Existence Results for Nonlinear Fractional q-Difference Equations with Nonlocal Riemann-Liouville q-Integral Boundary Conditions

Wengui Yang[a]

[a]*Department of Mathematics, Southeast University, Nanjing 210096, China*

Abstract. This paper deals with the existence and uniqueness of solutions for a class of nonlinear fractional q-difference equations boundary value problems involving four-point nonlocal Riemann-Liouville q-integral boundary conditions of different order. Our results are based on some well-known tools of fixed point theory such as Banach contraction principle, Krasnoselskii fixed point theorem, and the Leray-Schauder nonlinear alternative. As applications, some interesting examples are presented to illustrate the main results.

1. Introduction

This paper considers the following nonlinear fractional q-difference equations with nonlocal Riemann-Liouville q-integral boundary conditions of different order given by

$$
\begin{cases}
(^cD_q^\alpha u)(t) = f(t, u(t)), \quad t \in [0,1], \quad \alpha \in (1,2], \\
a_1 u(0) - b_1 (D_q u)(0) = c_1 (I_q^\beta u)(\eta) = c_1 \int_0^\eta \dfrac{(\eta - s)^{(\beta-1)}}{\Gamma_q(\beta)} u(s) d_q s, \quad \beta \in (0,1], \\
a_2 u(1) + b_2 (D_q u)(1) = c_2 (I_q^\gamma u)(\sigma) = c_2 \int_0^\sigma \dfrac{(\sigma - s)^{(\gamma-1)}}{\Gamma_q(\gamma)} u(s) d_q s, \quad \gamma \in (0,1],
\end{cases}
\tag{1.1}
$$

where $^cD_q^\alpha$ denotes the Caputo fractional q-derivative of order α, f is a given continuous function, and $a_i, b_i, c_i, \eta, \sigma$ are real constants with $0 < \eta, \sigma < 1$, for $i = 1, 2$.

Due to the fact that the tools of fractional calculus has numerous applications in various disciplines of science and engineering such as physics, mechanics, chemistry, biology, engineering, etc, the subject of fractional differential equations has gained a considerable attentions by a great deal of researchers. Therefore, there have been many papers and books dealing with the theoretical development of fractional calculus and the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations; for examples and details, one can see [3, 4, 10, 11, 25, 26, 38] and references along these lines. Ahmad and Alsaedi [7] considered the existence of solutions for a class of nonlinear Caputo type fractional

boundary value problems with nonlocal fractional integro-differential boundary conditions by applying some fixed point principles and Leray-Schauder degree theory. Ahmad and Nieto [9] studied some existence results for boundary value problems involving a nonlinear integrodifferential equation of fractional order with integral boundary conditions based on contraction mapping principle and Krasnoselskii fixed point theorem. Ahmad et al. [12] investigated the existence and uniqueness of solutions for a class of Caputo type fractional boundary value problem involving four-point nonlocal Riemann-Liouville integral boundary conditions of different order by means of standard tools of fixed point theory and Leray-Schauder nonlinear alternative.

The early work on q-difference calculus or quantum calculus appeared already in Jackson's papers [22, 23] at the beginning of the twentieth century, basic definitions and properties of quantum calculus can be found in the book [24]. For some recent existence results on q-difference equations, we refer to [5, 6, 8] and the references therein. Later, the fractional q-difference calculus has been proposed by Al-Salam [14] and Agarwal [2]. Recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus have been addressed extensively by several researchers. For example, some researcher obtained q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula, q-Mittage-Leffler function [1, 15, 29, 30], and so on.

Recently, many people pay attention to boundary value problems involving nonlinear fractional q-difference equations. There have been some papers dealing with the existence and multiplicity of solutions or positive solutions for boundary value problems involving nonlinear fractional q-difference equations by the use of some well-known fixed point theorems. For some recent developments on the subject, see [16, 17, 18, 28, 32, 33, 34, 35, 37] and the references therein. For example, Ferreira [19] considered the nonlinear fractional q-difference boundary value problem as follows.

$$\begin{cases} (D_q^\alpha u)(t) + f(u(t)) = 0, & t \in [0,1], \quad \alpha \in (2,3], \\ u(0) = (D_q u)(0) = 0, & (D_q u)(1) = \beta \geq 0, \end{cases}$$

where D_q^α is the q-derivative of Riemann-Liouville type of order α. By applying a fixed point theorem in cones, sufficient conditions for the existence of positive solutions were enunciated.

In [27], Liang and Zhang discussed the following nonlinear q-fractional three-point boundary value problem

$$\begin{cases} (D_q^\alpha u)(t) + f(t, u(t)) = 0, & t \in [0,1], \quad \alpha \in (2,3], \\ u(0) = (D_q u)(0) = 0, & (D_q u)(1) = \beta(D_q u)(\eta). \end{cases}$$

By using a fixed point theorem in partially ordered sets, the authors obtained sufficient conditions for the existence and uniqueness of positive and nondecreasing solutions to the above boundary value problem.

In [20], Graef and Kong investigated the following boundary value problem with fractional q-derivatives:

$$\begin{cases} (D_q^\alpha u)(t) + f(t, u(t)) = 0, & t \in [0,1], \quad \alpha \in (n-1, n], \quad n \in \mathbb{N}, \\ (D_q^i u)(0) = 0, \quad i = 0, 1, \ldots, n-2, \quad b(D_q u)(1) = \sum_{j=1}^{m} a_j(D_q u)(t_j) + \lambda. \end{cases}$$

where D_q^α is the q-derivative of Riemann-Liouville type of order α. The uniqueness, existence, and nonexistence of positive solutions are considered in terms of different ranges of λ.

In [13], Ahmad et al. studied the following nonlocal boundary value problems of nonlinear fractional q-difference equations:

$$\begin{cases} (^cD_q^\alpha u)(t) = f(t, u(t)), & t \in [0, 1], \quad \alpha \in (1, 2], \\ a_1 u(0) - b_1 (D_q u)(0) = c_1 u(\eta_1), & a_2 u(1) + b_2 (D_q u)(1) = c_2 u(\eta_2), \end{cases}$$

where $^cD_q^\alpha$ denotes the Caputo fractional q-derivative of order α, and $a_i, b_i, c_i, \eta_i \in \mathbb{R}$ ($i = 1, 2$). The existence of solutions for the problem were shown by applying some well-known tools of fixed point theory such as Banach contraction principle, Krasnoselskii fixed point theorem, and the Leray-Schauder nonlinear alternative.

In [36], the authors investigated the following nonlocal q-integral boundary value problem of nonlinear fractional q-derivatives equation:

$$\begin{cases} (D_q^\alpha u)(t) + f(t, u(t)) = 0, & t \in [0, 1], \quad \alpha \in (1, 2], \\ u(0) = 0, \quad u(1) = \mu(I_q^\beta)(\eta) = \mu \displaystyle\int_0^\eta \frac{(\eta - qs)^{(\beta-1)}}{\Gamma_q(\beta)} u(s) d_q s. \end{cases}$$

By using the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskiis fixed point theorem, some existence results of positive solutions to the above boundary value problems were obtained.

Motivated greatly by the above mentioned works, we establish the existence and uniqueness of solutions for boundary value problem (1.1). In Section 2, we shall give some definitions and lemmas to prove our main results. In Section 3, we establish the existence and uniqueness of solutions for boundary value problem (1.1) by applying some well-known tools of fixed point theory such as Banach contraction principle, Krasnoselskii fixed point theorem, and the Leray-Schauder nonlinear alternative. As applications, some interesting examples are presented to illustrate the main results in Section 4.

2. Preliminaries on Fractional q-Calculus

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of problem (1.1). These details can be found in the recent literature; see [24] and references therein.

Let $q \in (0, 1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a - b)^n$ with $n \in \mathbb{N}_0$ is

$$(a - b)^{(0)} = 1, \quad (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^\infty \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x + 1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)d_q t = x(1 - q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [24]. We now point out five formulas that will be used later ($_i D_q$ denotes the derivative with respect to variable i)

$$\int_a^b f(s)(D_q g)(s)d_q s = [f(s)g(s)]_{s=a}^{s=b} - \int_a^b (D_q f)(s)g(qs)d_q s \quad (q-\text{integration by parts}),$$

$$[a(t - s)]^{(\alpha)} = a^{\alpha}(t - s)^{(\alpha)}, \quad {}_t D_q(t - s)^{(\alpha)} = [\alpha]_q(t - s)^{(\alpha-1)},$$

$${}_s D_q(t - s)^{(\alpha)} = -[\alpha]_q(t - qs)^{(\alpha-1)}, \quad \left({}_x D_q \int_0^x f(x, t)d_q t\right)(x) = \int_0^x {}_x D_q f(x, t)d_q t + f(qx, x).$$

Denote that if $\alpha > 0$ and $a \le b \le t$, then $(t - a)^{(\alpha)} \ge (t - b)^{(\alpha)}$ [18].

Definition 2.1. (see [2]). Let $\alpha \ge 0$ and f be function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t)d_q t, \quad \alpha > 0, x \in [0, 1].$$

Definition 2.2. (see [29]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 2.3. (see [29]). *The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by*

$$({}^{c}D_{q}^{\alpha}f)(x) = (I_{q}^{m-\alpha}D_{q}^{m}f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.4. (see [30]). *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*

(1) $(I_{q}^{\beta}I_{q}^{\alpha}f)(x) = I_{q}^{\alpha+\beta}f(x)$,
(2) $(D_{q}^{\alpha}I_{q}^{\alpha}f)(x) = f(x)$.

Lemma 2.5. (see [29]). *Let $\alpha > 0$ and $\alpha \in \mathbb{R}^{+} \setminus \mathbb{N}$. Then the following equality holds:*

$$(I_{q}^{\alpha}{}^{c}D_{q}^{\alpha}f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^{k}}{\Gamma_{q}(k+1)}(D_{q}^{k}f)(0),$$

where m is the smallest integer greater than or equal to α.

Lemma 2.6. (see [30]). *For $\alpha \in \mathbb{R}^{+}$, $\lambda \in (-1, \infty)$, the following is valid:*

$$I_{q}^{\alpha}((x-a)^{(\lambda)}) = \frac{\Gamma_{q}(\lambda+1)}{\Gamma_{q}(\alpha+\lambda+1)}(x-a)^{(\alpha+\lambda)}, \quad 0 < a < x < b.$$

In particular, for $\lambda = 0$, $a = 0$, using q-integration by parts, we have [13]

$$\begin{aligned}
(I_{q}^{\alpha}1)(x) &= \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{x} (x - qt)^{(\alpha-1)}d_{q}t = \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{x} \frac{D_{q}((x-t)^{(\alpha)})}{-[\alpha]_{q}}d_{q}t \\
&= -\frac{1}{\Gamma_{q}(\alpha+1)} \int_{0}^{x} D_{q}((x-t)^{(\alpha)})d_{q}t = \frac{x^{(\alpha)}}{\Gamma_{q}(\alpha+1)}.
\end{aligned} \tag{2.1}$$

In additional, using q-integration by parts and (2.1), we obtain

$$\begin{aligned}
(I_{q}^{\alpha}t)(x) &= \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{x} (x - qt)^{(\alpha-1)}t\,d_{q}t = -\frac{1}{\Gamma_{q}(\alpha+1)} \int_{0}^{x} t D_{q}((x-t)^{(\alpha)})d_{q}t \\
&= -\frac{1}{\Gamma_{q}(\alpha+1)} \left([t(x-t)^{(\alpha)}]_{t=0}^{t=x} - \int_{0}^{x} (x - qt)^{(\alpha)}d_{q}t \right) = \frac{x^{(\alpha+1)}}{\Gamma_{q}(\alpha+2)}.
\end{aligned} \tag{2.2}$$

Lemma 2.7. *For any $y \in C[0,1]$, the unique solution of the linear fractional q-difference boundary value problem*

$$\begin{cases}
({}^{c}D_{q}^{\alpha}u)(t) = y(t), \quad t \in [0,1], \quad \alpha \in (1,2], \\
a_{1}u(0) - b_{1}(D_{q}u)(0) = c_{1}(I_{q}^{\beta}u)(\eta) = c_{1} \int_{0}^{\eta} \frac{(\eta-s)^{(\beta-1)}}{\Gamma_{q}(\beta)}u(s)d_{q}s, \quad \beta \in (0,1], \\
a_{2}u(1) + b_{2}(D_{q}u)(1) = c_{2}(I_{q}^{\gamma}u)(\sigma) = c_{2} \int_{0}^{\sigma} \frac{(\sigma-s)^{(\gamma-1)}}{\Gamma_{q}(\gamma)}u(s)d_{q}s, \quad \gamma \in (0,1],
\end{cases} \tag{2.3}$$

is given by

$$u(t) = (I_{q}^{\alpha}y)(t) + (v_{4} - v_{3}t)(I_{q}^{\alpha+\beta}y)(\eta) + (v_{2} + v_{1}t)\left[c_{2}(I_{q}^{\alpha+\gamma}y)(\sigma) - a_{2}(I_{q}^{\alpha}y)(1) - b_{2}(I_{q}^{\alpha-1}y)(1)\right]. \tag{2.4}$$

where

$$v_{1} = \frac{1}{v}\left(a_{1} - \frac{c_{1}\eta^{(\beta)}}{\Gamma_{q}(\beta+1)}\right), \quad v_{2} = \frac{1}{v}\left(b_{1} + \frac{c_{1}\eta^{(\beta+1)}}{\Gamma_{q}(\beta+2)}\right),$$

$$v_{3} = \frac{c_{1}}{v}\left(a_{2} - \frac{c_{2}\sigma^{(\gamma)}}{\Gamma_{q}(\gamma+1)}\right), \quad v_{4} = \frac{c_{1}}{v}\left(a_{2} + b_{2} - \frac{c_{2}\sigma^{(\gamma+1)}}{\Gamma_{q}(\gamma+2)}\right),$$

$$v = \left(a_{1} - \frac{c_{1}\eta^{(\beta)}}{\Gamma_{q}(\beta+1)}\right)\left(a_{2} + b_{2} - \frac{c_{2}\sigma^{(\gamma+1)}}{\Gamma_{q}(\gamma+2)}\right) + \left(b_{1} + \frac{c_{1}\eta^{(\beta+1)}}{\Gamma_{q}(\beta+2)}\right)\left(a_{2} - \frac{c_{2}\sigma^{(\gamma)}}{\Gamma_{q}(\gamma+1)}\right) \neq 0.$$

Proof. We may apply Lemma 2.4 and Lemma 2.5; we see that

$$u(t) = (I_q^\alpha y)(t) + d_1 + d_2 t = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s + d_1 + d_2 t. \tag{2.5}$$

Differentiating both sides of (2.5), we obtain

$$(D_q u)(t) = (I_q^{\alpha-1} y)(t) + d_2 = \int_0^t \frac{(t - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} y(s) d_q s + d_2. \tag{2.6}$$

Applying (2.1)-(2.2) and the boundary conditions for the problem (2.3) in (2.5) and (2.6), we find a system of equations

$$\left(a_1 - \frac{c_1 \eta^{(\beta)}}{\Gamma_q(\beta + 1)} \right) d_1 - \left(b_1 + \frac{c_1 \eta^{(\beta+1)}}{\Gamma_q(\beta + 2)} \right) d_2 = c_1 (I_q^{\alpha+\beta} y)(\eta),$$

$$\left(a_2 - \frac{c_2 \sigma^{(\gamma)}}{\Gamma_q(\gamma + 1)} \right) d_1 + \left(a_2 + b_2 - \frac{c_2 \sigma^{(\gamma+1)}}{\Gamma_q(\gamma + 2)} \right) d_2 = c_2 (I_q^{\alpha+\gamma} y)(\sigma) - a_2 (I_q^\alpha y)(1) - b_2 (I_q^{\alpha-1} y)(1).$$

Solving the above system of equations for d_1, d_2, we get

$$d_1 = \frac{1}{v} \left\{ \left(b_1 + \frac{c_1 \eta^{(\beta+1)}}{\Gamma_q(\beta + 2)} \right) \left[c_2 (I_q^{\alpha+\gamma} y)(\sigma) - a_2 (I_q^\alpha y)(1) - b_2 (I_q^{\alpha-1} y)(1) \right] + c_1 \left(a_2 + b_2 - \frac{c_2 \sigma^{(\gamma+1)}}{\Gamma_q(\gamma + 2)} \right) (I_q^{\alpha+\beta} y)(\eta) \right\}$$

$$= v_2 \left[c_2 (I_q^{\alpha+\gamma} y)(\sigma) - a_2 (I_q^\alpha y)(1) - b_2 (I_q^{\alpha-1} y)(1) \right] + v_4 (I_q^{\alpha+\beta} y)(\eta),$$

$$d_2 = \frac{1}{v} \left\{ \left(a_1 - \frac{c_1 \eta^{(\beta)}}{\Gamma_q(\beta + 1)} \right) \left[c_2 (I_q^{\alpha+\gamma} y)(\sigma) - a_2 (I_q^\alpha y)(1) - b_2 (I_q^{\alpha-1} y)(1) \right] - c_1 \left(a_2 - \frac{c_2 \sigma^{(\gamma)}}{\Gamma_q(\gamma + 1)} \right) (I_q^{\alpha+\beta} y)(\eta) \right\}$$

$$= v_1 \left[c_2 (I_q^{\alpha+\gamma} y)(\sigma) - a_2 (I_q^\alpha y)(1) - b_2 (I_q^{\alpha-1} y)(1) \right] - v_3 (I_q^{\alpha+\beta} y)(\eta).$$

Substituting the values of d_1, d_2 in (2.3), we get (2.2). This completes the proof. □

In view of Lemma 2.6, we define an operator $F : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ as

$$(Fu)(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s + (v_4 - v_3 t) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} f(s, u(s)) d_q s$$

$$+ (v_2 + v_1 t) \left(c_2 \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} f(s, u(s)) d_q s \right.$$

$$\left. - a_2 \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s - b_2 \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} f(s, u(s)) d_q s \right), \tag{2.7}$$

where v_1, v_2, v_3 and v_4 are given in Lemma 2.6. In view of Lemma 2.6 and (2.7), we obtain a fixed point problem $u = Fu$. Thus it follows that problem (1.1) has a solution if and only if we show that the operator F has a fixed point.

3. Main Results

Let $\mathscr{C} = C([0, 1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$.

Now we are in a position to present the first main results of this paper.

Theorem 3.1. *Assume that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and that there exists a q-integrable function $L : [0, 1] \to \mathbb{R}$ such that*

(A_1) $|f(t, u) - f(t, v)| \leq L(t)|u - v|$, $t \in [0, 1]$, $u, v \in \mathbb{R}$.

Then the boundary value problem (1.1) has a unique solution on $[0, 1]$ *provided*

$$
\begin{aligned}
\rho =& (1 + |a_2|(|v_1| + |v_2|))(I_q^\alpha L)(1) + (|v_3| + |v_4|)(I_q^{\alpha+\beta} L)(\eta) \\
& + |c_2|(|v_1| + |v_2|)(I_q^{\alpha+\gamma} L)(\sigma) + |b_2|(|v_1| + |v_2|)(I_q^{\alpha-1} L)(1) < 1.
\end{aligned} \tag{3.1}
$$

Proof. Let us set $\sup_{t\in[0,1]} |f(t, 0)| = M$ and choose $k \geq MA/(1 - \rho)$, where

$$
A = \frac{1 + (|a_2| + [\alpha]_q |b_2|)(|v_1| + |v_2|)}{\Gamma_q(\alpha + 1)} + \frac{\eta^{(\alpha+\beta)}(|v_3| + |v_4|)}{\Gamma_q(\alpha + \beta + 1)} + \frac{|c_2|\sigma^{(\alpha+\gamma)}(|v_1| + |v_2|)}{\Gamma_q(\alpha + \gamma + 1)} < 1. \tag{3.2}
$$

We define $B_k = \{u \in \mathscr{C} : \|u\| \leq k\}$ and show that $FB_k \subset B_k$, where F is defined by (2.7). For $u \in B_k$, we observe that

$$
|f(t, u(t))| \leq |f(t, u(t)) - f(t, 0)| + |f(t, 0)| \leq L(t)k + M.
$$

Then $u \in B_k$, $t \in [0, 1]$, we have

$$
\begin{aligned}
|(Fu)(t)| \leq& \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [L(s)k + M]d_q s + |v_4 - v_3 t| \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} [L(s)k + M]d_q s \\
& + |v_2 + v_1 t| \left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} [L(s)k + M]d_q s \right. \\
& \left. + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [L(s)k + M]d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} [L(s)k + M]d_q s \right) \\
\leq& M \left\{ \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} d_q s \right. \\
& + (|v_1| + |v_2|) \left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} d_q s + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \right. \\
& \left. \left. + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} d_q s \right) \right\} + k \left\{ \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L(s)d_q s + (|v_3| + |v_4|) \right. \\
& \times \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} L(s)d_q s + (|v_1| + |v_2|) \left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} L(s)d_q s \right. \\
& \left. \left. + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L(s)d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} L(s)d_q s \right) \right\} \\
\leq& M \left(\frac{1 + (|a_2| + [\alpha]_q |b_2|)(|v_1| + |v_2|)}{\Gamma_q(\alpha + 1)} + \frac{\eta^{(\alpha+\beta)}(|v_3| + |v_4|)}{\Gamma_q(\alpha + \beta + 1)} + \frac{|c_2|\sigma^{(\alpha+\gamma)}(|v_1| + |v_2|)}{\Gamma_q(\alpha + \gamma + 1)} \right) \\
& + k \left((1 + |a_2|(|v_1| + |v_2|)) \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L(s)d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} L(s)d_q s \right. \\
& \left. + |c_2|(|v_1| + |v_2|) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} L(s)d_q s + |b_2|(|v_1| + |v_2|) \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L(s)d_q s \right),
\end{aligned}
$$

which, in view of (3.1) and (3.2), implies that

$$
\|Fu\| \leq MA + k\rho \leq k.
$$

This shows that $FB_k \subset B_k$.

Now, for $u, v \in \mathscr{C}$, we obtain

$$\|(Fu) - (Fv)\|$$

$$\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_q s + |v_4 - v_3 t| \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} \right.$$

$$\times |f(s, u(s)) - f(s, v(s))| d_q s + |v_2 + v_1 t| \left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} |f(s, u(s)) - f(s, v(s))| d_q s \right.$$

$$\left. + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} |f(s, u(s)) - f(s, v(s))| d_q s \right) \right\}$$

$$\leq \|u - v\| \left((1 + |a_2|(|v_1| + |v_2|)) \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L(s) d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} L(s) d_q s \right.$$

$$\left. + |c_2|(|v_1| + |v_2|) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} L(s) d_q s + |b_2|(|v_1| + |v_2|) \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} L(s) d_q s \right),$$

which, in view of (3.1), yields

$$\|(Fu) - (Fv)\| \leq \rho \|u - v\|.$$

Since $\rho \in (0, 1)$ by assumption (3.1), therefore, F is a contraction. Hence, it follows by Banachs contraction principle that the problem (1.1) has a unique solution. $\square$

In case $L(t) = L$ (L is a constant), the condition (3.1) becomes $LA < 1$ and Theorem 3.1 takes the form of the following result.

Corollary 3.2. *Assume that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and that there exists a constant $L \in (0, 1/A)$ such that $|f(t, u) - f(t, v)| \leq L|u - v|$, $t \in [0, 1]$, $u, v \in \mathbb{R}$, where A is given by (3.2). Then the boundary value problem (1.1) has a unique solution on $[0, 1]$.*

Our next existence results is based on Krasnoselskii's fixed-point theorem [31].

Lemma 3.3. (Krasnoselskii). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be two operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exis$z \in M$ such that $z = Az + Bz$.*

Theorem 3.4. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous function satisfying (A_1). In addition, we assume that*

(A_2) *there exists a function $\mu \in C([0, 1], \mathbb{R}^+)$ and a nondecreasing function $\phi \in C([0, 1], \mathbb{R}^+)$ with*

$$|f(t, u)| \leq \mu(t)\phi(|u|), \quad (t, u) \in [0, 1] \times \mathbb{R};$$

(A_3) *let $\|\mu\| = \sup_{t \in [0,1]} |\mu(t)|$, and there exists a constant $\bar{r}$ with*

$$\bar{r} \geq \|\mu\| \phi(\bar{r}) \left(\frac{1 + (|a_2| + [\alpha]_q |b_2|)(|v_1| + |v_2|)}{\Gamma_q(\alpha + 1)} + \frac{\eta^{(\alpha+\beta)}(|v_3| + |v_4|)}{\Gamma_q(\alpha + \beta + 1)} + \frac{|c_2|\sigma^{(\alpha+\gamma)}(|v_1| + |v_2|)}{\Gamma_q(\alpha + \gamma + 1)} \right). \tag{3.3}$$

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$ provided

$$|a_2|(|v_1| + |v_2|)(I_q^\alpha L)(1) + (|v_3| + |v_4|)(I_q^{\alpha+\beta} L)(\eta)$$

$$+ |c_2|(|v_1| + |v_2|)(I_q^{\alpha+\gamma} L)(\sigma) + |b_2|(|v_1| + |v_2|)(I_q^{\alpha-1} L)(1) < 1. \tag{3.4}$$

Proof. Consider the set $B_{\bar{r}} = \{u \in \mathscr{C} : \|u\| \leq \bar{r}\}$, where $\bar{r}$ is given in (3.3) and define the operators P and Q on $B_{\bar{r}}$ as

$$(Pu)(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s, \quad t \in [0, 1],$$

$$(Qu)(t) = (v_4 - v_3 t) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} f(s, u(s)) d_q s + (v_2 + v_1 t)\left(c_2 \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} f(s, u(s)) d_q s\right.$$

$$\left. -a_2 \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s - b_2 \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} f(s, u(s)) d_q s\right), \quad t \in [0, 1].$$

For $u, v \in B_{\bar{r}}$, we find that

$$|(Pu + Qv)(t)| \leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(t)\phi(|u(s)|) d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha + \beta)} \mu(t)\phi(|v(s)|) d_q s$$

$$+ (|v_1| + |v_2|)\left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha + \gamma)} \mu(t)\phi(|v(s)|) d_q s\right.$$

$$\left. +|a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(t)\phi(|v(s)|) d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} \mu(t)\phi(|v(s)|) d_q s\right)$$

$$\leq \|\mu\|\phi(\bar{r})\left(\frac{1 + (|a_2| + [\alpha]_q|b_2|)(|v_1| + |v_2|)}{\Gamma_q(\alpha + 1)} + \frac{\eta^{(\alpha+\beta)}(|v_3| + |v_4|)}{\Gamma_q(\alpha + \beta + 1)} + \frac{|c_2|\sigma^{(\alpha+\gamma)}(|v_1| + |v_2|)}{\Gamma_q(\alpha + \gamma + 1)}\right)$$

$$\leq \bar{r}.$$

Thus, $Px + Qy \in B_{\bar{r}}$. From (A_1) and (3.4) it follows that Q is a contraction mapping. Continuity of f implies that the operator P is continuous. Also, P is uniformly bounded on B_r as $\|Pu\| \leq \frac{\phi(\bar{r})}{\Gamma_q(\alpha+1)}\|\mu\|$.

Now, for any $u \in B_r$, and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we have

$$|(Pu)(t_2) - (Pu)(t_1)| = \left|\int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s\right|$$

$$\leq \left|\int_0^{t_1} \frac{(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s\right| + \left|\int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s\right|$$

$$\leq \|\mu\|\phi(\bar{r})\left(\int_0^{t_1} \frac{(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s\right),$$

which is independent of u and tends to zero as $t_2 \to t_1$. Thus, P is equicontinuous. So P is relatively compact on $B_{\bar{r}}$. Hence, by the Arzela-Ascoli theorem, P is compact on $B_{\bar{r}}$. Thus, all the assumptions of Lemma 3.3 are satisfied. So the conclusion of Lemma 3.3 implies that the boundary value problem (1.1) has at least one solution on $[0, 1]$. $\square$

In the special case when $\phi(u) \equiv 1$, we can see that there always exists a positive r so that (3.3) holds true, thus we have the following corollary.

Corollary 3.5. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous function satisfying (A_1). In addition, we assume that $|f(t, u)| \leq \mu(t)$, $(t, u) \in [0, 1] \times \mathbb{R}$, $\mu \in C([0, 1], \mathbb{R}^+)$. If (3.4) holds, then the boundary value problem (1.1) has at least one solution on $[0, 1]$.*

The next existence result is based on Leray-Schauder nonlinear alternative [21].

Lemma 3.6. (Nonlinear alternative for single valued maps). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C with $0 \in U$. Suppose that $F : \bar{U} \to C$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of C) map. Then either* (i) *F has a fixed point in $\bar{U}$, or* (ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

Theorem 3.7. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous function. In addition, we assume that*

(A_4) *there exist functions $p_1, p_2 \in L^1([0, 1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, u)| \leq p_1(t)\psi(|u|) + p_2(t)$, for $(t, u) \in [0, 1] \times \mathbb{R}$;*

(A_5) *there exists a number $M > 0$ such that $M > \psi(M)\omega_1 + \omega_2$, where*

$$\omega_i = (1 + |a_2|(|v_1| + |v_2|))(I_q^\alpha p_i)(1) + (|v_3| + |v_4|)(I_q^{\alpha+\beta} p_i)(\eta)$$
$$+ |c_2|(|v_1| + |v_2|)(I_q^{\alpha+\gamma} p_i)(\sigma) + |b_2|(|v_1| + |v_2|)(I_q^{\alpha-1} p_i)(1), \quad i = 1, 2.$$

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof. Consider the operator $F : \mathscr{C} \to \mathscr{C}$ defined by (2.7). It is easy to show that F is continuous. Next, we show that F maps bounded sets into bounded sets in $\mathscr{C}$. For a positive number r, let $B_r = \{u \in \mathscr{C} : \|u\| \leq r\}$ be a bounded set in $\mathscr{C}$. Then we have

$$
\begin{aligned}
|(Fu)(t)| \leq & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s))| d_q s + |v_4 - v_3 t| \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} |f(s, u(s))| d_q s \\
& + |v_2 + v_1 t| \Bigg(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} |f(s, u(s))| d_q s \\
& + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s))| d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |f(s, u(s))| d_q s \Bigg) \\
\leq & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [p_1(s)\psi(\|u\|) + p_2(s)] d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} [p_1(s)\psi(\|u\|) + p_2(s)] d_q s \\
& + (|v_1| + |v_2|) \Bigg(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} [p_1(s)\psi(\|u\|) + p_2(s)] d_q s \\
& + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [p_1(s)\psi(\|u\|) + p_2(s)] d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} [p_1(s)\psi(\|u\|) + p_2(s)] d_q s \Bigg) \\
\leq & \psi(r) \Bigg((1 + |a_2|(|v_1| + |v_2|)) \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p_1(s) d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} p_1(s) d_q s \\
& + |c_2|(|v_1| + |v_2|) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} p_1(s) d_q s + |b_2|(|v_1| + |v_2|) \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} p_1(s) d_q s \Bigg) \\
& + \Bigg((1 + |a_2|(|v_1| + |v_2|)) \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p_2(s) d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} p_2(s) d_q s \\
& + |c_2|(|v_1| + |v_2|) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} p_2(s) d_q s + |b_2|(|v_1| + |v_2|) \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} p_2(s) d_q s \Bigg).
\end{aligned}
$$

Thus, for any $u \in B_r$, then $\|Fu\| \leq \psi(r)\omega_1 + \omega_2$ holds, which proves our assertion.

Now we show that F maps bounded sets into equicontinuous sets of $\mathscr{C}$. Let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, and $u \in B_r$, where B_r is a bounded set of $\mathscr{C}$. Then taking into consideration the inequality $(t_2 - qs)^{(\alpha-1)} - (t_1 -$

$qs)^{(\alpha-1)}(t_2 - t_1)$ for $0 < t_1 < t_2$, we have

$$
\begin{aligned}
&|(Fu)(t_2) - (Fu)(t_1)| \\
&\leq \left| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s \right| \\
&\quad + |v_3|(t_2 - t_1) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} f(s, u(s)) d_q s + |v_1|(t_2 - t_1)\left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} f(s, u(s)) d_q s \right. \\
&\quad \left. + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} f(s, u(s)) d_q s \right) \\
&\leq \int_0^{t_1} \frac{(t_2 - t_1)}{\Gamma_q(\alpha)} [p_1(s)\psi(r) + p_2(s)] d_q s + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [p_1(s)\psi(r) + p_2(s)] d_q s \\
&\quad + |v_3|(t_2 - t_1) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} [p_1(s)\psi(r) + p_2(s)] d_q s \\
&\quad + |v_1|(t_2 - t_1)\left(|c_2| \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} [p_1(s)\psi(r) + p_2(s)] d_q s \right. \\
&\quad \left. + |a_2| \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [p_1(s)\psi(r) + p_2(s)] d_q s + |b_2| \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} [p_1(s)\psi(r) + p_2(s)] d_q s \right).
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 \to t_1$. Therefore, it follows by the Arzela-Ascoli theorem that $F : \mathscr{C} \to \mathscr{C}$ is completely continuous.

Thus, the operator F satisfies all the conditions of Lemma 3.6, and hence by its conclusion, either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible.

Let $U = \{u \in \mathscr{C} : \|u\| < M\}$ with $\psi(M)\omega_1 + \omega_2 < M$. Then it can be shown that $\|Fu\| < M$. Indeed, in view of (A_4), we have

$$
\begin{aligned}
|(Fu)(t)| \leq &\psi(\|u\|)\left((1 + |a_2|(|v_1| + |v_2|)) \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p_1(s) d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} p_1(s) d_q s \right. \\
&\left. + |c_2|(|v_1| + |v_2|) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} p_1(s) d_q s + |b_2|(|v_1| + |v_2|) \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} p_1(s) d_q s \right) \\
&+ \left((1 + |a_2|(|v_1| + |v_2|)) \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p_2(s) d_q s + (|v_3| + |v_4|) \int_0^\eta \frac{(\eta - qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} p_2(s) d_q s \right. \\
&\left. + |c_2|(|v_1| + |v_2|) \int_0^\sigma \frac{(\sigma - qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha+\gamma)} p_2(s) d_q s + |b_2|(|v_1| + |v_2|) \int_0^1 \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} p_2(s) d_q s \right) \\
\leq &\psi(M)\omega_1 + \omega_2 < M.
\end{aligned}
$$

Suppose there exists a $u \in \partial U$ and a $\lambda \in (0, 1)$ such that $u = \lambda Fu$. Then for such a choice of u and λ, we have

$$
M = \|u\| = \lambda \|Fu\| < \psi(\|u\|)\omega_1 + \omega_2 = \psi(M)\omega_1 + \omega_2 < M,
$$

which is a contradiction. Consequently, by the Leray-Schauder alternative, we deduce that F has a fixed point $u \in \bar{U}$ which is a solution of the problem (1.1). This completes the proof. $\square$

4. Some Examples

Example 4.1. Consider the following fractional q-difference equations with nonlocal Riemann-Liouville q-integral boundary conditions

$$
\begin{cases}
(^cD_q^{3/2}u)(t) = \dfrac{1}{(t + \cos t + 3)^2}\left(\sin^2 t + \arctan u + \dfrac{|u|}{1 + |u|}\right), \quad t \in [0,1], \\
u(0) - \dfrac{1}{2}(D_q u)(0) = (I_q^{1/2}u)\left(\dfrac{2}{3}\right), \quad \dfrac{1}{3}u(1) + \dfrac{2}{3}(D_q u)(1) = (I_q^{1/3}u)\left(\dfrac{3}{4}\right).
\end{cases}
\tag{4.1}
$$

Here, $q = 1/2$, $\alpha = 3/2$, $\beta = 1/2$, $\gamma = 1/3$, $a_1 = 1$, $b_1 = 1/2$, $c_1 = 1$, $a_2 = 1/3$, $b_2 = 2/3$, $c_2 = 1$, $\eta = 2/3$, $\sigma = 3/4$ and $f(t,u) = \frac{1}{(t+\cos t+3)^2}\left(\sin^2 t + \arctan u + \frac{|u|}{1+|u|}\right)$. Clearly, $L = 2/9$ as $|f(t,u) - f(t,v)| \le (2/9)|u - v|$. Furthermore, by simple computation, we have

$$
LA = L\left(\frac{1 + (|a_2| + [\alpha]_q|b_2|)(|v_1| + |v_2|)}{\Gamma_q(\alpha + 1)} + \frac{\eta^{(\alpha+\beta)}(|v_3| + |v_4|)}{\Gamma_q(\alpha + \beta + 1)} + \frac{|c_2|\sigma^{(\alpha+\gamma)}(|v_1| + |v_2|)}{\Gamma_q(\alpha + \gamma + 1)}\right) \approx 0.894022 < 1.
$$

Thus all the assumptions of Corollary 3.2 are satisfied. So, by the conclusion of Corollary 3.2, problem (4.1) has a unique solution on $[0, 1]$.

Example 4.2. Consider the following fractional q-difference equations with nonlocal Riemann-Liouville q-integral boundary conditions

$$
\begin{cases}
(^cD_q^{3/2}u)(t) = \dfrac{1}{4}\cos t^2 \ln\left(1 + \dfrac{|u|}{2}\right) + \dfrac{e^u(1 + t^2)}{2 + t^2} + \dfrac{1}{3}, \quad t \in [0,1], \\
u(0) - \dfrac{1}{2}(D_q u)(0) = (I_q^{1/2}u)\left(\dfrac{1}{3}\right), \quad \dfrac{1}{3}u(1) + \dfrac{2}{3}(D_q u)(1) = (I_q^{1/3}u)\left(\dfrac{3}{4}\right).
\end{cases}
\tag{4.2}
$$

Here, $q = 1/2$, $\alpha = 3/2$, $\beta = 1/2$, $\gamma = 1/3$, $a_1 = 1$, $b_1 = 1/2$, $c_1 = 1$, $a_2 = 1/3$, $b_2 = 2/3$, $c_2 = 1$, $\eta = 2/3$, $\sigma = 3/4$ and $f(t,u) = \frac{1}{4}\cos t^2 \ln\left(1 + \frac{|u|}{2}\right) + \frac{e^u(1+t^2)}{2+t^2} + \frac{1}{3}$. Furthermore, by simple computation, we have

$$
|f(t,u)| = \left|\frac{1}{4}\cos t^2 \ln\left(1 + \frac{|u|}{2}\right) + \frac{e^u(1 + t^2)}{2 + t^2} + \frac{1}{3}\right| \le \frac{1}{8}|u| + 1.
$$

Clearly, $p_1 = 1/8$, $p_2 = 1$, $\psi(M) = M$. Consequently, $\omega_1 \approx 0.50288745$, $\omega_2 \approx 4.02309959$, and the condition (A_5) implies that $M > 8.09293506$. Thus, all the assumptions of Theorem 3.7 are satisfied. Therefore, the conclusion of Theorem 3.7 applies to the problem (4.2), then boundary value problem (4.2) has at least one solution on $[0, 1]$.

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