphi(\overline{\Delta}) \subset R$, we have that $\varphi(\overline{\Delta}) \subset W \cap R = D$. Thus we proved that D satisfies the strong disk property in R. On the other hand, the set $R \setminus D = T \setminus W$ is compact. It follows that D is not Runge in R by Lemma 2.5. □

Problem 3.5. Let R be an open Riemann surface. Assume that for every connected open set D of R the condition that D satisfies the strong disk property in R implies the condition that D is Runge in R. Then, we ask whether R is planar.

4. Stein Space of Dimension 1

By Narasimhan [11], a complex space R of dimension 1 is Stein if and only if R has no compact irreducible component. We have the following theorem which generalizes Theorem 3.3.

Theorem 4.1. *Let R be a Stein space of pure dimension 1. Assume that every connected component of $\tilde{R}$ is planar, where $\pi : \tilde{R} \to R$ is the normalization of R. Then for every open set D of R the following two conditions are equivalent.*

(1) *D satisfies the strong disk property in R.*
(2) *Every irreducible component E of D is Runge in R.*

Proof. (1) → (2). Let $\tilde{D} := \pi^{-1}(D)$. Then $\tilde{D}$ satisfies the strong disk property in $\tilde{R}$. Let E be an arbitrary irreducible component of D. Since $\pi|_{\tilde{D}} : \tilde{D} \to D$ is the normalization of D, there exists a connected component $\tilde{E}$ of $\tilde{D}$ such that $\pi(\tilde{E}) = E$. Let $\tilde{P}$ be the connected component of $\tilde{R}$ such that $\tilde{E} \subset \tilde{P}$. Then $\tilde{E}$ satisfies the strong disk property in $\tilde{P}$. Since by assumption $\tilde{P}$ is a planar Riemann surface, the open set $\tilde{E}$ is Runge in $\tilde{P}$ by Theorem 3.3. Therefore $\tilde{E}$ is also Runge in $\tilde{R}$. Take an arbitrary compact set K of E and let $L := \hat{K}_E$. Then the set $M := (\pi|_{\tilde{E}})^{-1}(L) = \pi^{-1}(L) \cap \tilde{E}$ is compact and we have that $\hat{M}_{\tilde{R}} = \hat{M}_{\tilde{E}} = M$. Let F be the union of irreducible components of D except E. Since L is compact and $E \cap F$ is discrete, the set $L \cap F$ consists of at most finitely many points. Let $N := \pi^{-1}(L)$ and $Q := \pi^{-1}(L \cap F)$. Then we have that $N = M \cup Q$. Since Q is a finite set, we have that $\hat{N}_{\tilde{R}} = \hat{M}_{\tilde{R}} \cup Q = M \cup Q = N$ by Lemma 2.2. It follows that $\hat{L}_R = L \subset E$ by Mihalache [9, Theorem 3.1]. Thus we obtain that $\hat{K}_R \subset E$. Take moreover an arbitrary $f \in \mathscr{O}(E)$ and an arbitrary $\varepsilon > 0$. Since E is an analytic set of a Stein space D, there exists $g \in \mathscr{O}(D)$ such that $g|_E = f$. By the holomorphic approximation theorem, there exists $h \in \mathscr{O}(R)$ such that $\left\|g - h\right\|_{\hat{K}_R} < \varepsilon$. It follows that $\left\|f - h\right\|_K < \varepsilon$ and thus we proved that E is Runge in R.
(2) → (1). The assertion is a direct consequence of Lemma 4.2 below. □

Lemma 4.2. *Let R be a Stein space such that the set $N(R)$ of non-normal points of R is discrete in R. Let D be a Stein open set of R such that every irreducible component E of D is Runge in R. Then D satisfies the strong disk property in R.*

Proof. Let $\pi : \tilde{R} \to R$ be the normalization of R. Let $\varphi : \overline{\Delta} \to R$ be a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset D$. Let $P := (\varphi|_{\Delta})^{-1}(N(R))^{\circ}$. Take an arbitrary point $p \in \overline{P} \cap \Delta$. There exists an open neighborhood V of $\varphi(p)$ in R, $m \in \mathbb{N}$, and an injective holomorphic map $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) : V \to \mathbb{C}^m$. Take a connected open neighborhood U of p such that $U \subset \Delta \cap \varphi^{-1}(V)$. Since $P \cap U \neq \emptyset$ and $\#\varphi(P \cap U) \leq \#\varphi(P) \leq \#N(R) \leq \aleph_0$, we have that $\#(\lambda_\nu \circ \varphi)(P \cap U) \leq \aleph_0$ for every $\nu = 1, 2, \ldots, m$. If $\lambda_\nu \circ \varphi$ is not locally constant on $P \cap U$, then, by the open mapping theorem, we have that $\#(\lambda_\nu \circ \varphi)(P \cap U) = \aleph$, which is a contradiction. Therefore λ_ν is locally constant on $P \cap U$ for $\nu = 1, 2, \ldots, m$. Then, by the identity theorem, the holomorphic map φ is constant on U. Since $\varphi(U) = \{\varphi(p)\} = \varphi(P \cap U) \subset N(R)$, we have that $p \in P$. Thus we proved that $\overline{P} \cap \Delta = P$ and φ is locally constant on P. If $P = \emptyset$, then there exists a holomorphic map $\tilde{\varphi} : \Delta \to \tilde{R}$ such that $\pi \circ \tilde{\varphi} = \varphi$ on Δ (see Grauert-Remmert [7, p. 164]). If $P \neq \emptyset$, then the connectedness of Δ implies that φ is constant on $P = \Delta$ and therefore the existence of such $\tilde{\varphi}$ is clear. On the other hand, there exists a number $\rho \in (0, 1)$ such that $\varphi(\{t \in \mathbb{C} \mid \rho \leq |t| \leq 1\}) \subset D$. Let $\tilde{D} := \pi^{-1}(D)$. Then we have that $\tilde{\varphi}(\{t \in \mathbb{C} \mid \rho \leq |t| < 1\}) \subset \tilde{D}$. Let $\tilde{E}$ be the connected component of $\tilde{D}$ which contains the connected set $\tilde{\varphi}(\{t \in \mathbb{C} \mid \rho \leq |t| < 1\})$. Since $\pi|_{\tilde{D}} : \tilde{D} \to D$ is the normalization of D, the set $E := \pi(\tilde{E})$ is an irreducible component of D. Since by assumption E is Runge in R, the open set $\tilde{E}$ is Runge in $\tilde{R}$ by Lemma 2.4. By Proposition 2.6, the open set $\tilde{E}$ satisfies the strong disk property in $\tilde{R}$. Since $\tilde{\varphi}(\{t \in \mathbb{C} \mid |t| = \rho\}) \subset \tilde{E}$, we have that $\tilde{\varphi}(\{t \in \mathbb{C} \mid |t| \leq \rho\}) \subset \tilde{E} \subset \tilde{D}$. Therefore we have that $\tilde{\varphi}(\Delta) \subset \tilde{D}$ and thus $\varphi(\overline{\Delta}) = \pi(\tilde{\varphi}(\Delta)) \cup \varphi(\partial\Delta) \subset D$. It follows that D satisfies the strong disk property in R. □

Let R be a Stein space and D a Stein open set of R. If every connected component E of D is Runge in R, then every irreducible component of D is also Runge in R. However, the converse of this fact is not true in general even if R is an irreducible Stein space of dimension 1 whose normalization is planar. We have the following example.

Example 4.3. Let $R := \left\{(z, w) \in \mathbb{C}^2 \mid w^2 = z^2(z+1)\right\} \setminus \left\{(\frac{5}{4}, -\frac{15}{8})\right\}$ and $\tilde{R} := \mathbb{C} \setminus \left\{-\frac{3}{2}\right\}$. The holomorphic map $\pi : \tilde{R} \to R$, $t \mapsto (t^2 - 1, t^3 - t)$, gives a normalization of R. Therefore R is irreducible. Let $\tilde{E}_1 := \left\{t \in \mathbb{C} \mid 0 < \left|t + \frac{3}{2}\right| < 2\right\}$, $\tilde{E}_2 := \left\{t \in \mathbb{C} \mid \left|t + \frac{3}{2}\right| > 2\right\}$, and $\tilde{D} := \tilde{E}_1 \cup \tilde{E}_2$. Both $\tilde{E}_1$ and $\tilde{E}_2$ are Runge in $\tilde{R}$ but $\tilde{D}$ is not Runge in $\tilde{R}$. Let $E_1 := \pi(\tilde{E}_1)$, $E_2 := \pi(\tilde{E}_2)$, and $D := \pi(\tilde{D})$. Then D is a connected open set of R and we have that $\tilde{D} = \pi^{-1}(D)$. The sets E_1 and E_2 are the irreducible components of D. By the argument in the proof of Theorem 4.1, we see that both E_1 and E_2 are Runge in R. On the other hand, D is not Runge in R.

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