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On Relative (p, q)-th Order Based Growth Measure of Entire Functions

Sanjib Kumar Datta[a], Tanmay Biswas[b], Chinmay Ghosh[c]

[a]*Department of Mathematics, University of Kalyani P.O. Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.*
[b]*Rajbari, Rabindrapalli, R. N. Tagore Road P.O. Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.*
[c]*Guru Nanak Institute of Technology 157/F Nilgunj Road, Panihati, Sodepur Kolkata-700114, West Bengal, India.*

Abstract. In this paper we intend to find out relative (p, q) -th order (relative (p, q) -th lower order) of an entire function f with respect to another entire function g when relative (m, q) -th order (relative (m, q) -th lower order) of f and relative (m, p) -th order (relative (m, p) -th lower order) of g with respect to another entire function h are given with p, q, m are all positive integers.

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\left\{\left|f(z)\right| : |z| = r\right\}$. If f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1}(r) : \left(\left|f(0)\right|, \infty\right) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$. In the sequel we use the following notation :

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}$$
$$\log^{[0]} x = x;$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}$$
$$\exp^{[0]} x = x.$$

The following definitions are well known:

Definition 1.1. *The order ρ_f and the lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

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Email addresses: sanjib_kr_datta@yahoo.co.in (Sanjib Kumar Datta), tanmaybiswas_math@rediffmail.com (Tanmay Biswas), chinmayarp@gmail.com (Chinmay Ghosh)

Juneja, Kapoor and Bajpai [2] defined the (p,q)-th order and (p,q)-th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} ,$$

where p,q are any two positive integers with $p \geq q$.

If $p = 2$ and $q = 1$ we respectively denote $\rho_f(2,1)$ and $\lambda_f(2,1)$ by ρ_f and λ_f.

In this connection we just recall the following definition :

Definition 1.2. *[2] An entire function f is said to have index-pair (p,q), $p \geq q \geq 1$ if $b < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. Moreover if $0 < \rho_f(p,q) < \infty$, then*

$$\rho_f(p-n,q) = \infty \text{ for } n < p,\ \rho_f(p,q-n) = 0 \text{ for } n < q \text{ and}$$

$$\rho_f(p+n,q+n) = 1 \text{ for } n = 1,2,\ldots .$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n < p,\ \lambda_f(p,q-n) = 0 \text{ for } n < q \text{ and}$$

$$\lambda_f(p+n,q+n) = 1 \text{ for } n = 1,2,\ldots .$$

An entire function for which (p,q)-th order and (p,q)-th lower order are the same is said to be of regular (p,q)-growth. Functions which are not of regular (p,q)-growth are said to be of irregular (p,q)-growth.

Bernal [1] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0.\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} . \end{aligned}$$

The definition coincides with the classical one [6] if $g(z) = \exp z$.

Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .$$

In the case of relative order, it therefore seems reasonable to define suitably the relative (p,q) th order of entire functions. Lahiri and Banerjee [3] introduced such definition in the following manner:

Definition 1.3. *[3] Let p and q be any two positive integers with $p > q$. The relative (p,q)-th order of f with respect to g is defined by*

$$\rho_g^{(p,q)}(f) = \inf\left\{ \begin{array}{c} \mu > 0 : M_f(r) < M_g\left(\exp^{[p-1]}\left(\mu \log^{[q]} r\right)\right) \\ \text{for all } r > r_0(\mu) > 0 \end{array} \right\} = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

Sánchez Ruiz et. al. [5] gave a more improved definition of relative (p,q)-th order of an entire function in the light of index-pair which is the following:

Definition 1.4. *[5] Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where p,q,m,n are all positive integer such that $p \geq q$ and $m \geq n$. Then the relative (p,q) th order of f with respect to g is defined as*

$$\rho_g^{(p,q)}(f) = \inf\left\{ \begin{array}{c} \mu > 0 : M_f(r) < M_g\left(\exp^{[p]}\left\{\log^{[m]} \exp^{[m]}\left(\mu \log^{[q]} r\right)\right\}\right) \\ \text{for all } r > r_0(\mu) > 0 \end{array} \right\} = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

Similarly, one can define the relative (p,q)-th lower order of f with respect to g denoted by $\lambda_g^{(p,q)}(f)$ as follows :

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

In this connection, we intend to give a definition of relative index-pair of an entire function with respect to another entire function which is relevant in the sequel :

Definition 1.5. *Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where p,q,m,n are all positive integer such that $p \geq q$ and $m \geq n$. Then the entire function f is said to have relative index-pair (p,q) with respect to another entire function g, if $b < \rho_g^{(p,q)}(f) < \infty$ and $\rho_g(f)(p-1,q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q = m$ and $b = 0$ for otherwise. Moreover if $0 < \rho_g^{(p,q)}(f) < \infty$, then*

$\rho_g(f)(p-n,q) = \infty$ *for* $n < p$, $\rho_g(f)(p,q-n) = 0$ *for* $n < q$ *and*

$\rho_g(f)(p+n,q+n) = 1$ *for* $n = 1, 2, \ldots.$.

Similarly for $0 < \lambda_g^{(p,q)}(f) < \infty$, one can easily verify that

$\lambda_g(f)(p-n,q) = \infty$ *for* $n < p$, $\lambda_g(f)(p,q-n) = 0$ *for* $n < q$ *and*

$\lambda_g(f)(p+n,q+n) = 1$ *for* $n = 1, 2, \ldots.$.

An entire function f for which relative (p,q)-th order and relative (p,q)-th lower order with respect to another entire function g are the same is called a function of regular relative (p,q) growth with respect to g. Otherwise, f is said to be irregular relative (p,q) growth with respect to g.

Now a question may arise about the relative index-pair and the values of relative order (relative lower order) of f with respect to another entire function g when relative (m,q)-th order (relative (m,q)-th lower order) of f and relative (m,p)-th order (relative (m,p)-th lower order) of g with respect to another entire function h are given where p,q,m are all positive integers. In this paper we intend to provide this answer. We do not explain the standard definitions and notations of the theory of entire functions as those are available in [8].

2. Theorems.

In this section we present the main results of the paper.

Theorem 2.1. *Let f, g and h be any three entire functions and p, q, m are all positive integers. If the relative (m,q)-th order (relative (m,q)-th lower order) of f with respect to h and relative (m,p)-th order (relative (m,p)-th lower order) of g with respect to h are respectively denoted by $\rho_h^{(m,q)}(f)$ $\left(\lambda_h^{(m,q)}(f)\right)$ and $\rho_h^{(m,p)}(g)$ $\left(\lambda_h^{(m,p)}(g)\right)$, then*

$$\begin{aligned}\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} &\leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\} \\ &\leq \max\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.\end{aligned}$$

Proof. From the definitions of $\rho_h^{(m,q)}(f)$ and $\lambda_h^{(m,q)}(f)$ we have for all sufficiently large values of r that

$$\begin{aligned}M_h^{-1}M_f(r) &\leq \exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\} \\ \text{i.e., } M_f(r) &\leq M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\}\right], \qquad (1)\end{aligned}$$

$$\begin{aligned}M_h^{-1}M_f(r) &\geq \exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\} \\ \text{i.e., } M_f(r) &\geq M_h\left[\exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}\right]. \qquad (2)\end{aligned}$$

Also for a sequence of values of r tending to infinity we get that

$$\begin{aligned}M_h^{-1}M_f(r) &\geq \exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\} \\ \text{i.e., } M_f(r) &\geq M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}\right], \qquad (3) \\ M_h^{-1}M_f(r) &\leq \exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\} \\ \text{i.e., } M_f(r) &\leq M_h\left[\exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\}\right]. \qquad (4)\end{aligned}$$

Similarly from the definitions of $\rho_h^{(m,p)}(g)$ and $\lambda_h^{(m,p)}(g)$, it follows for all sufficiently large values of r that

$$\begin{aligned}M_h^{-1}M_g(r) &\leq \exp^{[m]}\left\{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)\log^{[p]} r\right\} \\ \text{i.e., } M_g(r) &\leq M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)\log^{[p]} r\right\}\right] \\ \text{i.e., } M_h(r) &\geq M_g\left[\exp^{[p]}\left[\frac{\log^{[m]} r}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)}\right]\right], \qquad (5)\end{aligned}$$

$$\begin{aligned} M_h^{-1}M_g(r) &\geq \exp^{[m]}\left\{\left(\lambda_h^{(m,p)}(g)-\varepsilon\right)\log^{[p]} r\right\} \\ i.e.,\ M_g(r) &\geq M_h\left[\exp^{[m]}\left\{\left(\lambda_h^{(m,p)}(g)-\varepsilon\right)\log^{[p]} r\right\}\right] \\ i.e.,\ M_h(r) &\leq M_g\left[\exp^{[p]}\left[\frac{\log^{[m]} r}{\left(\lambda_h^{(m,p)}(g)-\varepsilon\right)}\right]\right] \end{aligned} \tag{6}$$

and for a sequence of values of r tending to infinity we obtain that

$$\begin{aligned} M_h^{-1}M_g(r) &\geq \exp^{[m]}\left\{\left(\rho_h^{(m,p)}(g)-\varepsilon\right)\log^{[p]} r\right\} \\ i.e.,\ M_g(r) &\geq M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,p)}(g)-\varepsilon\right)\log^{[p]} r\right\}\right] \\ i.e.,\ M_h(r) &\leq M_g\left[\exp^{[p]}\left[\frac{\log^{[m]} r}{\left(\rho_h^{(m,p)}(g)-\varepsilon\right)}\right]\right], \end{aligned} \tag{7}$$

$$\begin{aligned} M_h^{-1}M_g(r) &\leq \exp^{[m]}\left\{\left(\lambda_h^{(m,p)}(g)+\varepsilon\right)\log^{[p]} r\right\} \\ i.e.,\ M_g(r) &\leq M_h\left[\exp^{[m]}\left\{\left(\lambda_h^{(m,p)}(g)+\varepsilon\right)\log^{[p]} r\right\}\right] \\ i.e.,\ M_h(r) &\geq M_g\left[\exp^{[p]}\left[\frac{\log^{[m]} r}{\left(\lambda_h^{(m,p)}(g)+\varepsilon\right)}\right]\right]. \end{aligned} \tag{8}$$

Now from (3) and in view of (5) we get for a sequence of values of r tending to infinity that

$$\log^{[p]} M_g^{-1}M_f(r) \geq \log^{[p]} M_g^{-1}M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}\right]$$

$$\begin{aligned} & i.e.,\ \log^{[p]} M_g^{-1}M_f(r) \\ \geq\ & \log^{[p]} M_g^{-1}M_g\left[\exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e.,\ \log^{[p]} M_g^{-1}M_f(r) &\geq \frac{\left(\rho_h^{(m,q)}(f)-\varepsilon\right)}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)}\log^{[q]} r \\ i.e.,\ \frac{\log^{[p]} M_g^{-1}M_f(r)}{\log^{[q]} r} &\geq \frac{\left(\rho_h^{(m,q)}(f)-\varepsilon\right)}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)}. \end{aligned}$$

As ε (> 0) is arbitrary, it follows that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{[p]} M_g^{-1}M_f(r)}{\log^{[q]} r} &\geq \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\\ i.e.,\ \rho_g^{(p,q)}(f) &\geq \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.\end{aligned} \tag{9}$$

Analogously from (2) and in view of (8) it follows for a sequence of values of r tending to infinity that

$$\log^{[p]} M_g^{-1}M_f(r) \geq \log^{[p]} M_g^{-1}M_h\left[\exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}\right]$$

$$\begin{aligned}&i.e.,\ \log^{[p]} M_g^{-1}M_f(r)\\ \geq\ &\log^{[p]} M_g^{-1}M_g\left[\exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}}{\left(\lambda_h^{(m,p)}(g)+\varepsilon\right)}\right]\right]\end{aligned}$$

$$\begin{aligned}i.e.,\ \log^{[p]} M_g^{-1}M_f(r) &\geq \frac{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)}{\left(\lambda_h^{(m,p)}(g)+\varepsilon\right)}\log^{[q]} r\\ i.e.,\ \frac{\log^{[p]} M_g^{-1}M_f(r)}{\log^{[q]} r} &\geq \frac{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)}{\left(\lambda_h^{(m,p)}(g)+\varepsilon\right)}.\end{aligned}$$

Since ε (> 0) is arbitrary, we get from above that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{[p]} M_g^{-1}M_f(r)}{\log^{[q]} r} &\geq \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}\\ i.e.,\ \rho_g^{(p,q)}(f) &\geq \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.\end{aligned} \tag{10}$$

Again in view of (6) we have from (1) for all sufficiently large values of r that

$$\log^{[p]} M_g^{-1}M_f(r) \leq \log^{[p]} M_g^{-1}M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\}\right]$$

$$\begin{aligned}&i.e.,\ \log^{[p]} M_g^{-1}M_f(r)\\ \leq\ &\log^{[p]} M_g^{-1}M_g\left[\exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\}}{\left(\lambda_h^{(m,p)}(g)-\varepsilon\right)}\right]\right]\end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[p]} M_g^{-1} M_f(r) &\leq \frac{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)}{\left(\lambda_h^{(m,p)}(g)-\varepsilon\right)} \log^{[q]} r \\ i.e., \ \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\leq \frac{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)}{\left(\lambda_h^{(m,p)}(g)-\varepsilon\right)} . \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \\ i.e., \ \rho_g^{(p,q)}(f) &\leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} . \end{aligned} \tag{11}$$

Again from (2) and in view of (5) we get for all sufficiently large values of r that

$$\log^{[p]} M_g^{-1} M_f(r) \geq \log^{[p]} M_g^{-1} M_h\left[\exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}\right]$$

$$\begin{aligned} & i.e., \ \log^{[p]} M_g^{-1} M_f(r) \\ \geq \ & \log^{[p]} M_g^{-1} M_g\left[\exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)\log^{[q]} r\right\}}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[p]} M_g^{-1} M_f(r) &\geq \frac{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)} \log^{[q]} r \\ i.e., \ \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\geq \frac{\left(\lambda_h^{(m,q)}(f)-\varepsilon\right)}{\left(\rho_h^{(m,p)}(g)+\varepsilon\right)} . \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\geq \frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \\ i.e., \ \lambda_g^{(p,q)}(f) &\geq \frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} . \end{aligned} \tag{12}$$

Also in view of (7) , we get from (1) for a sequence of values of r tending to infinity that

$$\log^{[p]} M_g^{-1} M_f(r) \leq \log^{[p]} M_g^{-1} M_h\left[\exp^{[m]}\left\{\left(\rho_h^{(m,q)}(f)+\varepsilon\right)\log^{[q]} r\right\}\right]$$

$$
\begin{aligned}
& i.e.,\ \log^{[p]} M_g^{-1} M_f(r) \\
\leq\ & \log^{[p]} M_g^{-1} M_g \left[\exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\rho_h^{(m,q)}(f) + \varepsilon \right) \log^{[q]} r \right\}}{\left(\rho_h^{(m,p)}(g) - \varepsilon \right)} \right] \right]
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\ \log^{[p]} M_g^{-1} M_f(r) &\leq \frac{\left(\rho_h^{(m,q)}(f) + \varepsilon \right)}{\left(\rho_h^{(m,p)}(g) - \varepsilon \right)} \log^{[q]} r \\
i.e.,\ \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\leq \frac{\left(\rho_h^{(m,q)}(f) + \varepsilon \right)}{\left(\rho_h^{(m,p)}(g) - \varepsilon \right)} .
\end{aligned}
$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$
\begin{aligned}
\liminf_{r \to \infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\leq \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \\
i.e.,\ \lambda_g^{(p,q)}(f) &\leq \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} . \qquad (13)
\end{aligned}
$$

Similarly from (4) and in view of (6) , it follows for a sequence of values of r tending to infinity we get that

$$
\log^{[p]} M_g^{-1} M_f(r) \leq \log^{[p]} M_g^{-1} M_h \left[\exp^{[m]} \left\{ \left(\lambda_h^{(m,q)}(f) + \varepsilon \right) \log^{[q]} r \right\} \right]
$$

$$
\begin{aligned}
& i.e.,\ \log^{[p]} M_g^{-1} M_f(r) \\
\leq\ & \log^{[p]} M_g^{-1} M_g \left[\exp^{[p]} \left[\frac{\log^{[m]} \exp^{[m]} \left\{ \left(\lambda_h^{(m,q)}(f) + \varepsilon \right) \log^{[q]} r \right\}}{\left(\lambda_h^{(m,p)}(g) - \varepsilon \right)} \right] \right]
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\ \log^{[p]} M_g^{-1} M_f(r) &\leq \frac{\left(\lambda_h^{(m,q)}(f) + \varepsilon \right)}{\left(\lambda_h^{(m,p)}(g) - \varepsilon \right)} \log^{[q]} r \\
i.e.,\ \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\leq \frac{\left(\lambda_h^{(m,q)}(f) + \varepsilon \right)}{\left(\lambda_h^{(m,p)}(g) - \varepsilon \right)} .
\end{aligned}
$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$
\begin{aligned}
\liminf_{r \to \infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\leq \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \\
i.e.,\ \lambda_g^{(p,q)}(f) &\leq \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} . \qquad (14)
\end{aligned}
$$

Thus the theorem follows from (9), (10), (11), (12), (13) and (14). □

Remark 2.2. *Theorem 2.1 can be verified by the following example and in fact the equality sign can not be removed from the concluding statement of the theorem.*

Example 2.3. *Let*

$$f = \exp^{[25]} z^9,\ g = \exp^{[25]} z^2 \text{ and } h = \exp^{[25]} z^7 .$$

Then

$$\rho_h^{(1,1)}(f) = \lambda_h^{(1,1)}(f) = \frac{9}{7} \text{ and } \rho_h^{(1,1)}(g) = \lambda_h^{(1,1)}(g) = \frac{2}{7} .$$

Now

$$\rho_g^{(1,1)}(f) = \frac{9}{2} = \frac{\rho_h^{(1,1)}(f)}{\lambda_h^{(m,p)}(g)} = \frac{\lambda_h^{(1,1)}(f)}{\lambda_h^{(m,p)}(g)} = \frac{\rho_h^{(1,1)}(f)}{\rho_h^{(m,p)}(g)} .$$

Similarly

$$\lambda_g^{(1,1)}(f) = \frac{9}{2} = \frac{\lambda_h^{(1,1)}(f)}{\rho_h^{(m,p)}(g)} = \frac{\lambda_h^{(1,1)}(f)}{\lambda_h^{(m,p)}(g)} = \frac{\rho_h^{(1,1)}(f)}{\rho_h^{(m,p)}(g)} .$$

In view of Theorem 2.1, one can easily verify the following corollaries:

Corollary 2.4. *Let f be an entire function with regular relative (m,q) growth with respect to entire function h and g be entire having relative index-pair (m,p) with respect to another entire function h where p,q,m are all positive integers. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} .$$

In addition, if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, then

$$\lambda_g^{(p,q)}(f) = \rho_f^{(q,p)}(g) = 1 .$$

Corollary 2.5. *Let f be an entire function with relative index-pair (m,q) with respect to entire function h and g be entire of regular relative (m,p) growth with respect to another entire function h where p,q,m are all positive integers. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} .$$

In addition, if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, then

$$\rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = 1 .$$

Corollary 2.6. *Let f and g be any two entire functions with regular relative (m,q) growth and regular relative (m,p) growth with respect to entire function h respectively where p,q,m are all positive integers . Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} .$$

Corollary 2.7. *Let f and g be any two entire functions with regular relative (m,q) growth and regular relative (m,p) growth with respect to entire function h respectively where p,q,m are all positive integers . Also suppose that $\rho_h^{(m,q)}(f)=\rho_h^{(m,p)}(g)$. Then*

$$\lambda_g^{(p,q)}(f)=\rho_g^{(p,q)}(f)=\lambda_f^{(q,p)}(g)=\rho_f^{(q,p)}(g)=1\ .$$

Corollary 2.8. *Let f and g be any two entire functions with relative index-pairs (m,q) and (m,p) with respect to entire function h respectively where p,q,m are all positive integers and either f is not of regular relative (m,q) growth or g is not of regular relative (m,p) growth, then*

$$\rho_g^{(p,q)}(f)\,.\rho_f^{(q,p)}(g)\geq 1\ .$$

If f and g are both of regular relative (m,q) growth and regular relative (m,p) growth respectively with respect to entire function h respectively, then

$$\rho_g^{(p,q)}(f)\,.\rho_f^{(q,p)}(g)=1\ .$$

Corollary 2.9. *Let f and g be any two entire functions with relative index-pairs (m,q) and (m,p) with respect to entire function h respectively where p,q,m are all positive integers and either f is not of regular relative (m,q) growth or g is not of regular relative (m,p) growth, then*

$$\lambda_g^{(p,q)}(f)\,.\lambda_f^{(q,p)}(g)\leq 1\ .$$

If f and g are both of regular relative (m,q) growth and regular relative (m,p) growth with respect to entire function h respectively, then

$$\lambda_g^{(p,q)}(f)\,.\lambda_f^{(q,p)}(g)=1\ .$$

Corollary 2.10. *Let f be an entire function with relative index-pair (m,q) where m,q are positive integers . Then for any entire function g,*

$$\begin{aligned}(i)\ \lambda_g^{(p,q)}(f) &= \infty \text{ when } \rho_h^{(m,p)}(g)=0\ ,\\ (ii)\ \rho_g^{(p,q)}(f) &= \infty \text{ when } \lambda_h^{(m,p)}(g)=0\ ,\\ (iii)\ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \rho_h^{(m,p)}(g)=\infty\end{aligned}$$

and

$$(iv)\ \rho_g^{(p,q)}(f)=0 \text{ when } \lambda_h^{(m,p)}(g)=\infty,$$

where p is any positive integer.

Corollary 2.11. *Let g be an entire function with relative index-pair (m,p) where m,p are positive integers . Then for any entire function f,*

$$\begin{aligned}(i)\ \rho_g^{(p,q)}(f) &= 0 \text{ when } \rho_h^{(m,q)}(f)=0\ ,\\ (ii)\ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \lambda_h^{(m,q)}(f)=0\ ,\\ (iii)\ \rho_g^{(p,q)}(f) &= \infty \text{ when } \rho_h^{(m,q)}(f)=\infty\end{aligned}$$

and

$$(iv)\ \lambda_g^{(p,q)}(f)=\infty \text{ when } \lambda_h^{(m,q)}(f)=\infty\ ,$$

where q is any positive integer .

The following examples strengthens the conclusions of the above corollaries:

Example 2.12. *Let*

$$f = z^{11},\ g = 5z^3 \text{ and } h = 9z^4 .$$

Then

$$\rho_h^{(1,1)}(f) = \lambda_h^{(1,1)}(f) = \frac{11}{4} \text{ and } \rho_h^{(1,1)}(g) = \lambda_h^{(1,1)}(g) = \frac{3}{4} .$$

Now

$$\rho_g^{(1,1)}(f) = \frac{11}{3} = \frac{\rho_h^{(1,1)}(f)}{\lambda_h^{(1,1)}(g)} = \frac{\rho_h^{(1,1)}(f)}{\rho_h^{(1,1)}(g)} .$$

Similarly,

$$\lambda_g^{(1,1)}(f) = \frac{11}{3} = \frac{\rho_h^{(1,1)}(f)}{\lambda_h^{(1,1)}(g)} = \frac{\rho_h^{(1,1)}(f)}{\rho_h^{(1,1)}(g)} .$$

Hence

$$\lambda_g^{(1,1)}(f) = \rho_g^{(1,1)}(f) = \frac{\rho_h^{(1,1)}(f)}{\rho_h^{(1,1)}(g)} .$$

Again

$$\begin{aligned}\rho_f^{(1,1)}(g) &= \limsup_{r\to\infty}\frac{\log M_f^{-1}M_g(r)}{\log r} = \limsup_{r\to\infty}\frac{\log\left(5r^3\right)^{\frac{1}{11}}}{\log r}\\ &= \frac{3}{11} = \frac{\frac{3}{4}}{\frac{11}{4}}\end{aligned}$$

and

$$\begin{aligned}\lambda_f^{(1,1)}(g) &= \liminf_{r\to\infty}\frac{\log M_f^{-1}M_g(r)}{\log r} = \liminf_{r\to\infty}\frac{\log\left(5r^3\right)^{\frac{1}{11}}}{\log r}\\ &= \frac{3}{11} = \frac{\frac{3}{4}}{\frac{11}{4}} .\end{aligned}$$

Thus

$$\rho_g^{(1,1)}(f)\cdot\rho_f^{(1,1)}(g) = 1$$

and

$$\lambda_g^{(1,1)}(f)\cdot\lambda_f^{(1,1)}(g) = 1 .$$

Example 2.13. *Let*

$$f = \exp z^{13},\ g = \exp(13z)^{13} \text{ and } h = \exp\left(9z^4\right) .$$

Then

$$\rho_h^{(1,1)}(f) = \lambda_h^{(1,1)}(f) = \frac{13}{4} \text{ and } \rho_h^{(1,1)}(g) = \lambda_h^{(1,1)}(g) = \frac{13}{4}$$

and

$$\lambda_g^{(1,1)}(f) = \rho_g^{(1,1)}(f) = \lambda_f^{(1,1)}(g) = \rho_f^{(1,1)}(g) = 1 .$$

Example 2.14. *Let*

$$f = \exp z^{31},\ g = z^{25} \text{ and } h = \exp z^2 .$$

So

$$\rho_h^{(1,1)}(f) = \lambda_h^{(1,1)}(f) = \frac{31}{2} \text{ and } \rho_h^{(1,1)}(g) = \lambda_h^{(1,1)}(g) = 0 .$$

Now

$$\rho_g^{(1,1)}(f) = \limsup_{r\to\infty} \frac{\log\left(\exp r^{31}\right)^{\frac{1}{25}}}{\log r} = \infty$$

and

$$\lambda_g^{(1,1)}(f) = \liminf_{r\to\infty} \frac{\log\left(\exp r^{31}\right)^{\frac{1}{25}}}{\log r} = \infty .$$

Similarly,

$$\rho_f^{(1,1)}(g) = \limsup_{r\to\infty} \frac{\log\left(\log r^{25}\right)^{\frac{1}{31}}}{\log r} = 0$$

and

$$\lambda_f^{(1,1)}(g) = \liminf_{r\to\infty} \frac{\log\left(\log r^{25}\right)^{\frac{1}{31}}}{\log r} = 0 .$$

Example 2.15. *Let*

$$f = \exp z^3,\ g = \exp^{[2]} z^{45} \text{ and } h = \exp z^5 .$$

So

$$\rho_h^{(1,1)}(f) = \lambda_h^{(1,1)}(f) = \frac{3}{5} \text{ and } \rho_h^{(1,1)}(g) = \lambda_h^{(1,1)}(g) = \infty .$$

Now

$$\rho_g^{(1,1)}(f) = \limsup_{r\to\infty} \frac{\log\left(\log^{[2]} \exp(r)^3\right)^{\frac{1}{45}}}{\log r} = 0$$

and

$$\lambda_g^{(1,1)}(f) = \liminf_{r\to\infty} \frac{\log\left(\log^{[2]} \exp(r)^3\right)^{\frac{1}{45}}}{\log r} = 0 .$$

Similarly,

$$\rho_f^{(1,1)}(g) = \limsup_{r\to\infty} \frac{\log\left[\log \exp^{[2]} r^{45}\right]^{\frac{1}{3}}}{\log r} = \infty$$

and

$$\lambda_f^{(1,1)}(g) = \liminf_{r\to\infty} \frac{\log\left[\log \exp^{[2]} r^{45}\right]^{\frac{1}{3}}}{\log r} = \infty .$$

3. Conclusion

The main aim of the present paper is to revisit some growth properties of entire functions on the basis of their relative (p, q)-th order (relative (p, q)-th lower order) for any two positive integers p and q. Recently, Qiao and Deng [7] have studied the generalizations of the growth properties of analytic functions, harmonic functions and superharmonic functions. Further, Madych [4] have investigated some problems regarding the bounds on the growth of functions with their derivatives of certain finite order k in the space $L^p(R^n)$ where $k > \frac{n}{p}$. The notion involved in our paper may be reinvestigated in the light of the theories employed in [7] and [4].

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