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Extending the CGLS Method for Finding the Least Squares Solutions of General Discrete-Time Periodic Matrix Equations

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Abstract. The periodic matrix equations are strongly related to analysis of periodic control systems for various engineering and mechanical problems. In this work, a matrix form of the conjugate gradient for least squares (MCGLS) method is constructed for obtaining the least squares solutions of the general discrete-time periodic matrix equations

$$\sum_{j=1}^{t}(A_{i,j}X_{i,j}B_{i,j}+C_{i,j}X_{i+1,j}D_{i,j})=M_i,\quad i=1,2,....$$

It is shown that the MCGLS method converges smoothly in a finite number of steps in the absence of round-off errors. Finally two numerical examples show that the MCGLS method is efficient.

1. Introduction

The main aim of this work, is to find the least squares solutions of general discrete-time periodic matrix equations

$$\sum_{j=1}^{t}(A_{i,j}X_{i,j}B_{i,j}+C_{i,j}X_{i+1,j}D_{i,j})=M_i,\quad i=1,2,...,\tag{1.1}$$

where the coefficient matrices $A_{i,j},C_{i,j}\in\mathbf{R}^{p\times n_j}$, $B_{i,j},D_{i,j}\in\mathbf{R}^{m_j\times q}$, $M_i\in\mathbf{R}^{p\times q}$ and the solutions $X_{i,j}\in\mathbf{R}^{n_j\times m_j}$ are periodic with period ξ, i.e., $A_{i+\xi,j}=A_{i,j}$, $B_{i+\xi,j}=B_{i,j}$, $C_{i+\xi,j}=C_{i,j}$, $D_{i+\xi,j}=D_{i,j}$, $M_{i+\xi,j}=M_{i,j}$ and $X_{i+\xi,j}=X_{i,j}$ for $i=1,2,...$ and $j=1,2,...,t$.

The linear periodic systems are one of important topics in engineering [27–29]. In the last decades of the past century the discrete-time periodic matrix equations have been used as a main tool of analysis and design problems involving periodic systems [1, 5, 6, 8–10, 27]. For example, the periodic Lyapunov matrix equation serves as a fundamental tool in the analysis of cyclostationary and stochastic processes [27]. Based on the periodic Lyapunov matrix equation, the notion of the l_2-norm for the periodic system can be characterized in the time-domain [27]. The periodic Lyapunov matrix equations

$$A_i^TX_iA_i-X_{i+1}=B_i,\quad i=1,2,...,\tag{1.2}$$

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and

$$A_i^T X_{i+1} A_i - X_i = B_i, \quad i = 1, 2, ..., \tag{1.3}$$

have several important applications in the semi-global stabilization problem of discrete-time linear periodic system

$$x(i+1) = A_i x(i) + B_i u(i), \quad i = 1, 2, ..., \tag{1.4}$$

in the solution of state- and output-feedback optimal periodic control problems, in the stabilization by periodic state feedback and in the square-root balancing of discrete-time periodic systems [11, 12]. When dealing with Luenberger-type observers design problem for linear discrete-time periodic systems, the periodic Sylvester matrix equations is encountered [30]. The discrete-time periodic Sylvester matrix equation [13]

$$A_i^T X_i - X_{i+1} B_i = C_i, \quad i = 1, 2, ..., \tag{1.5}$$

has applications in computation and condition estimation of periodic invariant subspaces of square matrix products of the form

$$\mathcal{A}_\xi \mathcal{A}_{\xi-1} ... \mathcal{A}_1.$$

Also a class of periodic robust state-feedback pole assignment problems can be reduced to the solution of a discrete-time periodic Sylvester matrix equation [14, 15].
So far various methods for the solution of discrete-time periodic matrix equations have been considered in some studies [16–19]. Based on the squared Smith iteration and Krylov subspaces, two methods were respectively proposed for solving the discrete-time periodic Lyapunov matrix equations [20]. In [31], a gradient based iterative method was proposed to find the solutions of the general Sylvester discrete-time periodic matrix equations. Also Hajarian in [32] introduced a gradient based iterative algorithm to solve general coupled discrete-time periodic matrix equations over generalized reflexive matrices. Byers and Rhee developed the Bartels-Stewart and Hessenberg-Schur algorithms for solving the discrete-time periodic Lyapunov and Sylvester matrix equations. In [22], the least-squares QR-factorization (LSQR) methods were extended to solve the discrete-time periodic Sylvester matrix equations. Recently by extending the bi-conjugate gradients (Bi-CG), bi-conjugate residual (Bi-CR), conjugate gradients squared (CGS), bi-conjugate gradient stabilized (Bi-CGSTAB), biconjugate A-orthogonal residual (BiCOR) and conjugate A-orthogonal residual squared (CORS) methods, Hajarian obtained effective iterative algorithms for finding the solutions of periodic coupled matrix equations [33–35].
In this paper, we study the following both problems:

Problem 1. *For given ξ-periodic matrices $A_{i,j}, C_{i,j} \in \mathbf{R}^{p\times n_j}$, $B_{i,j}, D_{i,j} \in \mathbf{R}^{m_j\times q}$ and $M_i \in \mathbf{R}^{p\times q}$ where $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$, find the ξ-periodic matrix group $(X^*_{1,1}, X^*_{1,2}, ..., X^*_{1,t}, ..., X^*_{\xi,t})$ such that*

$$\begin{aligned} &\sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j} X^*_{i,j} B_{i,j} + C_{i,j} X^*_{i+1,j} D_{i,j})\|^2 \\ = &\min_{(X_{1,1}, X_{1,2}, ..., X_{1,t}, ..., X_{\xi,t})} \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j} X_{i,j} B_{i,j} + C_{i,j} X_{i+1,j} D_{i,j})\|^2. \end{aligned} \tag{1.6}$$

Problem 2. *Let the solution set of Problem 1 be denoted by S_r. For a given matrix group $(\overline{X}_{1,1}, \overline{X}_{1,2}, ..., \overline{X}_{1,t}, ..., \overline{X}_{\xi,t})$ find the matrix group $(\check{X}_{1,1}, \check{X}_{1,2}, ..., \check{X}_{1,t}, ..., \check{X}_{\xi,t}) \in S_r$ such that*

$$\sum_{i=1}^{\xi} \sum_{j=1}^{t} \|\check{X}_{i,j} - \overline{X}_{i,j}\|^2 = \min_{(X_{1,1}, X_{1,2}, ..., X_{1,t}, ..., X_{\xi,t}) \in S_r} \sum_{i=1}^{\xi} \sum_{j=1}^{t} \|X_{i,j} - \overline{X}_{i,j}\|^2. \tag{1.7}$$

The remainder of this paper is organized as follows. In Section 2, we propose the MCGLS method to solve Problems 1 and 2. The convergence results of the MCGLS method are presented in Section 3. In Section 4, two numerical examples are solved by the proposed method. Section 5 concludes this paper with a brief summary.

Throughout this paper, the notation $\mathbf{R}^{m\times n}$ stands for the set of all $m \times n$ real matrices. For a real matrix A, the symbols A^T and $\mathrm{tr}(A)$ stand for the transpose and trace of A, respectively. The inner product of $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{m\times n}$ is defined by $\langle A, B\rangle = \mathrm{tr}(B^T A)$. The associated norm is the well-known Frobenius norm denoted by $\|.\|$. For matrices R, A, B and X with appropriate dimension, a well-known property of the inner product is $\langle R, AXB\rangle = \langle A^T RB^T, X\rangle$. We will apply the he above property many times in the rest of this paper. For a matrix $A \in \mathbf{R}^{m\times n}$, the so-called vectorization operator $\mathrm{vec}(A)$ is defined by $\mathrm{vec}(A) = (a_1^T, a_2^T, ..., a_n^T)^T$, where a_k is the k-th column of A. The symbol $A \otimes B$ stands for the Kronecker product of matrices A and B. For matrices A, B and X with appropriate dimension, we have the following well-known results related to the vectorization operator and Kronecker product:

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X).$$

2. The MCGLS Method

In this section, we first give some preliminary results related to the least squares solutions of general discrete-time periodic matrix equations (1.1). Second we propose the MCGLS method for solving Problems 1 and 2.

Lemma 1. *[23, 26] Let U be an inner product space, V be a subspace of U, and $V^\perp$ be the orthogonal complement subspace of V. For a given $u \in U$, if there exists an $v_0 \in V$ such that $\|u - v_0\| \leq \|u - v\|$ holds for any $v \in V$, then v_0 is unique and $v_0 \in V$ is the unique minimization vector in V if and only if $(u - v_0)\perp V$, i.e., $(u - v_0) \in V^\perp$.*

Lemma 2. *Suppose that $\widehat{R}_i$, $i = 1, 2, ..., \xi$, are the residuals of (1.1) corresponding to the matrices $\widehat{X}_{i,j} \in \mathbf{R}^{n_j\times m_j}$ for $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$, that is,*

$$\widehat{R}_i = M_i - \sum_{j=1}^{t}(A_{i,j}\widehat{X}_{i,j}B_{i,j} + C_{i,j}\widehat{X}_{i+1,j}D_{i,j}), \quad i = 1, 2, ..., \xi.$$

Then the matrix group $(\widehat{X}_{1,1}, \widehat{X}_{1,2}, ..., \widehat{X}_{1,t}, ..., \widehat{X}_{\xi,t})$ is the least squares solution group of (1.1) if

$$A_{i,j}^T\widehat{R}_iB_{i,j}^T + C_{i-1,j}^T\widehat{R}_{i-1}D_{i-1,j}^T = 0, \quad j = 1, 2, ..., t, \quad i = 1, 2, ..., \xi,$$

where $R_0 = R_\xi$, $C_{0,j} = C_{\xi,j}$ and $D_{0,j} = D_{\xi,j}$ for $j = 1, 2, ..., t$.

Proof. In order to prove this lemma, first we present the following operator

$$L(X_{i,1}, ..., X_{i,t}) = \sum_{j=1}^{t}(A_{i,j}X_{i,j}B_{i,j} + C_{i,j}X_{i+1,j}D_{i,j}),$$

for all ξ-periodic matrices $X_{i,j} \in \mathbf{R}^{n_j\times m_j}$, $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$. Now we define the linear subspace

$$\mathbb{S} = \{S|S = \mathrm{diag}(L(X_{1,1}, ..., X_{1,t}), L(X_{2,1}, ..., X_{2,t}), ..., L(X_{\xi,1}, ..., X_{\xi,t}))\}.$$

For ξ-periodic matrices $\widehat{X}_{i,j} \in \mathbf{R}^{n_j\times m_j}$, $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$, we consider

$$\widehat{S} = \mathrm{diag}(L(\widehat{X}_{1,1}, ..., \widehat{X}_{1,t}), L(\widehat{X}_{2,1}, ..., \widehat{X}_{2,t}), ..., L(\widehat{X}_{\xi,1}, ..., \widehat{X}_{\xi,t})).$$

It is obvious that $\widehat{S} \in \mathbb{S}$. For any $S \in \mathbb{S}$, we have

$$\begin{aligned}\langle \mathrm{diag}(M_1, M_2, ..., M_\xi) - \widehat{S}, S\rangle &= \langle \mathrm{diag}(\widehat{R}_1, \widehat{R}_2, ..., \widehat{R}_\xi), S\rangle \\ &= \sum_{i=1}^{\xi} \langle \widehat{R}_i, \sum_{j=1}^{t} (A_{i,j}X_{i,j}B_{i,j} + C_{i,j}X_{i+1,j}D_{i,j})\rangle \\ &= \sum_{i=1}^{\xi}\sum_{j=1}^{t} \langle A_{i,j}^T \widehat{R}_i B_{i,j}^T + C_{i-1,j}^T \widehat{R}_{i-1} D_{i-1,j}^T, X_{i,j}\rangle.\end{aligned}$$

Now, if

$$A_{i,j}^T \widehat{R}_i B_{i,j}^T + C_{i-1,j}^T \widehat{R}_{i-1} D_{i-1,j}^T = 0, \quad j = 1, 2, ..., t, \quad i = 1, 2, ..., \xi,$$

then we deduce that

$$\langle \mathrm{diag}(M_1, M_2, ..., M_\xi) - \widehat{S}, S\rangle = 0.$$

From Lemma 1, we conclude that

$$(\mathrm{diag}(M_1, M_2, ..., M_\xi) - \widehat{S}) \in \mathbb{S}^\perp.$$

Therefore the matrix group $(\widehat{X}_{1,1}, \widehat{X}_{1,2}, ..., \widehat{X}_{1,t}, ..., \widehat{X}_{\xi,t})$ is the least squares solution group of (1.1). □

Lemma 3. *Suppose that that $(\widetilde{X}_{1,1}, \widetilde{X}_{1,2}, ..., \widetilde{X}_{1,t}, ..., \widetilde{X}_{\xi,t})$ is a solution group of Problem 1. Any arbitrary solution group $(\widehat{X}_{1,1}, \widehat{X}_{1,2}, ..., \widehat{X}_{1,t}, ..., \widehat{X}_{\xi,t})$ of Problem 1 can be expressed as*

$$(\widehat{X}_{1,1}, \widehat{X}_{1,2}, ..., \widehat{X}_{1,t}, ..., \widehat{X}_{\xi,t}) = (\widetilde{X}_{1,1} + Y_{1,1}, \widetilde{X}_{1,2} + Y_{1,2}, ..., \widetilde{X}_{1,t} + Y_{1,t}, ..., \widetilde{X}_{\xi,t} + Y_{\xi,t}),$$

where the ξ-periodic matrices $Y_{i,j} \in \mathbf{R}^{n_j \times m_j}$, $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$ satisfy

$$\sum_{j=1}^{t} (A_{i,j}Y_{i,j}B_{i,j} + C_{i,j}Y_{i+1,j}D_{i,j}) = 0, \quad i = 1, 2, ..., \xi. \tag{2.1}$$

Proof. For arbitrary solution group $(\widehat{X}_{1,1}, \widehat{X}_{1,2}, ..., \widehat{X}_{1,t}, ..., \widehat{X}_{\xi,t})$ of Problem 1, first we define the ξ-periodic matrices

$$Y_{i,j} = \widehat{X}_{i,j} - \widetilde{X}_{i,j}, \quad i = 1, 2, ..., \xi, \quad j = 1, 2, ..., t.$$

From Lemma 2 and its proof, we observe that

$$\begin{aligned}\sum_{i=1}^{\xi} \|\widetilde{R}_i\|^2 &= \sum_{i=1}^{\xi} \|\widehat{R}_i\|^2 \\ &= \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j}\widehat{X}_{i,j}B_{i,j} + C_{i,j}\widehat{X}_{i+1,j}D_{i,j})\|^2 \\ &= \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j}(\widetilde{X}_{i,j} + Y_{i,j})B_{i,j} + C_{i,j}(\widetilde{X}_{i+1,j} + Y_{i+1,j})D_{i,j})\|^2 \\ &= \sum_{i=1}^{\xi} \|\widetilde{R}_i - \sum_{j=1}^{t} (A_{i,j}Y_{i,j}B_{i,j} + C_{i,j}Y_{i+1,j}D_{i,j})\|^2\end{aligned}$$

$$= \sum_{i=1}^{\xi}[\|\widetilde{R}_i\|^2 + \|\sum_{j=1}^{t}(A_{i,j}Y_{i,j}B_{i,j} + C_{i,j}Y_{i+1,j}D_{i,j})\|^2$$

$$-2\langle\widetilde{R}_i, \sum_{j=1}^{t}(A_{i,j}Y_{i,j}B_{i,j} + C_{i,j}Y_{i+1,j}D_{i,j})\rangle]$$

$$= \sum_{i=1}^{\xi}[\|\widetilde{R}_i\|^2 + \|\sum_{j=1}^{t}(A_{i,j}Y_{i,j}B_{i,j} + C_{i,j}Y_{i+1,j}D_{i,j})\|^2$$

$$-2\sum_{j=1}^{t}\langle A_{i,j}^T\widetilde{R}_iB_{i,j}^T + C_{i-1,j}^T\widetilde{R}_{i-1}D_{i-1,j}^T, Y_{i,j}\rangle]$$

$$= \sum_{i=1}^{\xi}[\|\widetilde{R}_i\|^2 + \|\sum_{j=1}^{t}(A_{i,j}Y_{i,j}B_{i,j} + C_{i,j}Y_{i+1,j}D_{i,j})\|^2].$$

Now the proof is finished. □

So far several iterative algorithms have been introduced to solve nonsymmetric linear system of equations $Ax = b$ [24, 38–41]. One of these algorithms is the CGLS method [24]. The CGLS method determines a sequence of approximate solutions of the linear system of equations $Ax = b$ without explicitly forming the matrix A^TA, but instead multiplying vectors with A and A^T separately. The CGLS method is one of the important methods for solving large non square linear systems. The CGLS method for solving $Ax = b$ can be summarized as following [24]:

CGLS method

For the initial vector $x(1)$, compute $r(1) = b - Ax(1)$, $z(1) = A^Tr(1)$, $p(1) = z(1)$,
For $i = 1, 2, ...$ until convergence Do:
$u(i) = Ap(i)$, $\alpha(i) = \frac{\|z(i)\|^2}{\|u(i)\|^2}$, $x(i+1) = x(i) + \alpha(i)p(i)$, $r(i+1) = r(i) - \alpha(i)u(i)$,
$z(i+1) = A^Tr(i+1)$, $\beta(i) = \frac{\|z(i+1)\|^2}{\|z(i)\|^2}$, $p(i+1) = z(i+1) + \beta(i)p(i)$.

In recent years, the CGLS method was developed to find least squares solutions of several linear matrix equations [26, 42–44]. Our purpose in the current paper is to develop the CGLS method to solve Problems 1 and 2. First we obtain a Sylvester matrix equation and a linear system of equations equivalent to the general discrete-time periodic matrix equations (1.1). We can easily show that the general discrete-time periodic matrix equations (1.1) can be transformed into the following Sylvester matrix equation:

$$\sum_{j=1}^{t}(\mathcal{A}_j\mathcal{X}_j\mathcal{B}_j + \mathcal{C}_j\mathcal{X}_j\mathcal{D}_j) = \mathcal{M}, \tag{2.2}$$

where

$$\mathcal{A}_j = \begin{pmatrix} 0 & \cdots & 0 & A_{1,j} \\ A_{2,j} & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{\xi,j} & 0 \end{pmatrix}, \quad \mathcal{B}_j = \begin{pmatrix} 0 & B_{2,j} & & 0 \\ \vdots & & \ddots & \\ 0 & & & B_{\xi,j} \\ B_{1,j} & 0 & \cdots & 0 \end{pmatrix}, \quad \mathcal{C}_j = \operatorname{diag}\left(C_{1,j}, C_{2,j}, ..., C_{\xi,j}\right),$$

$$\mathcal{D}_j = \operatorname{diag}\left(D_{1,j}, D_{2,j}, ..., D_{\xi,j}\right), \quad \mathcal{M} = \operatorname{diag}\left(M_1, M_2, ..., M_{\xi}\right), \quad \mathcal{X}_j = \operatorname{diag}\left(X_{1,j}, X_{2,j}, ..., X_{\xi,j}\right),$$

for $j = 1, 2, ..., t$. Also by means of Kronecker product, vectorization operator and the Sylvester matrix equation (2.2), we can transform the general discrete-time periodic matrix equations (1.1) into the following linear system:

$$\underbrace{\Big((\mathcal{B}_1^T \otimes \mathcal{A}_1) + (\mathcal{D}_1^T \otimes C_1) \quad (\mathcal{B}_2^T \otimes \mathcal{A}_2) + (\mathcal{D}_2^T \otimes C_2) \quad \dots \quad (\mathcal{B}_t^T \otimes \mathcal{A}_t) + (\mathcal{D}_t^T \otimes C_t)\Big)}_{A} \underbrace{\begin{pmatrix} \mathrm{vec}(\mathcal{X}_1) \\ \mathrm{vec}(\mathcal{X}_2) \\ \vdots \\ \mathrm{vec}(\mathcal{X}_t) \end{pmatrix}}_{x} = \underbrace{\mathrm{vec}(\mathcal{M})}_{b}. \qquad (2.3)$$

Obviously the sizes of coefficient matrices of the Sylvester matrix equation (2.2) and the linear system (2.3) are very larger than sizes of coefficient matrices of the general discrete-time periodic matrix equations (1.1). Iterative algorithms such as CGLS method and algorithms proposed in [2–4, 25, 26, 36, 37] are slow and inefficient to solve the large-scale Sylvester matrix equation (2.2) and large-scale linear system (2.3). In order to overcome this challenge, we directly develop the CGLS method to solve Problems 1 and 2 as follows.

Algorithm 1. Choose the initial matrices $X_{i,j}(1) \in \mathbf{R}^{n_j \times m_j}$ for $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$,
Set $X_{\xi+1,j}(1) = X_{1,j}(1)$ for $j = 1, 2, ..., t$;
Compute

$$R_i(1) = M_i - \sum_{j=1}^{t} (A_{i,j} X_{i,j}(1) B_{i,j} + C_{i,j} X_{i+1,j}(1) D_{i,j}), \quad i = 1, 2, ..., \xi;$$

Set $R_0(1) = R_\xi(1)$, $C_{0,j} = C_{\xi,j}$ and $D_{0,j} = D_{\xi,j}$ for $j = 1, 2, ..., t$;
Compute

$$Z_{i,j}(1) = A_{i,j}^T R_i(1) B_{i,j}^T + C_{i-1,j}^T R_{i-1}(1) D_{i-1,j}^T \quad i = 1, 2, ..., \xi, \quad j = 1, 2, ..., t,$$

$$P_{i,j}(1) = Z_{i,j}(1), \quad i = 1, 2, ..., \xi, \quad j = 1, 2, ..., t;$$

Set $P_{\xi+1,j}(1) = P_{1,j}(1)$ for $j = 1, 2, ..., t$;
For $k = 1, 2, 3, ...,$ repeat the following;
If $\sum_{i=1}^{\xi} \|R_i(k)\|^2 = 0$, then $(X_{1,1}(k), X_{1,2}(k), ..., X_{1,t}(k), ..., X_{\xi,t}(k))$ is the solution group of (1.1), break;
If $\sum_{i=1}^{\xi} \|R_i(k)\|^2 \neq 0$ and $\sum_{j=1}^{t} \sum_{i=1}^{\xi} \|Z_{i,j}(k)\|^2 = 0$, then $(X_{1,1}(k), X_{1,2}(k), ..., X_{1,t}(k), ..., X_{\xi,t}(k))$ is the solution group of Problem 1, break;
Compute

$$U_i(k) = \sum_{j=1}^{t} (A_{i,j} P_{i,j}(k) B_{i,j} + C_{i,j} P_{i+1,j}(k) D_{i,j}), \quad i = 1, 2, ..., \xi,$$

Set $U_0(k) = U_\xi(k)$;

$$\alpha(k) = \frac{\sum_{j=1}^{t} \sum_{i=1}^{\xi} \|Z_{i,j}(k)\|^2}{\sum_{i=1}^{\xi} \|U_i(k)\|^2},$$

$$X_{i,j}(k+1) = X_{i,j}(k) + \alpha(k) P_{i,j}(k), \quad i = 1, 2, ..., \xi, \quad j = 1, 2, ..., t;$$

Set $X_{\xi+1,j}(k+1) = X_{1,j}(k+1)$, $j = 1, 2, ..., t$;

$$\begin{aligned} R_i(k+1) =& M_i - \sum_{j=1}^{t} (A_{i,j} X_{i,j}(k+1) B_{i,j} + C_{i,j} X_{i+1,j}(k+1) D_{i,j}) \\ =& R_i(k) - \alpha(k) U_i(k), \quad i = 1, 2, ..., \xi; \end{aligned}$$

Set $R_0(k+1) = R_\xi(k+1)$;

$$\begin{aligned} Z_{i,j}(k+1) =& A_{i,j}^T R_i(k+1) B_{i,j}^T + C_{i-1,j}^T R_{i-1}(k+1) D_{i-1,j}^T \\ =& Z_{i,j}(k) - \alpha(k)[A_{i,j}^T U_i(k) B_{i,j}^T + C_{i-1,j}^T U_{i-1}(k) D_{i-1,j}^T], \quad i = 1,2,...,\xi, \quad j = 1,2,....,t, \end{aligned}$$

$$\beta(k) = \frac{\sum_{j=1}^t \sum_{i=1}^\xi \|Z_{i,j}(k+1)\|^2}{\sum_{j=1}^t \sum_{i=1}^\xi \|Z_{i,j}(k)\|^2},$$

$$P_{i,j}(k+1) = Z_{i,j}(k+1) + \beta(k) P_{i,j}(k), \quad i = 1,2,...,\xi, \quad j = 1,2,....,t;$$

Set $P_{\xi+1,j}(k+1) = P_{1,j}(k+1)$.

Remark 1. *Algorithm 1 implies that if $\sum_{i=1}^\xi \|R_i(k)\|^2 = 0$, then $(X_{1,1}(k), X_{1,2}(k), ..., X_{1,t}(k), ..., X_{\xi,t}(k))$ is the solution group of (1.1).*

Remark 2. *Because of the influence of the error of calculation, we regard the arbitrary matrix M as a zero matrix if $\|M\| < \varepsilon$ where ε is a small positive number.*

3. Convergence Results

In this section, we establish convergence results of Algorithm 1. To this purpose, we first propose some properties of sequences generated by Algorithm 1.

Lemma 4. *For the sequences $\{Z_{i,j}(k)\}$, $\{P_{i,j}(k)\}$ $\{U_i(k)\}$, $i = 1,2,...,\xi$, $j = 1,2,....,t$ generated by Algorithm 1, if there exists a positive number r such that $\sum_{j=1}^t \sum_{i=1}^\xi \|Z_{i,j}(u)\|^2 \neq 0$, $\alpha(u) \neq 0$ and $\alpha(u) \neq \infty$ $\forall u = 1,2,...,r$, then the following statements hold for $u, v = 1,2,...,r$ and $u \neq v$*

(I) $\sum_{j=1}^t \sum_{i=1}^\xi \langle Z_{i,j}(u), Z_{i,j}(v)\rangle = 0$,

(II) $\sum_{i=1}^\xi \langle U_i(u), U_i(v)\rangle = 0$,

(III) $\sum_{j=1}^t \sum_{i=1}^\xi \langle P_{i,j}(u), Z_{i,j}(v)\rangle = 0$.

The proof of Lemma 4 is derived in the Appendix.

Lemma 5. *In Algorithm* 1*, if there exists a positive number l such that $\alpha(l) = 0$ or $\alpha(l) = \infty$ then $(X_{1,1}(l), X_{1,2}(l), ..., X_{1,t}(l), ..., X_{\xi,t}(l))$ is the solution group of Problem* 1*.*

Proof. We can see if $\alpha(l) = 0$ then $\sum_{j=1}^t \sum_{i=1}^\xi \|Z_{i,j}(l)\|^2 = 0$. Also if $\alpha(l) = \infty$, then we have $\sum_{i=1}^\xi \|U_i(l)\|^2 = 0$. It follows from $\sum_{i=1}^\xi \|U_i(l)\|^2 = 0$ that

$$\sum_{j=1}^t \sum_{i=1}^\xi \|Z_{i,j}(l)\|^2 = \sum_{j=1}^t \sum_{i=1}^\xi \langle Z_{i,j}(l), Z_{i,j}(l)\rangle$$

$$= \sum_{j=1}^t \sum_{i=1}^\xi \langle Z_{i,j}(l) + \beta(l-1) Z_{i,j}(l-1) + \beta(l-1)\beta(l-2) Z_{i,j}(l-2) + ... + \beta(l-1)...\beta(1) Z_{i,j}(1), Z_{i,j}(l)\rangle$$

$$= \sum_{j=1}^t \sum_{i=1}^\xi \langle P_{i,j}(l), Z_{i,j}(l)\rangle = \sum_{j=1}^t \sum_{i=1}^\xi \langle P_{i,j}(l), A_{i,j}^T R_i(l) B_{i,j}^T + C_{i-1,j}^T R_{i-1}(l) D_{i-1,j}^T\rangle$$

$$= \sum_{i=1}^\xi \langle \sum_{j=1}^t (A_{i,j} P_{i,j}(l) B_{i,j} + C_{i,j} P_{i+1,j}(l) D_{i,j}), R_i(l)\rangle = \sum_{i=1}^\xi \langle U_i(l), R_i(l)\rangle = 0.$$

Hence in two cases $\alpha(l) = 0$ or $\alpha(l) = \infty$, we conclude that

$$Z_{i,j}(l) = A_{i,j}^T R_i(l) B_{i,j}^T + C_{i-1,j}^T R_{i-1}(l) D_{i-1,j}^T = 0 \quad i = 1,2,...,\xi, \quad j = 1,2,....,t.$$

Now Lemma 2 implies that $(X_{1,1}(l), X_{1,2}(l), ..., X_{1,t}(l), ..., X_{\xi,t}(l))$ is the solution group of Problem 1. □

Here we introduce the convergence theorem of Algorithm 1. In this theorem, we show that Algorithm 1 can obtain the solution group of Problem 1 in a finite number of steps in the absence of round-off errors.

Theorem 1. *Algorithm 1 with any arbitrary initial matrix group $(X_{1,1}(1), X_{1,2}(1), ..., X_{1,t}(1), ..., X_{\xi,t}(1))$, $X_{i,j}(1) \in \boldsymbol{R}^{n_j\times m_j}$, $i = 1,2,...,\xi$, $j = 1,2,....,t$ can obtain the solution group $(X_{1,1}^*, X_{1,2}^*, ..., X_{1,t}^*, ..., X_{\xi,t}^*)$ of Problem 1 within a finite number of iterations in the absence of round-off errors.*

Proof. By using the sequences $\{Z_{i,j}(i)\}$ generated by Algorithm 1, first we define the following sequences

$$F_i(k) := \text{diag}(Z_{i,1}(k), Z_{i,2}(k), ..., Z_{i,t}(k)) \in \mathbf{R}^{(\sum_{j=1}^t m_j)\times(\sum_{j=1}^t n_j)}, \quad \text{for} \quad i = 1,2,...,\xi. \tag{3.1}$$

From (3.1) we have

$$\sum_{i=1}^{\xi}\langle F_i(u), F_i(v)\rangle = \sum_{i=1}^{\xi}\sum_{j=1}^{t}\langle Z_{i,j}(u), Z_{i,j}(v)\rangle \quad \text{for any } u \text{ and } v. \tag{3.2}$$

Using Lemma 2 and (3.2) gives us that the set $(F_1(k), F_2(k), ..., F_\xi(k))$, $k = 1,2,3,...,\xi(\sum_{j=1}^t m_j)\times(\sum_{j=1}^t n_j)$ is an orthogonal basis of the inner product space $\mathbf{R}^{(\sum_{j=1}^t m_j)\times(\sum_{j=1}^t n_j)} \times ... \times \mathbf{R}^{(\sum_{j=1}^t m_j)\times(\sum_{j=1}^t n_j)}$ with dimension $\xi(\sum_{j=1}^t m_j)\times(\sum_{j=1}^t n_j) := m$. This implies that $(F_1(m+1), F_2(m+1), ..., F_\xi(m+1)) = 0$. Then from (3.2), we have

$$(Z_{1,1}(m+1), Z_{1,2}(m+1), ..., Z_{1,t}(m+1), ..., Z_{\xi,t}(m+1)) = 0.$$

This implies that $(X_{1,1}(m+1), X_{1,2}(m+1), ..., X_{1,t}(m+1), ..., X_{\xi,t}(m+1))$ is the solution group of Problem 1 in the absence of round-off errors. □

In the next theorem, we obtain the least norm solution group of Problem 1 by Algorithm 1.

Theorem 2. *If we take the initial matrices*

$$X_{i,j}(1) = A_{i,j}^T Z_i(1) B_{i,j}^T + C_{i-1,j}^T Z_{i-1}(1) D_{i-1}^T, \quad for \quad i = 1,2,...,\xi, \;\; j = 1,2,....,t, \tag{3.3}$$

where $Z_i(1) \in \boldsymbol{R}^{p\times q}$ are arbitrary ξ-periodic matrices for $i = 1,2,...,\xi$ ($Z_0(1) = Z_\xi(1)$), or especially $X_{i,j}(1) = 0$ for $i = 1,2,...,\xi$ and $j = 1,2,....,t$, then the solution group $(X_{1,1}^*, X_{1,2}^*, ..., X_{1,t}^*, ..., X_{\xi,t}^*)$ obtained by Algorithm 1, is the least norm solution group of Problem 1.

Proof. By taking the initial matrices (3.3), we can easily observe that the generated matrices $X_{i,j}(k)$ by Algorithm 1 can be expressed as

$$X_{i,j}(k) = A_{i,j}^T Z_i(k) B_{i,j}^T + C_{i-1,j}^T Z_{i-1}(k) D_{i-1,j}^T, \quad for \quad i = 1,2,...,\xi, \;\; j = 1,2,....,t, \tag{3.4}$$

for certain matrices $Z_i(k) \in \mathbf{R}^{p\times q}$ for $i = 1,2,...,\xi$. This concludes that there exist matrices $Z_i^* \in \mathbf{R}^{p\times q}$ for $i = 1,2,...,\xi$ such that

$$X_{i,j}^* = A_{i,j}^T Z_i^* B_{i,j}^T + C_{i-1,j}^T Z_{i-1}^* D_{i-1,j}^T, \quad for \quad i = 1,2,...,\xi, \;\; j = 1,2,....,t. \tag{3.5}$$

Now we suppose that $(\widetilde{X}_{1,1}^*, \widetilde{X}_{1,2}^*, ..., \widetilde{X}_{1,t}^*, ..., \widetilde{X}_{\xi,t}^*)$ is an arbitrary solution group of Problem 1. By using Lemma 3, there exists the ξ-periodic matrices $Y_{i,j} \in \mathbf{R}^{n_j\times m_j}$ for $i = 1,2,...,\xi$ and $j = 1,2,....,t$ such that

$$(\widetilde{X}_{1,1}^*, \widetilde{X}_{1,2}^*, ..., \widetilde{X}_{1,t}^*, ..., \widetilde{X}_{\xi,t}^*) = (X_{1,1}^* + Y_{1,1}^*, X_{1,2}^* + Y_{1,2}^*, ..., X_{1,t}^* + Y_{1,t}^*, ..., X_{\xi,t}^* + Y_{\xi,t}^*),$$

and

$$\sum_{j=1}^{t}(A_{i,j}Y^*_{i,j}B_{i,j}+C_{i,j}Y^*_{i+1,j}D_{i,j})=0,\quad i=1,2,...,\xi. \tag{3.6}$$

We can obtain

$$\sum_{j=1}^{t}\langle X^*_{i,j},Y^*_{i,j}\rangle=\sum_{j=1}^{t}\langle A^T_{i,j}Z^*_iB^T_{i,j}+C^T_{i-1,j}Z^*_{i-1}D^T_{i-1,j},Y^*_{i,j}\rangle$$

$$=\langle Z^*_i,\sum_{j=1}^{t}(A_{i,j}Y^*_{i,j}B_{i,j}+C_{i,j}Y^*_{i+1,j}D_{i,j})\rangle=0,\quad for\quad i=1,2,...,\xi,\quad j=1,2,....,t. \tag{3.7}$$

Using (3.7) gives us

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\|\widetilde{X}^*_{i,j}\|^2=\sum_{j=1}^{t}\sum_{i=1}^{\xi}\|X^*_{i,j}+Y^*_{i,j}\|^2$$

$$=\sum_{j=1}^{t}\sum_{i=1}^{\xi}[\|X^*_{i,j}\|^2+\|Y^*_{i,j}\|^2+2\langle X^*_{i,j},Y^*_{i,j}\rangle]\geq\sum_{j=1}^{t}\sum_{i=1}^{\xi}\|X^*_{i,j}\|^2.$$

This implies that the solution group $(X^*_{1,1},X^*_{1,2},...,X^*_{1,t},...,X^*_{\xi,t})$ is the least norm solution group of Problem 1. □

Now we obtain the residual reducing property of Algorithm 1, which ensures that Algorithm 1 possesses smoothly convergence.

Theorem 3. *For any arbitrary initial matrix group $(X_{1,1}(1),X_{1,2}(1),...,X_{1,t}(1),...,X_{\xi,t}(1))$ with $X_{i,j}(1)\in\mathbf{R}^{n_j\times m_j}$, $i=1,2,...,\xi$, $j=1,2,....,t$ we have*

$$\sum_{i=1}^{\xi}\|M_i-\sum_{j=1}^{t}(A_{i,j}X_{i,j}(k+1)B_{i,j}+C_{i,j}X_{i+1,j}(k+1)D_{i,j})\|^2$$

$$=\min_{(X_{1,1},X_{1,2},...,X_{1,t},...,X_{\xi,t})\in\Upsilon_k}\sum_{i=1}^{\xi}\|M_i-\sum_{j=1}^{t}(A_{i,j}X_{i,j}B_{i,j}+C_{i,j}X_{i+1,j}D_{i,j})\|^2,$$

where $(X_{1,1}(k+1),X_{1,2}(k+1),...,X_{1,t}(k+1),...,X_{\xi,t}(k+1))$ is generated by Algorithm 1 at the $k+1$-th iteration and Υ_k presents an affine subspace which has the following form:

$$\Upsilon_k=(X_{1,1}(1),X_{1,2}(1),...,X_{1,t}(1),...,X_{\xi,t}(1))$$
$$+span\{(P_{1,1}(1),P_{1,2}(1),...,P_{1,t}(1),...,P_{\xi,t}(1)),(P_{1,1}(2),P_{1,2}(2),...,P_{1,t}(2),...,P_{\xi,t}(2)),...,$$
$$(P_{1,1}(k),P_{1,2}(k),...,P_{1,t}(k),...,P_{\xi,t}(k))\}. \tag{3.8}$$

Proof. First from (3.8), we can get that for any matrix group $(X_{1,1},X_{1,2},...,X_{1,t},...,X_{\xi,t})\in\Upsilon_k$ there exist numbers δ_s for $s=1,2,...,k$ such that

$$X_{i,j}=X_{i,j}(1)+\sum_{s=1}^{k}\delta_sP_{i,j}(s),\quad for\quad i=1,2,...,\xi,\quad j=1,2,....,t. \tag{3.9}$$

We define the continuous and differentiable function f with respect to the variable $\delta_1,\delta_2,...,\delta_k$ as

$$f(\delta_1,\delta_2,...,\delta_k)=\sum_{i=1}^{\xi}\|M_i-\sum_{j=1}^{t}(A_{i,j}X_{i,j}B_{i,j}+C_{i,j}X_{i+1,j}D_{i,j})\|^2$$

$$= \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} [A_{i,j}(X_{i,j}(1) + \sum_{s=1}^{k} \delta_s P_{i,j}(s))B_{i,j} + C_{i,j}(X_{i+1,j}(1) + \sum_{s=1}^{k} \delta_s P_{i+1,j}(s))D_{i,j}]\|^2.$$

Here we can write

$$f(\delta_1, \delta_2, ..., \delta_k) = \sum_{i=1}^{\xi} \|R_i(1) - \sum_{s=1}^{k} \delta_s U_i(s)\|^2$$

$$= \sum_{i=1}^{\xi} [\|R_i(1)\|^2 + \sum_{s=1}^{k} \delta_s^2 \|U_i(s)\|^2 - 2 \sum_{s=1}^{k} \delta_s \langle U_i(s), R_i(1)\rangle].$$

Now we consider the problem of minimizing the function $f(\delta_1, \delta_2, ..., \delta_k)$. It is obvious that

$$\min_{\delta_s} f(\delta_1, \delta_2, ..., \delta_k) = \min_{(X_{1,1}, X_{1,2}, ..., X_{1,t}, ..., X_{\xi,t}) \in \Upsilon_k} \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j} X_{i,j} B_{i,j} + C_{i,j} X_{i+1,j} D_{i,j})\|^2.$$

For this function, the minimum occurs when

$$\frac{\partial f(\delta_1, \delta_2, ..., \delta_k)}{\partial \delta_s} = 0, \quad \text{for} \quad s = 1, 2, ..., k. \tag{3.10}$$

This implies that

$$\delta_s = \frac{\sum_{i=1}^{\xi} \langle U_i(s), R_i(1)\rangle}{\sum_{i=1}^{\xi} \|U_i(s)\|^2}.$$

From Algorithm 1, we can get

$$R_i(1) = R_i(s) + \alpha(s-1)U_i(s-1) + \alpha(s-2)U_i(s-2) + ... + \alpha(1)U_i(1), \quad \text{for} \quad i = 1, 2, ..., \xi. \tag{3.11}$$

By applying Lemma 4 and (3.11), we have

$$\delta_s = \frac{\sum_{i=1}^{\xi} \langle U_i(s), R_i(1)\rangle}{\sum_{i=1}^{\xi} \|U_i(s)\|^2} = \frac{\sum_{i=1}^{\xi} \langle U_i(s), R_i(s)\rangle}{\sum_{i=1}^{\xi} \|U_i(s)\|^2} = \frac{\sum_{i=1}^{\xi} \sum_{j=1}^{t} \langle P_{i,j}(s), Z_{i,j}(s)\rangle}{\sum_{i=1}^{\xi} \|U_i(s)\|^2}$$

$$= \frac{\sum_{i=1}^{\xi} \sum_{j=1}^{t} \langle Z_{i,j}(s) + \beta(s-1)P_{i,j}(s-1), Z_{i,j}(s)\rangle}{\sum_{i=1}^{\xi} \|U_i(s)\|^2} = \frac{\sum_{i=1}^{\xi} \sum_{j=1}^{t} \|Z_{i,j}(s)\|^2}{\sum_{i=1}^{\xi} \|U_i(s)\|^2} = \alpha(s).$$

This completes the proof. □

Now we study Problem 2. For a given matrix group $(\overline{X}_{1,1}, \overline{X}_{1,2}, ..., \overline{X}_{1,t}, ..., \overline{X}_{\xi,t})$, we can write

$$\min_{(X_{1,1}, X_{1,2}, ..., X_{1,t}, ..., X_{\xi,t})} \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j} X_{i,j} B_{i,j} + C_{i,j} X_{i+1,j} D_{i,j})\|^2$$

$$= \min_{(X_{1,1}, X_{1,2}, ..., X_{1,t}, ..., X_{\xi,t})} \sum_{i=1}^{\xi} \|M_i - \sum_{j=1}^{t} (A_{i,j} \overline{X}_{i,j} B_{i,j} + C_{i,j} \overline{X}_{i+1,j} D_{i,j})$$

$$- \sum_{j=1}^{t} [A_{i,j}(X_{i,j} - \overline{X}_{i,j})B_{i,j} + C_{i,j}(X_{i+1,j} - \overline{X}_{i+1,j})D_{i,j}]\|^2.$$

Set

$$\hat{M}_i = M_i - \sum_{j=1}^{t}(A_{i,j}\overline{X}_{i,j}B_{i,j} + C_{i,j}\overline{X}_{i+1,j}D_{i,j}), \quad \hat{X}_{i,j} = X_{i,j} - \overline{X}_{i,j},$$

for $i = 1, 2, ..., \xi$. Now Problem 2 is equivalent to find the least norm solution group $(\hat{X}^*_{1,1}, \hat{X}^*_{1,2}, ..., \hat{X}^*_{1,t}, ..., \hat{X}^*_{\xi,t})$ of

$$\min_{(\hat{X}_{1,1},\hat{X}_{1,2},...,\hat{X}_{1,t},...,\hat{X}_{\xi,t})} \sum_{i=1}^{\xi} \|\hat{M}_i - \sum_{j=1}^{t}(A_{i,j}\hat{X}_{i,j}B_{i,j} + C_{i,j}\hat{X}_{i+1,j}D_{i,j})\|^2,$$

which can be computed applying Algorithm 1 with the initial matrices

$$\hat{X}_{i,j}(1) = A^T_{i,j}Z_i(1)B^T_{i,j} + C^T_{i-1,j}Z_{i-1}(1)D^T_{i-1}, \quad for \quad i = 1, 2, ..., \xi, \quad j = 1, 2, ..., t,$$

where $Z_i(1) \in \mathbf{R}^{p\times q}$ are arbitrary ξ-periodic matrices for $i = 1, 2, ..., \xi$ ($Z_0(1) = Z_\xi(1)$), or especially $\hat{X}_{i,j}(1) = 0$ for $i = 1, 2, ..., \xi$ and $j = 1, 2, ..., t$. Hence the solution of Problem 2 can be presented as

$$(\check{X}_{1,1}, \check{X}_{1,2}, ..., \check{X}_{1,t}, ..., \check{X}_{\xi,t}) = (\hat{X}^*_{1,1} + \overline{X}_{1,1}, \hat{X}^*_{1,2} + \overline{X}_{1,2}, ..., \hat{X}^*_{1,t} + \overline{X}_{1,t}, ..., \hat{X}^*_{\xi,t} + \overline{X}_{\xi,t}). \tag{3.12}$$

4. Numerical Examples

This section gives two numerical examples to illustrate the validity of the results presented in the previous sections. The numerical results are carried out using MATLAB with machine precision around 10^{-16}.

Example 1. *As the first example we study the periodic discrete-time matrix equations*

$$A_iX_i + X_{i+1} + Y_iB_i + Y_{i+1} = C_i \quad for \quad i = 1, 2, 3,$$

with

$$A_1 = \begin{pmatrix} 4 & 2 & 6 & 15 & 15 & 3 \\ -2 & 1 & 0 & 1 & 1 & 7 \\ 5 & 5 & 0 & 9 & 8 & 4 \\ 10 & 12 & 1 & 4 & 0 & -4 \\ -2 & -10 & 7 & 1 & -4 & 10 \\ 8 & 4 & 3 & 4 & 5 & 7 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -5 & 1 & 0 & 2 & 1 \\ 2 & -3 & 0 & -10 & 7 & 10 \\ 3 & 0 & 0 & 0 & 0 & -1 \\ 3 & 7 & -10 & 0 & 0 & -2 \\ -9 & -10 & 0 & -6 & 5 & 2 \\ 1 & -7 & -5 & 10 & -3 & 10 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -5 & -7 & -5 & -15 & -13 & -2 \\ 4 & -4 & 0 & -11 & 6 & 3 \\ -2 & -5 & 0 & -9 & -8 & -5 \\ -7 & -5 & -11 & -4 & 0 & 2 \\ -7 & 0 & -7 & -7 & 9 & -8 \\ -7 & -11 & -8 & 6 & -8 & 3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 7 & 0 & -1 & 9 & 15 & 3 \\ -4 & 3 & 2 & 2 & 2 & 0 \\ -4 & 2 & 2 & 7 & 8 & -3 \\ -3 & -7 & 8 & 5 & 0 & -12 \\ 3 & 3 & 7 & 11 & -4 & 5 \\ 2 & 3 & -1 & -9 & 11 & -5 \end{pmatrix},$$

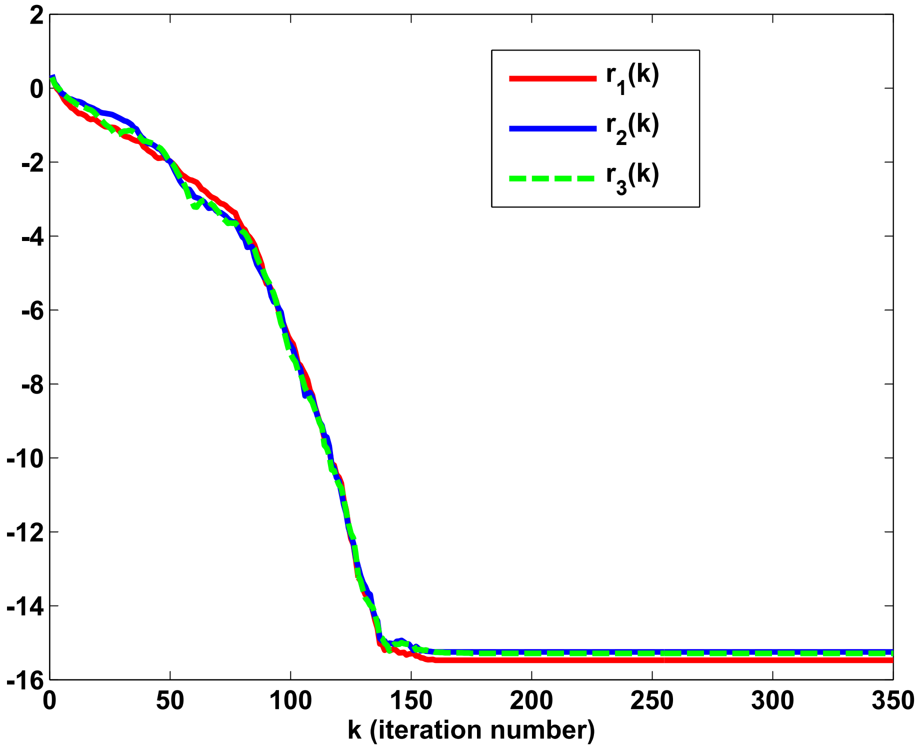
$$B_2 = \begin{pmatrix} 11 & 2 & 5 & 24 & 30 & 6 \\ -6 & 4 & 2 & 3 & 3 & 7 \\ 1 & 7 & 2 & 16 & 16 & 1 \\ 7 & 5 & 9 & 9 & 0 & -16 \\ 1 & -7 & 14 & 12 & -8 & 15 \\ 10 & 7 & 2 & -5 & 16 & 2 \end{pmatrix}, B_3 = \begin{pmatrix} 6 & -5 & 0 & 9 & 17 & 4 \\ -2 & 0 & 2 & -8 & 9 & 10 \\ -1 & 2 & 2 & 7 & 8 & -4 \\ 0 & 0 & -2 & 5 & 0 & -14 \\ -6 & -7 & 7 & 5 & 1 & 7 \\ 3 & -4 & -6 & 1 & 8 & 5 \end{pmatrix},$$

$$C_1 = C_2 = C_3 = \text{rand}(6).$$

We apply Algorithm 1 with the initial matrices $X_1(1) = X_2(1) = X_3(1) = Y_1(1) = Y_2(1) = Y_3(1) = 0$ *to obtain the sequences* $\{X_1(k)\}$, $\{X_2(k)\}$, $\{X_3(k)\}$, $\{Y_1(k)\}$, $\{Y_2(k)\}$ *and* $\{Y_3(k)\}$. *The numerical results are depicted in Figure 1 where*

$$r_i(k) = \log\|C_i - A_iX_i(k) - X_{i+1}(k) - Y_i(k)B_i - Y_{i+1}(k)\|, \quad for \quad i = 1, 2, 3.$$

Figure 1: The residuals for Example 1



Example 2. *Here we consider the periodic discrete-time matrix equations*

$$A_iX_i + X_{i+1}B_i = C_i \quad for \quad i = 1, 2, 3,$$

with

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & -4 & 0 & -3 & -1 \\ -2 & -6 & 1 & 0 & 0 & -7 & 0 \\ -3 & 7 & 0 & 0 & 0 & 0 & -1 \\ -3 & -4 & 7 & 0 & 0 & -2 & -1 \\ 10 & 11 & -3 & 3 & -4 & 0 & 4 \\ 0 & 4 & 5 & -10 & -2 & -13 & -3 \\ 2 & 4 & -1 & 1 & -3 & -8 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 8 & -1 & -4 & 3 & -1 & -1 \\ -2 & 3 & 0 & 10 & -7 & -10 & -2 \\ -3 & 0 & 0 & 0 & 0 & 1 & 3 \\ -3 & -7 & 10 & 0 & 0 & 2 & 3 \\ 9 & 10 & 0 & 6 & -5 & -2 & 0 \\ -1 & 7 & 5 & -10 & -1 & -10 & -3 \\ 4 & 1 & 0 & 0 & -6 & -1 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & -7 & 2 & 0 & -3 & -2 & 0 \\ 0 & -9 & 1 & -10 & 7 & 3 & 2 \\ 0 & 7 & 0 & 0 & 0 & -1 & -4 \\ 0 & 3 & -3 & 0 & 0 & -4 & -4 \\ 1 & 1 & -3 & -3 & 1 & 2 & 4 \\ 1 & -3 & 0 & 0 & -1 & -3 & 0 \\ -2 & 3 & -1 & 1 & 3 & -7 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -6 & -3 & 1 & -5 & -15 & -3 & -5 \\ -8 & 7 & -2 & -2 & -2 & 0 & -5 \\ 4 & -2 & -2 & -7 & -8 & 3 & -5 \\ 3 & 7 & -8 & 9 & 0 & 12 & -6 \\ -3 & -3 & -7 & -11 & 4 & -5 & -4 \\ -2 & -5 & 1 & 9 & -7 & 5 & -4 \\ -4 & -8 & -1 & -8 & -4 & -2 & 0 \end{pmatrix},$$

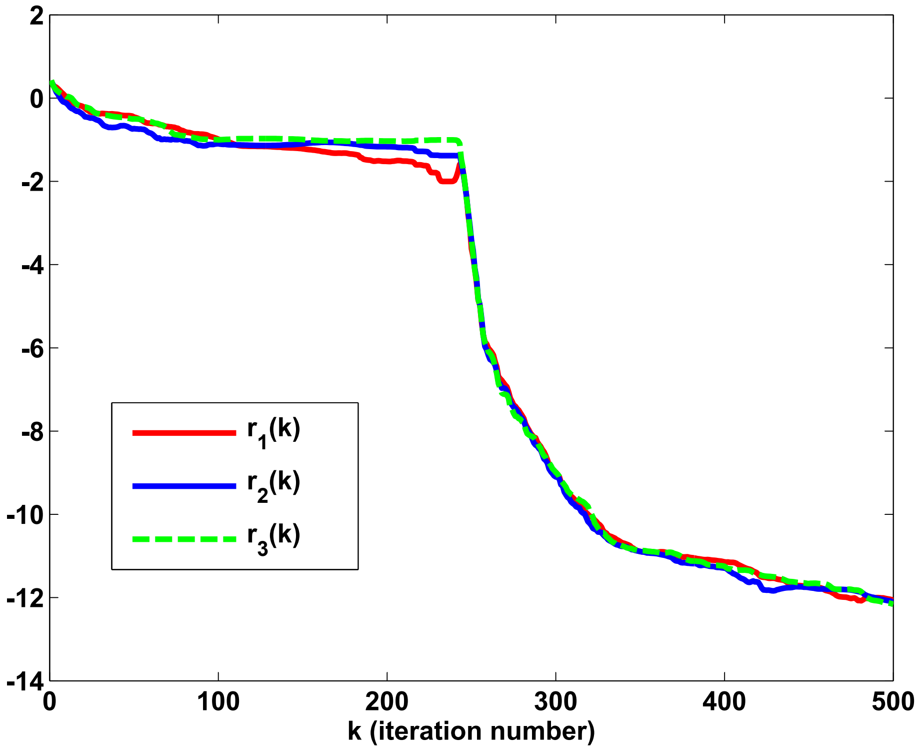
$$B_2 = \begin{pmatrix} -7 & -4 & 0 & -1 & -15 & 0 & -4 \\ -6 & 13 & -3 & -2 & -2 & 7 & -5 \\ 7 & -9 & -2 & -7 & -8 & 3 & -4 \\ 6 & 11 & -15 & 9 & 0 & 14 & -5 \\ -13 & -14 & -4 & -14 & 8 & -5 & -8 \\ -2 & -9 & -4 & 19 & -5 & 18 & -1 \\ -6 & -12 & 0 & -9 & -1 & 6 & -3 \end{pmatrix}, B_3 = \begin{pmatrix} 6 & 11 & -2 & 1 & 18 & 2 & 4 \\ 6 & -4 & 2 & 12 & -5 & -10 & 3 \\ -7 & 2 & 2 & 7 & 8 & -2 & 8 \\ -6 & -14 & 18 & -9 & 0 & -10 & 9 \\ 12 & 13 & 7 & 17 & -9 & 3 & 4 \\ 1 & 12 & 4 & -19 & 6 & -15 & 1 \\ 8 & 9 & 1 & 8 & -2 & 1 & 1 \end{pmatrix},$$

$$C_1 = C_2 = C_3 = \text{rand}(7).$$

By using Algorithm 1 with the initial matrices $X_1(1) = X_2(1) = X_3(1) = 0$, *we compute the sequences* $\{X_1(k)\}$, $\{X_2(k)\}$ *and* $\{X_3(k)\}$. *The numerical results are presented in Figure 2 where*

$$r_i(k) = \log ||C_i - A_iX_i(k) - X_{i+1}(k)B_i||, \quad for \quad i = 1, 2, 3.$$

Figure 2: The residuals for Example 2



The above numerical examples demonstrate the good accuracy of Algorithm 1.

5. Conclusions

In this paper, the least squares solutions of general discrete-time periodic matrix equations (1.1) have been discussed. Algorithm 1 has been introduced to find the solutions of Problems 1 and 2 corresponding to the general discrete-time periodic matrix equations (1.1). We have proven that Algorithm 1 converges in a finite number of steps in the absence of round-off errors and the norms of the residual matrices of this algorithm decrease monotonically during its iteration. Two numerical examples have been given to illustrate the efficiency of Algorithm 1.

Acknowledgments

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Appendix

The proof of Lemma 4

Step 1. Obviously it is enough to prove three statements of Lemma 4 for $1 \leq u < v \leq r$. For $u = 1$ and $v = 2$, we have

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(1), Z_{i,j}(2)\rangle = \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(1), Z_{i,j}(1) - \alpha(1)[A_{i,j}^T U_i(1)B_{i,j}^T + C_{i-1,j}^T U_{i-1}(1)D_{i-1,j}^T]\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}[\|Z_{i,j}(1)\|^2 - \alpha(1)\langle Z_{i,j}(1), A_{i,j}^T U_i(1)B_{i,j}^T + C_{i-1,j}^T U_{i-1}(1)D_{i-1,j}^T\rangle]$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(1)\|^2 - \alpha(1)\sum_{i=1}^{\xi}\langle\sum_{j=1}^{t}(A_{i,j}P_{i,j}(1)B_{i,j} + C_{i,j}P_{i+1,j}(1)D_{i,j}), U_i(1)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(1)\|^2 - \alpha(1)\sum_{i=1}^{\xi}\langle U_i(1), U_i(1)\rangle = 0,$$

$$\sum_{i=1}^{\xi}\langle U_i(1), U_i(2)\rangle$$

$$= \sum_{i=1}^{\xi}\langle U_i(1), \sum_{j=1}^{t}[A_{i,j}(Z_{i,j}(2) + \beta(1)P_{i,j}(1))B_{i,j} + C_{i,j}(Z_{i+1,j}(2) + \beta(1)P_{i+1,j}(1))D_{i,j}]\rangle$$

$$= \beta(1)\sum_{i=1}^{\xi}\|U_i(1)\|^2 + \sum_{i=1}^{\xi}\langle U_i(1), \sum_{j=1}^{t}(A_{i,j}Z_{i,j}(2)B_{i,j} + C_{i,j}Z_{i+1,j}(2)D_{i,j})\rangle$$

$$= \beta(1)\sum_{i=1}^{\xi}\|U_i(1)\|^2 + \frac{1}{\alpha(1)}\sum_{i=1}^{\xi}\langle R_i(1) - R_i(2), \sum_{j=1}^{t}(A_{i,j}Z_{i,j}(2)B_{i,j} + C_{i,j}Z_{i+1,j}(2)D_{i,j})\rangle$$

$$= \beta(1)\sum_{i=1}^{\xi}\|U_i(1)\|^2 + \frac{1}{\alpha(1)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle A_{i,j}(R_i(1) - R_i(2))B_{i,j} + C_{i-1,j}(R_{i-1}(1) - R_{i-1}(2))D_{i-1,j}, Z_{i,j}(2)\rangle$$

$$= \beta(1)\sum_{i=1}^{\xi}\|U_i(1)\|^2 + \frac{1}{\alpha(1)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(1) - Z_{i,j}(2), Z_{i,j}(2)\rangle$$

$$= \beta(1)\sum_{i=1}^{\xi}\|U_i(1)\|^2 - \frac{1}{\alpha(1)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(2)\|^2 = 0,$$

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(1), Z_{i,j}(2)\rangle = \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(1), Z_{i,j}(2)\rangle = 0.$$

Hence (I)-(III) hold for $u = 1$ and $v = 2$.
Step 2. In this step, for $u < w < r$ we suppose that

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(u), Z_{i,j}(w)\rangle = 0, \quad \sum_{i=1}^{\xi}\langle U_i(u), U_i(w)\rangle = 0,$$

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(u), Z_{i,j}(w)\rangle = 0.$$

Now we can obtain

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(u), Z_{i,j}(w+1)\rangle$$

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(u), Z_{i,j}(w) - \alpha(w)[A_{i,j}^{T}U_{i}(w)B_{i,j}^{T} + C_{i-1,j}^{T}U_{i-1}(w)D_{i-1,j}^{T}]\rangle$$

$$= -\alpha(w)\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(u), A_{i,j}^{T}U_{i}(w)B_{i,j}^{T} + C_{i-1,j}^{T}U_{i-1}(w)D_{i-1,j}^{T}\rangle$$

$$= -\alpha(w)\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle A_{i,j}Z_{i,j}(u)B_{i,j} + C_{i,j}Z_{i+1,j}(u)D_{i,j}, U_{i}(w)\rangle$$

$$= -\alpha(w)\sum_{i=1}^{\xi}\langle\sum_{j=1}^{t}[A_{i,j}(P_{i,j}(u) - \beta(u-1)P_{i,j}(u-1))B_{i,j} + C_{i,j}(P_{i+1,j}(u) - \beta(u-1)P_{i+1,j}(u-1))D_{i,j}], U_{i}(w)\rangle$$

$$= -\alpha(w)\sum_{i=1}^{\xi}\langle U_{i}(u) - \beta(u-1)U_{i}(u-1), U_{i}(w)\rangle = 0,$$

$$\sum_{i=1}^{\xi}\langle U_{i}(u), U_{i}(w+1)\rangle$$

$$= \sum_{i=1}^{\xi}\langle U_{i}(u), \sum_{j=1}^{t}[A_{i,j}(Z_{i,j}(w+1) + \beta(w)P_{i,j}(w))B_{i,j} + C_{i,j}(Z_{i+1,j}(w+1) + \beta(w)P_{i+1,j}(w))D_{i,j}]\rangle$$

$$= \sum_{i=1}^{\xi}\langle U_{i}(u), \sum_{j=1}^{t}(A_{i,j}Z_{i,j}(w+1)B_{i,j} + C_{i,j}Z_{i+1,j}(w+1)D_{i,j})\rangle$$

$$= \frac{1}{\alpha(u)}\sum_{i=1}^{\xi}\langle R_{i}(u) - R_{i}(u+1), \sum_{j=1}^{t}(A_{i,j}Z_{i,j}(w+1)B_{i,j} + C_{i,j}Z_{i+1,j}(w+1)D_{i,j})\rangle$$

$$= \frac{1}{\alpha(u)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle A_{i,j}(R_{i}(u) - R_{i}(u+1))B_{i,j} + C_{i-1,j}(R_{i-1}(u) - R_{i-1}(u+1))D_{i-1,j}, Z_{i,j}(w+1)\rangle$$

$$= \frac{1}{\alpha(u)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(u) - Z_{i,j}(u+1), Z_{i,j}(w+1)\rangle = -\frac{1}{\alpha(u)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(u+1), Z_{i,j}(w+1)\rangle, \quad (5.1)$$

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(u), Z_{i,j}(w+1)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(u), Z_{i,j}(w) - \alpha(w)[A_{i,j}^{T}U_{i}(w)B_{i,j}^{T} + C_{i-1,j}^{T}U_{i-1}(w)D_{i-1,j}^{T}]\rangle$$

$$= -\alpha(w)\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(u), A_{i,j}^T U_i(w)B_{i,j}^T + C_{i-1,j}^T U_{i-1}(w)D_{i-1,j}^T\rangle$$

$$= -\alpha(w)\sum_{i=1}^{\xi}\langle\sum_{j=1}^{t}(A_{i,j}P_{i,j}(u)B_{i,j} + C_{i,j}P_{i+1,j}(u)D_{i,j}), U_i(w)\rangle$$

$$= -\alpha(w)\sum_{i=1}^{\xi}\langle U_i(u), U_i(w)\rangle = 0.$$

Also for $u = w$, we deduce that

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(w), Z_{i,j}(w+1)\rangle = \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(w), Z_{i,j}(w) - \alpha(w)[A_{i,j}^T U_i(w)B_{i,j}^T + C_{i-1,j}^T U_{i-1}(w)D_{i-1,j}^T]\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}[\|Z_{i,j}(w)\|^2 - \alpha(w)\langle Z_{i,j}(w), A_{i,j}^T U_i(w)B_{i,j}^T + C_{i-1,j}^T U_{i-1}(w)D_{i-1,j}^T\rangle]$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w)\|^2 - \alpha(w)\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle A_{i,j}Z_{i,j}(w)B_{i,j} + C_{i,j}Z_{i+1,j}(w)D_{i,j}, U_i(w)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w)\|^2 - \alpha(w)\sum_{i=1}^{\xi}\langle\sum_{j=1}^{t}[A_{i,j}(P_{i,j}(w) - \beta(w-1)P_{i,j}(w-1))B_{i,j}$$
$$+C_{i,j}(P_{i+1,j}(w) - \beta(w-1)P_{i+1,j}(w-1))D_{i,j}], U_i(w)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w)\|^2 - \alpha(w)\sum_{i=1}^{\xi}\langle U_i(w) - \beta(w-1)U_i(w-1), U_i(w)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w)\|^2 - \alpha(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 = 0,$$

$$\sum_{i=1}^{\xi}\langle U_i(w), U_i(w+1)\rangle$$

$$= \sum_{i=1}^{\xi}\langle U_i(w), \sum_{j=1}^{t}[A_{i,j}(Z_{i,j}(w+1) + \beta(w)P_{i,j}(w))B_{i,j} + C_{i,j}(Z_{i+1,j}(w+1) + \beta(w)P_{i+1,j}(w))D_{i,j}]\rangle$$

$$= \beta(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 + \sum_{i=1}^{\xi}\sum_{i=1}^{\xi}\langle U_i(w), \sum_{j=1}^{t}(A_{i,j}Z_{i,j}(w+1)B_{i,j} + C_{i,j}Z_{i+1,j}(w+1)D_{i,j})\rangle$$

$$= \beta(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 + \frac{1}{\alpha(w)}\sum_{i=1}^{\xi}\langle R_i(w) - R_i(w+1), \sum_{j=1}^{t}(A_{i,j}Z_{i,j}(w+1)B_{i,j} + C_{i,j}Z_{i+1,j}(w+1)D_{i,j})\rangle$$

$$= \beta(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 + \frac{1}{\alpha(w)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle A_{i,j}(R_i(w) - R_i(w+1))B_{i,j} + C_{i-1,j}(R_{i-1}(w) - R_{i-1}(w+1))D_{i-1,j}, Z_{i,j}(w+1)\rangle$$

$$= \beta(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 + \frac{1}{\alpha(w)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(w) - Z_{i,j}(w+1), Z_{i,j}(w+1)\rangle$$

$$= \beta(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 - \frac{1}{\alpha(w)}\sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w+1)\|^2 = 0,$$

and

$$\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(w), Z_{i,j}(w+1)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(w), Z_{i,j}(w) - \alpha(w)[A_{i,j}^T U_i(w) B_{i,j}^T + C_{i-1,j}^T U_{i-1}(w) D_{i-1,j}^T]\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(w) + \beta(w-1)Z_{i,j}(w-1) + \beta(w-1)\beta(w-2)Z_{i,j}(w-2) + ... + \beta(w-1)...\beta(1)Z_{i,j}(1), Z_{i,j}(w)\rangle$$

$$-\alpha(w)\sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle P_{i,j}(w), A_{i,j}^T U_i(w) B_{i,j}^T + C_{i-1,j}^T U_{i-1}(w) D_{i-1,j}^T\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w)\|^2 - \alpha(w)\sum_{i=1}^{\xi}\langle\sum_{j=1}^{t}(A_{i,j}P_{i,j}(w)B_{i,j} + C_{i,j}P_{i+1,j}(w)D_{i,j}), U_i(w)\rangle$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{\xi}\|Z_{i,j}(w)\|^2 - \alpha(w)\sum_{i=1}^{\xi}\|U_i(w)\|^2 = 0.$$

By taking into account that

$$\sum_{i=1}^{\xi}\langle U_i(u), U_i(w)\rangle = 0, \qquad \sum_{j=1}^{t}\sum_{i=1}^{\xi}\langle Z_{i,j}(w), Z_{i,j}(w+1)\rangle = 0,$$

and (5.1), we have

$$\sum_{i=1}^{\xi}\langle U_i(u), U_i(w+1)\rangle = 0.$$

By considering Steps 1 and 2, three statements of Lemma 4 hold by the principle of induction.

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