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On Maps between Stone-Čech Compactifications Induced by Lattice Homomorphisms

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Abstract. Broverman has shown that if X and Y are Tychonoff spaces and $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ is a lattice homomorphism between the lattices of their zero-sets, then there is a continuous map $\tau\colon \beta X \to \beta Y$ induced by t. In this note we expound this idea and supplement Broverman's results by first showing that this phenomenon holds in the category of completely regular frames. Among results we obtain, which were not considered by Broverman, are necessary and sufficient conditions (in terms of properties of the map t) for the induced map τ to be (i) the inclusion of a subspace, (ii) surjective, and (iii) irreducible. We show that if X and Y are pseudocompact then t pulls back z-ultrafilters to z-ultrafilters if and only if $\mathrm{cl}_{\beta X}\, t(Z) = \tau^{\leftarrow}[\mathrm{cl}_{\beta Y} Z]$ for every $Z \in \mathbf{Z}(Y)$ if and only if t is σ-homomorphism.

1. Introduction

Let X and Y be Tychonoff spaces and $\mathbf{Z}(X)$ and $\mathbf{Z}(Y)$ be the lattices of their zero-sets. In [7], Broverman shows that any lattice homomorphism (throughout understood to preserve the bottom and the top elements) $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ induces a continuous map $\tau\colon \beta X \to \beta Y$. This is how he does it. Recall that, in the notation of Gillman and Jerison [9], for any $p \in \beta X$, $\boldsymbol{A}^p$ is the z-ultrafilter on X given by

$$Z \in \boldsymbol{A}^p \quad \iff \quad p \in \mathrm{cl}_{\beta X} Z.$$

Now $t^{\leftarrow}[\boldsymbol{A}^p]$ is a prime z-filter in Y, and is therefore contained in some unique z-ultrafilter $\boldsymbol{A}^q$ on Y. Broverman shows that the function $\tau\colon \beta X \to \beta Y$ defined by $\tau(p) = q$ is a continuous map.

Our approach in obtaining such an induced map between Stone-Čech compactifications of frames will be different. We will take the following categorical path. Let **DLat** denote the category of bounded distributive lattices and their homomorphisms. Recall that the ideal-lattice functor $\mathfrak{J}\colon \mathbf{DLat} \to \mathbf{Frm}$ sends $A \in \mathbf{DLat}$ to

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the frame $\mathfrak{I}A$ of ideals of A, and sends a lattice homomorphism $\phi\colon A \to B$ to the frame homomorphism $\mathfrak{I}\phi\colon \mathfrak{I}A \to \mathfrak{I}B$ given by

$$\mathfrak{I}\phi(I) = \{b \in B \mid b \leq \phi(a) \text{ for some } a \in I\}.$$

Now let L and M be completely regular frames and $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ be a lattice homomorphism. Then ϕ preserves the completely below relation, $\lll$, and hence $\mathfrak{I}\phi(I) \in \beta M$ whenever $I \in \beta L$. Since βL and βM are subframes of $\mathfrak{I}(\operatorname{Coz} L)$ and $\mathfrak{I}(\operatorname{Coz} M)$ respectively, it follows that the restriction of $\mathfrak{I}\phi$ to βL is a frame homomorphism into βM. We denote it by $\bar{\phi}$. The aim of this note is to explore some properties of the frame homomorphism $\bar{\phi}$ with the view to obtaining new results concerning the map $\tau\colon \beta X \to \beta Y$ induced by a lattice homomorphism $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$.

Here is a brief summary of the main results. We borrow the adjectives "dense" and "codense" from frames and use them to describe lattice homomorphisms in the same way as they describe frame homomorphisms. We show in Proposition 3.1 that $\bar{\phi}$ is one-one if and only if ϕ is dense. The topological upshot is that $\tau\colon \beta X \to \beta Y$ is surjective if and only if t is codense (Corollary 3.3). Dualising the term "uplifting" as used in [1], we say $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ is "deflating" if disjoint zero-sets in X are separated by images of disjoint zero-sets in Y. After proving that $\bar{\phi}$ is surjective precisely when ϕ is uplifting (Proposition 3.5), we deduce that $\tau\colon \beta X \to \beta Y$ is the inclusion of a subspace if and only if t is deflating (Corollary 3.6).

We close the first part of the paper by showing that $\bar{\phi}$ is $*$-dense (we will recall the definition at the appropriate time) exactly when ϕ is "inverse-dense", in the sense that the only ideal J for which $\phi^{\leftarrow}[J] = \{0\}$ is the zero ideal (Proposition 3.8). The resulting topological deduction is that when $\tau\colon \beta X \to \beta Y$ is surjective, then it is irreducible if and only if for every z-filter $\mathcal{F}$ in X which is not a singleton, there is a zero-set $Z \neq Y$ of Y such that $t(Z) \in \mathcal{F}$ (Corollary 3.9).

The second part of the paper has a more pronounced categorical flavour. A number of naturally occurring diagrams are shown to be commutative, culminating in the result that t pulls back z-ultrafilters to z-ultrafilters if and only if $\operatorname{cl}_{\beta X} t(Z) = \tau^{\leftarrow}[\operatorname{cl}_{\beta Y} Z]$ for every $Z \in \mathbf{Z}(Y)$ (Corollary 4.10). In particular, if X and Y are pseudocompact, then the result just stated holds when and only when t is a σ-homomorphism in the sense of Broverman [7] (Corollary 4.15).

2. Frame-Theoretic Resources

We collect a few facts we shall need regarding frames and σ-frames, and refer to [11] and [13] for the general theory of frames, and [6] for some information about σ-frames. We denote the top element and the bottom element of a lattice L by 1_L and 0_L respectively, dropping the subscripts if L is clear from the context. Our notation is fairly standard. We write $\mathfrak{O}X$ for the frame of open sets of a topological space X. All our spaces are Tychonoff, and our reference for notions such as zero-set, and so on, is [9]. Also, all our frames are completely regular.

An element p of a frame is called a *point* if $p \neq 1$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We write $\operatorname{Pt}(L)$ for the set of all points of L. The points of a regular frame are precisely those elements which are maximal strictly below the top. Any compact regular frame has *enough points*, in the sense that every element is the meet of the points above it.

An element a of a frame L is a *cozero element* if there is a sequence (a_n) in L such that $a_n \lll a_{n+1}$ for every n, and $a = \bigvee a_n$. The set of all cozero elements of L is called the *cozero part* of L and is denoted by $\operatorname{Coz} L$. It is a σ-frame which generates L if and only if L is completely regular. If $a \lll b$ in L, then there is an $s \in \operatorname{Coz} L$ such that $a \wedge s = 0$ and $s \vee b = 1$. For further properties of the cozero part of a frame we urge the reader to consult [4]. $\operatorname{Coz}\colon \mathbf{CRFrm} \to \mathbf{Reg\sigma Frm}$ is a functor with a left adjoint, $\mathfrak{H}\colon \mathbf{Reg\sigma Frm} \to \mathbf{CRFrm}$. See [5] for details concerning these functors.

A frame homomorphism is *dense* if it maps only the bottom element to the bottom element. We write h_* for the right adjoint of a frame homomorphism h. We view βL, the Stone-Čech compactification of L, as the frame of regular ideals of $\operatorname{Coz} L$. The right adjoint of the join map $j_L\colon \beta L \to L$ is denoted by r_L. By the way we have realised βL, $r_L(a) = \{c \in \operatorname{Coz} L \mid c \lll a\}$, for any $a \in L$. Every frame homomorphism $h\colon L \to M$ has the *Stone extension*, $\beta h \colon \beta L \to \beta M$, which is the unique frame homomorphism making the following square commute.

$$\begin{array}{ccc} \beta L & \xrightarrow{\ \beta h\ } & \beta M \\ {\scriptstyle j_L}\downarrow & & \downarrow{\scriptstyle j_M} \\ L & \xrightarrow{\ \ h\ \ } & M \end{array}$$

We shall use the following notation. If $g \colon L \to M$ is a frame homomorphism, we write g' for the σ-frame homomorphism $g' \colon \operatorname{Coz} L \to \operatorname{Coz} M$ which maps as g. This accords with Broverman's usage of the prime on a continuous map.

3. Some Properties of the Map $\bar{\phi}$

Throughout this section L and M are completely regular frames, $\phi \colon \operatorname{Coz} L \to \operatorname{Coz} M$ is a lattice homomorphism, and $\bar{\phi} \colon \beta L \to \beta M$ is the frame homomorphism defined above. We give necessary and sufficient conditions, in terms of properties of ϕ, for the map $\bar{\phi}$ to be (i) one-one, (ii) onto, and (iii) $*$-dense. Let us recall what this last term means. In their study of patch-generated frames, Hager and Martínez [10] call a frame homomorphism $h \colon L \to M$ *$*$-dense* if, for any $b \in M$,

$$h_*(b) = 0 \quad \Longrightarrow \quad b = 0.$$

Continuous maps $f \colon X \to Y$ for which $\mathfrak{O}f \colon \mathfrak{O}Y \to \mathfrak{O}X$ is $*$-dense occur quite naturally. Indeed, recall that a surjective continuous map is called *irreducible* if it sends no proper closed subset of its domain onto its codomain. Since, for any continuous map $f \colon X \to Y$ and any $U \in \mathfrak{O}X$,

$$(\mathfrak{O}f)_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]},$$

it follows that

a closed continuous surjection $f \colon X \to Y$ is irreducible iff the frame map $\mathfrak{O}f \colon \mathfrak{O}Y \to \mathfrak{O}X$ is $*$-dense.

We extend the meaning of the term "dense" by defining a lattice homomorphism to be *dense* if the zero of its domain is the only element mapped to zero. We remind the reader that a frame homomorphism between compact regular frames is dense precisely when it is one-one.

Proposition 3.1. *The homomorphism $\bar{\phi} \colon \beta L \to \beta M$ is one-one iff ϕ is dense.*

Proof. We start by calculating the right adjoint of $\bar{\phi}$. We claim that, for any $J \in \beta M$,

$$\begin{aligned} \bar{\phi}_*(J) &= \bigvee \{r_L(c) \mid c \in \operatorname{Coz} L \text{ and } \phi(c) \in J\} \\ &= \bigcup \{r_L(c) \mid c \in \operatorname{Coz} L \text{ and } \phi(c) \in J\}. \end{aligned}$$

Observe that the join is a union because the collection whose join is displayed is directed as ϕ is a lattice homomorphism. Denote the displayed join by T. Let $c \in \operatorname{Coz} L$ be such that $\phi(c) \in J$. Consider any $z \in \bar{\phi}(r_L(c))$. There is a $d \ll c$ such that $z \le \phi(d)$. Thus, $z \le \phi(c) \in J$, which implies $z \in J$. Therefore $\bar{\phi}(T) \le J$. Now let I be any element of βL with $\bar{\phi}(I) \le J$. Let $a \in I$, and pick $b \in I$ with $a \ll b$. Then $a \in r_L(b)$, and since $\phi(a) \le \phi(b) \in J$, it follows that $I \le T$. In all then this proves that $T = \bar{\phi}_*(J)$. Thus, in particular,

$$\bar{\phi}_*(0_{\beta M}) = \bigvee \{r_L(c) \mid \phi(c) = 0\},$$

so that

$$\bar{\phi}_*(0_{\beta M}) = 0_{\beta L} \quad \Longleftrightarrow \quad \phi(c) = 0 \text{ implies } c = 0.$$

Therefore $\bar{\phi}$ is dense if and only if ϕ is dense. This proves the result. □

The description of the right adjoint of $\bar{\phi}$ in the foregoing proof enables us to show that, for a lattice homomorphism $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$, the map $\tau\colon \beta X \to \beta Y$ is obtainable from some $\bar{\phi}$ via the spectrum functor $\Sigma\colon \mathbf{Frm} \to \mathbf{Top}$. To do this we shall view βX (and also βY) as the space $\Sigma\beta(\mathfrak{O}X)$, where the spectrum is taken as the set of prime elements of $\beta(\mathfrak{O}X)$, that is, the maximal regular ideals of $\mathrm{Coz}(\mathfrak{O}X)$.

A lattice homomorphism $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ induces a lattice homomorphism

$$\hat{t}\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X) \quad \text{by} \quad U \mapsto X \smallsetminus t(Y \smallsetminus U);$$

and a lattice homomorphism $s\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X)$ induces a lattice homomorphism

$$\tilde{s}\colon \mathbf{Z}(Y) \to \mathbf{Z}(X) \quad \text{by} \quad F \mapsto X \smallsetminus s(Y \smallsetminus F).$$

The correspondences $t \mapsto \hat{t}$ and $s \mapsto \tilde{s}$ are one-one onto, and are inverses to each other.

Proposition 3.2. *Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ be a lattice homomorphism. Write h for the map $\bar{\hat{t}}\colon \beta(\mathfrak{O}Y) \to \beta(\mathfrak{O}X)$. Then $\tau = \Sigma h$.*

Proof. Let $p \in \beta X$ and pick the unique $q \in \beta Y$ with $t^{\leftarrow}[\boldsymbol{A}^p] \subseteq \boldsymbol{A}^q$. Note that, for any $Z \in \mathbf{Z}(X)$,

$$Z \in \boldsymbol{A}^p \quad \Longleftrightarrow \quad r_{\mathfrak{O}X}(X \smallsetminus Z) \le p.$$

Now

$$h_*(p) = \bigvee \{r_{\mathfrak{O}Y}(V) \mid V \in \mathrm{Coz}(\mathfrak{O}Y) \text{ and } \hat{t}(V) \in p\}.$$

Consider any $V \in \mathrm{Coz}(\mathfrak{O}Y)$ with $\hat{t}(V) \in p$. Then $r_{\mathfrak{O}X}\big(\hat{t}(V)\big) \le p$; that is,

$$r_{\mathfrak{O}X}\big(X \smallsetminus t(Y \smallsetminus V)\big) \le p,$$

and hence $t(Y \smallsetminus V) \in \boldsymbol{A}^p$. Consequently,

$$Y \smallsetminus V \in t^{\leftarrow}[\boldsymbol{A}^p] \subseteq \boldsymbol{A}^q.$$

Thus, $r_{\mathfrak{O}Y}(V) \le q$, whence $h_*(p) \le q$, and therefore $h_*(p) = q$ because $h_*(p)$ is a maximal element in $\beta(\mathfrak{O}Y)$. □

We say a lattice homomorphism $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ is *codense* if, for any $K \in \mathbf{Z}(Y)$, $t(K) = X$ implies $K = Y$. Clearly, t is codense if and only if the associated $\hat{t}\colon \mathrm{Coz}(\mathfrak{O}Y) \to \mathrm{Coz}(\mathfrak{O}X)$ is dense.

Corollary 3.3. *Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ be a lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is onto iff t is codense.*

In [8] we showed that, for any surjective frame homomorphism $h\colon L \to M$, the Stone extension $\beta h\colon \beta L \to \beta M$ is surjective precisely when h is a C^*-quotient map, as defined in [2]. A closer look at the proof of the implication (1) ⇒ (2) in [8, Proposition 2.1] reveals that the property of the map h used in the proof is that, in the language of Ball, Hager and Walters-Wayland [1], $h'\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$ is uplifting. Using the same term (we will define it formally shortly) for a lattice homomorphism $\phi\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$, it is reasonable to expect that the surjectivity of the map $\bar{\phi}\colon \beta L \to \beta M$ should have something to do with the uplifting property. We show that indeed it does.

Definition 3.4. *A lattice homomorphism $\phi\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$ is* uplifting *if whenever $u \vee v = 1$ in $\mathrm{Coz}\, M$, there exist $a, b \in \mathrm{Coz}\, L$ such that $a \vee b = 1$, $\phi(a) \le u$ and $\phi(b) \le v$. We say a lattice homomorphism $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ is* deflating *if, for any disjoint zero-sets E and F of X, there are disjoint zero-sets G and H of Y such that $E \subseteq t(G)$ and $F \subseteq t(H)$.*

It should be clear that t is deflating if and only if the associated map $\hat{t}\colon \operatorname{Coz}(\mathfrak{O}Y) \to \operatorname{Coz}(\mathfrak{O}X)$ is uplifting. In the proof that follows we shall use the fact that if $c \vee b = 1$ in $\operatorname{Coz} L$, then $a \vee c = 1$ for some $a \in \operatorname{Coz} L$ with $a \ll b$.

Proposition 3.5. *The following conditions are equivalent for a lattice homomorphism* $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$.

1. $\bar{\phi}\colon \beta L \to \beta M$ *is onto.*
2. ϕ *is uplifting.*
3. *Whenever* $u \ll v$ *in* $\operatorname{Coz} M$, *there are elements* $a \ll b$ *in* $\operatorname{Coz} L$ *such that* $u \le \phi(a)$ *and* $\phi(b) \le v$.

Proof. (2) ⇔ (3): Suppose ϕ is uplifting, and let $u \ll v$ in $\operatorname{Coz} M$. Pick $s \in \operatorname{Coz} M$ such that $u \wedge s = 0$ and $s \vee v = 1$. By the uplifting property, there exist $c, b \in \operatorname{Coz} L$ such that $c \vee b = 1$, $\phi(c) \le s$ and $\phi(b) \le v$. Find $a \ll b$ in $\operatorname{Coz} L$ such that $c \vee a = 1$. Now, $u \wedge \phi(c) = 0$ since $\phi(c) \le s$. Since $\phi(c) \vee \phi(a) = 1$, this implies $u \le \phi(a)$. Therefore a and b are elements of $\operatorname{Coz} L$ with the required property.

Conversely, suppose the stated condition holds, and let $u \vee v = 1$ in $\operatorname{Coz} M$. Take $w \in \operatorname{Coz} M$ such that $w \ll u$ and $w \vee v = 1$. By the current hypothesis, there are elements $a \ll b$ in $\operatorname{Coz} L$ such that $w \le \phi(a)$ and $\phi(b) \le u$. Pick $s \in \operatorname{Coz} L$ such that $a \wedge s = 0$ and $s \vee b = 1$. Then $\phi(s) \wedge \phi(a) = 0$, which implies $\phi(s) \wedge w = 0$, and hence $\phi(s) \le v$. So the cover $\{b, s\}$ witnesses the uplifting of $\{u, v\}$.

(1) ⇔ (2): Assume ϕ is uplifting, and let $J \in \beta M$. We show that $J \le \bar{\phi}\bar{\phi}_*(J)$, which will imply equality, and hence that $\bar{\phi}$ is onto. Let $u \in J$, and pick $v \in J$ such that $u \ll v$. Since ϕ is uplifting, there are elements $a \ll b$ in $\operatorname{Coz} L$ such that $u \le \phi(a)$ and $\phi(b) \le v$. Now recall what $\bar{\phi}_*(J)$ looks like. Since $a \in r_L(b)$ and $\phi(b) \in J$, as $\phi(b) \le v \in J$, it follows that $a \in \bar{\phi}_*(J)$, and hence $u \in \bar{\phi}\bar{\phi}_*(J)$. Therefore $J \le \bar{\phi}\bar{\phi}_*(J)$, and hence equality.

Conversely, assume $\bar{\phi}$ is onto, and let $u \ll v$ in $\operatorname{Coz} M$. Since $\bar{\phi}$ is onto, $\bar{\phi}\bar{\phi}_*(r_M(v)) = r_M(v)$. Since $u \in r_M(v)$, there is a $t \in \bar{\phi}_*(r_M(v))$ such that $u \le \phi(t)$. Now, since

$$\begin{aligned}\bar{\phi}_*(r_M(v)) &= \bigcup\{r_L(c) \mid c \in \operatorname{Coz} L \text{ and } \phi(c) \in r_M(v)\}\\ &= \bigcup\{r_L(c) \mid c \in \operatorname{Coz} L \text{ and } \phi(c) \ll v\},\end{aligned}$$

it follows that $t \ll c$ for some c with $\phi(c) \ll v$. Consequently, t and c are elements of $\operatorname{Coz} L$ such that $t \ll c$, $u \le \phi(t)$ and $\phi(c) \le v$. Therefore ϕ is uplifting. □

Recall from [11, Lemma II 2.1] that if $f\colon X \to Y$ is a continuous map between Tychonoff spaces, then $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is onto if and only if f is the inclusion of a subspace.

Corollary 3.6. *Let* $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ *be a lattice homomorphism. The induced map* $\tau\colon \beta X \to \beta Y$ *is the inclusion of a subspace iff* t *is deflating.*

Remark 3.7. *Conspicuous by its absence in Proposition 3.5 is an assertion that an equivalent condition is that* ϕ *be onto. That would of course be false because if* $S \subseteq X$ *is a z-embedded subspace which is not* C^**-embedded, and* $h\colon \mathfrak{O}X \to \mathfrak{O}S$ *is the frame homomorphism* $U \mapsto U \cap S$, *then the lattice homomorphism (actually,* σ*-frame homomorphism)* $h'\colon \operatorname{Coz}(\mathfrak{O}X) \to \operatorname{Coz}(\mathfrak{O}S)$ *is surjective but not uplifting, in view of this proposition and [8, Proposition 2.1]. In fact, Ball, Hager and Walters-Wayland show in [1, Proposition 1.1] that any uplifting* σ*-frame homomorphism into a regular* σ*-frame is surjective, and give an example ([1, Example 1.2]) to show that the converse fails.*

We now turn to $*$-density of the map $\bar{\phi}$. Let us call a lattice homomorphism $\psi\colon A \to B$ *inverse-dense* if, for any ideal J of B, $\psi^{\leftarrow}[J] = \{0\}$ implies $J = \{0\}$.

Proposition 3.8. *The map* $\bar{\phi}\colon \beta L \to \beta M$ *is* $*$*-dense iff* ϕ *is inverse-dense.*

Proof. Suppose $\bar{\phi}$ is $*$-dense. Let J be an ideal in $\operatorname{Coz} M$ such that $\phi^{\leftarrow}[J] = \{0\}$. The set

$$\tilde{J} = \{z \in \operatorname{Coz} M \mid z \ll u \text{ for some } u \in J\}$$

is easily checked to be an element of βM with $\bar{\phi}_*(\tilde{J}) = 0_{\beta L}$. Therefore, by $*$-density, $\tilde{J} = 0_{\beta L}$, which, by complete regularity, implies $J = \{0\}$. Therefore ϕ is inverse-dense.

Conversely, let $I \in \beta M$ be such that $\bar{\phi}_*(I) = 0_{\beta L}$; that is,

$$\bigvee\{r_L(c) \mid c \in \operatorname{Coz} L \text{ and } c \in \phi^{\leftarrow}[J]\} = 0_{\beta L}.$$

This implies $\phi^{\leftarrow}[J] = \{0\}$, whence $J = \{0\}$ by inverse-density. Therefore $\bar{\phi}$ is $*$-dense. □

By a *nontrivial* z-filter we mean one which is not a singleton. Observe that a lattice homomorphism $s\colon \operatorname{Coz}(\mathfrak{O}Y) \to \operatorname{Coz}(\mathfrak{O}X)$ is inverse-dense if and only if the corresponding $\tilde{s}\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ has the property that, for every nontrivial z-filter $\mathcal{F}$ in X, there is a $Z \in \mathbf{Z}(Y)$ with $Z \neq Y$ such that $\tilde{s}(Z) \in \mathcal{F}$. The following result then follows by all that has come before it.

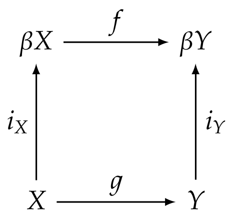
Corollary 3.9. *Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ be a codense lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is irreducible iff for every nontrivial z-filter $\mathcal{F}$ in X, there is a zero-set $Z \neq Y$ of Y such that $t(Z) \in \mathcal{F}$.*

4. Other Properties

Broverman observes in [7] that distinct homomorphisms from $\mathbf{Z}(Y)$ to $\mathbf{Z}(X)$ may induce the same continuous map $\beta X \to \beta Y$. We will see, however, that distinct σ-homomorphisms cannot induce the same map. Among other results we prove in this section is the following. Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ be a lattice homomorphism, and suppose that the induced map $\tau\colon \beta X \to \beta Y$ is such that $\tau[X] \subseteq Y$, so that we have a continuous map $\tau|_X\colon X \to Y$ (the restriction of τ to X), and the lattice homomorphism $\mathbf{Z}(Y) \to \mathbf{Z}(X)$, given by $Z \mapsto (\tau|_X)^{\leftarrow}[Z]$, it induces. Then $t = (\tau|_X)^{\leftarrow}$ if and only if t is a σ-homomorphism.

Because we want this result to be a corollary of a more general result in **CRFrm**, we start by observing the following categorical way of saying the restriction to X of a continuous map $\beta X \to \beta Y$ maps into Y.

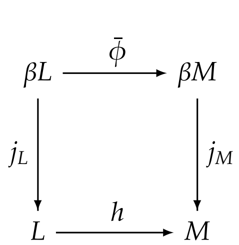
Lemma 4.1. *Let $f\colon \beta X \to \beta Y$ be a continuous map. Then $f[X] \subseteq Y$ iff there is a continuous map $g\colon X \to Y$ such that the square below commutes.*



Proof. If such a map g exists, then, for any $x \in X$, $f(x) = g(x) \in Y$, so that $f[X] \subseteq Y$. Conversely, take $g = f|_X$. □

Recall our notation that if $h\colon L \to M$ is a frame homomorphism, then $h'\colon \operatorname{Coz} L \to \operatorname{Coz} M$ is the restriction of h to $\operatorname{Coz} L$ as mapping into $\operatorname{Coz} M$.

Proposition 4.2. *Let $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ be a lattice homomorphism. Suppose that there is a frame homomorphism $h\colon L \to M$ which makes the diagram*



commute. Then $h' = \phi$ iff ϕ is a σ-frame homomorphism.

Proof. The left-to-right implication is trivial. Conversely, assume ϕ is a σ-frame homomorphism. Since the Stone extension of h, βh, is uniquely determined by h, it follows from the commutativity of the diagram that $\beta h = \bar{\phi}$. Let $a \in \operatorname{Coz} L$. Then $(\beta h)(r_L(a)) = \bar{\phi}(r_L(a))$. Take a sequence of cozero elements (a_n) such that $a_n \ll a_{n+1}$ for every n and $a = \bigvee a_n$. Since for any n we have $\phi(a_n) \leq \phi(a_{n+1})$ and $a_{n+1} \in r_L(a)$, it follows that

$$\begin{aligned} \phi(a) = \bigvee \phi(a_n) &\leq \bigvee \{c \in \operatorname{Coz} M \mid c \leq \phi(t) \text{ for some } t \in r_L(a)\} \\ &= \bigvee \bar{\phi}(r_L(a)) \\ &= \bigvee (\beta h)(r_L(a)) \\ &= h(a). \end{aligned}$$

Therefore $\phi = h'$. □

Corollary 4.3. *Let $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ be a lattice homomorphism such that the induced map $\tau\colon \beta X \to \beta Y$ maps X into Y. Then $t = (\tau|_X)^{\leftarrow}$ iff t is a σ-homomorphism.*

Remark 4.4. *The calculation in the proof of Proposition 4.2 enables us to show that distinct σ-homomorphisms mapping $\mathbf{Z}(Y)$ to $\mathbf{Z}(X)$ cannot induce the same continuous map $\beta X \to \beta Y$. Indeed, suppose $\bar{\phi} = \bar{\psi}$ for two σ-frame homomorphisms $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ and $\psi\colon \operatorname{Coz} L \to \operatorname{Coz} M$. Then, for any $a \in \operatorname{Coz} L$,*

$$\phi(a) = \bigvee \bar{\phi}(r_L(a)) = \bigvee \bar{\psi}(r_L(a)) = \psi(a),$$

so that $\phi = \psi$.

Remark 4.5. *Observe that if a is a complemented element in L, and $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ is a lattice homomorphism, then $\bigvee \bar{\phi}(r_L(a)) = \phi(a)$. Consequently, if Y is a P-space then distinct lattice homomorphisms mapping $\mathbf{Z}(Y)$ to $\mathbf{Z}(X)$ induce distinct continuous maps $\beta X \to \beta Y$, for any Tychonoff space X.*

We now consider another result concerning commuting squares which holds exactly when the lattice homomorphism $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ is a σ-frame homomorphism. Let $h\colon L \to M$ be a frame homomorphism. Applying the functor $\operatorname{Coz}\colon \mathbf{CRFrm} \to \mathbf{Reg\sigma Frm}$ to the commutative square

$$\begin{array}{ccc} \beta L & \xrightarrow{\ \beta h\ } & \beta M \\ {\scriptstyle j_L}\downarrow & & \downarrow{\scriptstyle j_M} \\ L & \xrightarrow{\ h\ } & M \end{array}$$

we obtain the following commutative square in $\mathbf{Reg\sigma Frm}$.

$$\begin{array}{ccc} \operatorname{Coz} \beta L & \xrightarrow{\ (\beta h)'\ } & \operatorname{Coz} \beta M \\ {\scriptstyle j'_L}\downarrow & & \downarrow{\scriptstyle j'_M} \\ \operatorname{Coz} L & \xrightarrow{\ h'\ } & \operatorname{Coz} M \end{array}$$

Now let $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ be a lattice homomorphism, and consider the square

$$\begin{array}{ccc} \operatorname{Coz} \beta L & \xrightarrow{\ \bar{\phi}'\ } & \operatorname{Coz} \beta M \\ {\scriptstyle j'_L}\downarrow & & \downarrow{\scriptstyle j'_M} \\ \operatorname{Coz} L & \xrightarrow{\ \phi\ } & \operatorname{Coz} M \end{array}$$

There is no reason why this square should always commute. Indeed, the following proposition shows that it commutes exactly when ϕ is a σ-frame homomorphism.

Proposition 4.6. *Let $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ be a lattice homomorphism. The square immediately above commutes iff ϕ is a σ-frame homomorphism.*

Proof. Suppose the diagram commutes, and let (s_n) be an increasing sequence in $\operatorname{Coz} L$. Define the set

$$J = \{c \in \operatorname{Coz} L \mid c \ll s_n \text{ for some } n\}.$$

It is routine to check that $J \in \beta L$, and that, as an ideal of $\operatorname{Coz} L$, J is countably generated. Therefore, by [3, Lemma 1], $J \in \operatorname{Coz} \beta L$. Now, the commutativity of the square implies

$$\bigvee \bar{\phi}\big(J\big) = \phi\Big(\bigvee J\Big).$$

Clearly, $\bigvee J = \bigvee s_n$; and hence, since

$$\bar{\phi}\big(J\big) = \{d \in \operatorname{Coz} M \mid d \le \phi(z) \text{ for some } z \in J\},$$

it is easy to see that $\bigvee \bar{\phi}(J) = \bigvee \phi(s_n)$. Therefore ϕ preserves directed countable joins, and hence all countable joins because it preserves finite ones. Thus, ϕ is a σ-frame homomorphism.

Conversely, suppose ϕ is a σ-frame homomorphism. Using [3, Lemma 1] again, a straightforward diagram chase, taking into cognisance how $\bar{\phi}$ maps, shows that the diagram commutes. □

Remark 4.7. *An alternative (albeit rather long-winded) way of seeing that if ϕ is a σ-frame homomorphism then the diagram commutes is categorical, and goes as follows. Apply the functor $\mathfrak{H}$: **RegσFrm** → **CRFrm** to obtain the frame homomorphism $\mathfrak{H}\phi\colon \mathfrak{H}(\operatorname{Coz} L) \to \mathfrak{H}(\operatorname{Coz} M)$, then lift this map to the Stone-Čech compactifications to obtain the following commutative square.*

$$\begin{array}{ccc} \beta(\mathfrak{H}(\operatorname{Coz} L)) & \longrightarrow & \beta(\mathfrak{H}(\operatorname{Coz} M)) \\ \downarrow & & \downarrow \\ \mathfrak{H}(\operatorname{Coz} L) & \longrightarrow & \mathfrak{H}(\operatorname{Coz} M) \end{array}$$

Since $\beta(\mathfrak{H}(\operatorname{Coz} L)) \cong \beta L$, applying the functor Coz *to this square yields the result.*

We now turn to the following question. If $t\colon \mathbf{Z}(Y) \to \mathbf{Z}(X)$ is a lattice homomorphism, when do we have that

$$t^{\leftarrow}[\boldsymbol{A}^p] = \boldsymbol{A}^{\tau(p)} \text{ for every } p \in \beta X?$$

This is not addressed in [7]. As has been the practice throughout, the answer to this question will be a corollary to a result in frames. We start by recalling from [12] the description of maximal ideals of $\operatorname{Coz} L$ in terms of points of βL. For each $I \in \beta L$ define the subset L^I of $\operatorname{Coz} L$ by

$$L^I = \{c \in \operatorname{Coz} L \mid r_L(c) \le I\}.$$

We then have the following result from [12].

Lemma 4.8. *An ideal of* $\operatorname{Coz} L$ *is maximal iff it is of the form L^I, for some $I \in \operatorname{Pt}(\beta L)$.*

Observe that if $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ is a lattice homomorphism, then

$$\bar{\phi}(r_L(c)) \subseteq r_M(\phi(c)) \text{ for every } c \in \operatorname{Coz} L.$$

Indeed, for any $u \in \bar{\phi}(r_L(c))$, $u \ll \phi(d)$ for some $d \in r_L(c)$. Thus $u \ll \phi(c)$, and so $u \in r_M(\phi(c))$. Given a lattice homomorphism $\psi\colon A \to B$ and an ideal I in B, we adopt ring-theoretic nomenclature and say the ideal $\psi^{\leftarrow}[I]$ is the *contraction* of I by ψ.

Proposition 4.9. *A lattice homomorphism $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ contracts maximal ideals to maximal ideals iff $\bar{\phi}r_L = r_M\phi$ on $\operatorname{Coz} L$.*

Proof. Let us first show that, for any $J \in \mathrm{Pt}(\beta M)$, if $\phi^{\leftarrow}[M^J]$ is a maximal ideal in $\operatorname{Coz} L$, then $\phi^{\leftarrow}[M^J] = L^{\bar{\phi}_*(J)}$. Let $a \in \phi^{\leftarrow}[M^J]$. Then $r_M(\phi(a)) \leq J$, and hence $\bar{\phi}(r_L(a)) \leq J$ by what we observed above. Thus $r_L(a) \leq \bar{\phi}_*(J)$, so that $a \in L^{\bar{\phi}_*(J)}$, and consequently $\phi^{\leftarrow}[M^J] \subseteq L^{\bar{\phi}_*(J)}$. Since $L^{\bar{\phi}_*(J)}$ is a proper ideal, it follows by maximality that $\phi^{\leftarrow}[M^J] = L^{\bar{\phi}_*(J)}$.

Now suppose $\bar{\phi}r_L = r_M\phi$ on $\operatorname{Coz} L$, and let $I \in \mathrm{Pt}(\beta M)$. We must show that $\phi^{\leftarrow}[M^I] = L^{\bar{\phi}_*(I)}$. As already seen, the inclusion $\phi^{\leftarrow}[M^I] \subseteq L^{\bar{\phi}_*(I)}$ always holds. For the reverse inclusion, let $v \in L^{\bar{\phi}_*(I)}$. Then $r_L(v) \leq \bar{\phi}_*(I)$. Thus,

$$\bar{\phi}(r_L(v)) \leq \bar{\phi}(\bar{\phi}_*(I)) \leq I,$$

which, by hypothesis, implies $r_M(\phi(v)) \leq I$, so that $\phi(v) \in M^I$, and hence $v \in \phi^{\leftarrow}[M^I]$. This establishes the other inclusion. Therefore ϕ contracts maximal ideals to maximal ideals.

Conversely, suppose ϕ contracts maximal ideals to maximal ideals, and let $c \in \operatorname{Coz} L$. It suffices to show that $r_M(\phi(c)) \leq \bar{\phi}(r_L(c))$. If $\bar{\phi}(r_L(c)) = 1_{\beta L}$, there is nothing to prove. So suppose $\bar{\phi}(r_L(c)) \neq 1_{\beta L}$, and consider any point I of βM such that $\bar{\phi}(r_L(c)) \leq I$. Then $r_L(c) \leq \bar{\phi}_*(I)$, and hence $c \in L^{\bar{\phi}_*(I)} = \phi^{\leftarrow}[M^I]$. Thus, $\phi(c) \in M^I$, and so $r_M(\phi(c)) \leq I$. In view of the fact that

$$\bar{\phi}(r_L(c)) = \bigwedge\{J \mid J \text{ is a point of } \beta M \text{ with } \bar{\phi}(r_L(c)) \leq J\},$$

it follows that $r_M(\phi(c)) \leq \bar{\phi}(r_L(c))$, and hence equality. □

Let $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ be a lattice homomorphism and $p \in \beta X$. If $t^{\leftarrow}[\boldsymbol{A}^p]$ is a z-ultrafilter, then it is $\boldsymbol{A}^{\tau(p)}$. Therefore we have the following corollary.

Corollary 4.10. *Let $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ be a lattice homomorphism. Then $t^{\leftarrow}[\boldsymbol{A}^p] = \boldsymbol{A}^{\tau(p)}$ for every $p \in \beta X$ iff $\mathrm{cl}_{\beta X}\, t(Z) = \tau^{\leftarrow}[\mathrm{cl}_{\beta Y} Z]$ for every $Z \in \boldsymbol{Z}(Y)$.*

Remark 4.11. *If in the latter part of [7, Proposition 3.2] one takes $Z = Y$ and assumes τ is surjective, then one obtains the result in the foregoing corollary. It will be noted that in this corollary we did not impose a condition on t that would ensure τ is surjective, so the corollary sharpens the case $Z = Y$ in [7, Proposition 3.2].*

In certain instances the lattice homomorphisms $\operatorname{Coz} L \to \operatorname{Coz} M$ which contract maximal ideals to maximal ideals are precisely the σ-frame homomorphisms. To present such an instance we recall that a frame is *pseudocompact* precisely when every countable cover by cozero elements has a finite subcover. We need two lemmas.

Lemma 4.12. *Let L be a pseudocompact frame and $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ be a lattice homomorphism. If ϕ contracts maximal ideals to maximal ideals, then ϕ is a σ-frame homomorphism.*

Proof. Let (a_n) be a sequence in $\operatorname{Coz} L$, and put $a = \bigvee a_n$. We shall be done if we can show that $\phi(a) \leq \bigvee \phi(a_n)$. Since ϕ contracts maximal ideals to maximal ideals, we have $r_M(\phi(a)) = \bar{\phi}(r_L(a))$, by Proposition 4.9, so that, on taking joins in M, we get

$$\phi(a) = \bigvee \bar{\phi}(r_L(a)) = \bigvee \Big\{c \in \operatorname{Coz} M \mid c \leq \phi(u) \text{ for some } u \ll \bigvee a_n\Big\}.$$

Consider any $c \in \operatorname{Coz} M$ with $c \leq \phi(u)$ for some $u \ll \bigvee a_n$. Pick $s \in \operatorname{Coz} L$ such that $u \wedge s = 0$ and $s \vee \bigvee a_n = 1$. Since L is pseudocompact, there are finitely many indices $n_1, \ldots, n_k$ such that

$$s \vee a_{n_1} \vee \cdots \vee a_{n_k} = 1.$$

Then $u \leq a_{n_1} \vee \cdots \vee a_{n_k}$, and hence $c \leq \phi(a_{n_1}) \vee \cdots \vee \phi(a_{n_k})$. This shows that $\phi(a) \leq \bigvee \phi(a_n)$, and hence ϕ is a σ-frame homomorphism. □

Lemma 4.13. *Let M be a pseudocompact frame. If* $\phi\colon \operatorname{Coz} L \to \operatorname{Coz} M$ *is a* σ*-frame homomorphism, then* ϕ *contracts maximal ideals to maximal ideals.*

Proof. We show that $\bar{\phi} r_L = r_M \phi$ on $\operatorname{Coz} L$. Let $a \in \operatorname{Coz} L$, and take cozero elements a_n such that $a_n \ll a_{n+1}$ and $a = \bigvee a_n$. Let $u \in r_M(\phi(a))$. Then $u \ll \bigvee \phi(a_n)$. Pick $s \in \operatorname{Coz} M$ such that $u \wedge s = 0$ and $s \vee \bigvee \phi(a_n)$. The pseudocompactness of M yields an index n such that $s \vee \phi(a_n) = 1$. Thus,

$$u \leq \phi(a_n) \leq \phi(a_{n+1}) \quad \text{and} \quad a_{n+1} \in r_L(a),$$

which then shows that $u \in \bar{\phi}(r_L(a))$. Therefore $\bar{\phi}(r_L(a)) = r_M(\phi(a))$, and hence ϕ contracts maximal ideals to maximal ideals by Proposition 4.9. □

Proposition 4.14. *A lattice homomorphism between cozero parts of pseudocompact frames contracts maximal ideals to maximal ideals iff it is a* σ*-frame homomorphism.*

Pseudocompactness is a "conservative" notion. That is, a Tychonoff space is pseudocompact if and only if the frame of its open sets is pseudocompact. Therefore we have the following corollary.

Corollary 4.15. *Let X and Y be pseudocompact Tychonoff spaces. The following conditions are equivalent for a lattice homomorphism* $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$.

1. $t^{\leftarrow}[\boldsymbol{A}^p] = \boldsymbol{A}^{\tau(p)}$ *for every* $p \in \beta X$.
2. $\operatorname{cl}_{\beta X} t(Z) = \tau^{\leftarrow}[\operatorname{cl}_{\beta Y} Z]$ *for every* $Z \in \boldsymbol{Z}(Y)$.
3. t *is a* σ*-homomorphism.*

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