Negative Dependence for Fuzzy Random Variables:
Basic Definitions and Some Limit Theorems

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Abstract.

The concept of negative dependence for fuzzy random variables is introduced. The basic properties of such random variables are investigated. Some results on weak and strong convergence for sums and weighted sums of pairwise negatively dependent fuzzy random variables are derived. As a direct extension of classical methods, some limit theorems are established based on the concept of variance and covariance.

1. Introduction

A fuzzy random variable has been extended as a vague perception of a real valued random variable and subsequently redefined as a particular random set, see e.g. [15], [20], [28], [33], and [29]. Over the last years, fuzzy random variable has been extensively applied in areas of stochastic process and probability theory. For the purposes of this study, we review some works on this topic. By using a certain distance on the space of fuzzy numbers, Miyakoshi and Shimbo [19] obtained a strong law of large numbers for independent fuzzy random variables. Klement et al. [14] established a strong law of large numbers for fuzzy random variables, based on embedding theorem as well as certain probability techniques in Banach spaces. Taylor et al. [34] proved a weak law of large numbers for fuzzy random variables in separable Banach spaces. Joo et al. [13] obtained Chung's type strong law of large numbers for fuzzy random variables based on isomorphic isometric embedding theorem. Guan and Li [9] presented weak and strong law of large numbers for weighted sums of independent(not necessarily identical distributed) fuzzy random variables in the sense of the extended Hausdorff metric. Based on the strong law of large numbers for fuzzy random variables with respect to the uniform metric, Wang [36] established some asymptotical properties of point estimation with fuzzy random samples. Fu and Zhang [8] obtained some strong limit theorems for fuzzy random variables with slowly varying weight. Li and Zhung [18] and Hong [10] obtained a general method for certain convergence theorems of fuzzy random variables based on Hausdorff metric. It should be mentioned that, although the concept of variance has been found very convenient in studying

limit theorems, but, as the authors know, it has not been developed the limit theorems for fuzzy random variables based on the concept of variance, except the work by Feng [6]. Also, several authors studied the concept of variance for fuzzy random variables, for instance [7, 24]. Based on a natural extension of the concept of variance, he extended the Kolmogorov's inequality to independent fuzzy random variables and obtained some limit theorems. His method is a direct application of classical methods in probability theory to fuzzy random variables. Hong and Kim [11] obtained weak laws of large numbers for sums of independent and identically distributed fuzzy random variables. As an application in probability theory, Yang and Liu [39] investigated some known inequalities such as Minkowski, Chebyshev and Jensen's inequalities for fuzzy random variables. It is mentioned that Mohiuddine et al.[21] studied the concepts of statistically convergent and statistically Cauchy double sequences in the framework of fuzzy norm spaces which provide better tool to study a more general class of sequences. Ahmadzade et al. [1] established some limit theorems for independent fuzzy random variables. As everyone knows, selecting of suitable metric spaces plays important role in studying of convergence theorems. Thus, Ahmadzade et al. [2] derived several convergence theorems for fuzzy martingales based on $D_{p,q}$-metric.

On the other hand, in many practical stochastic models, the assumption of independence among the random variables is not plausible. In fact, increases in some random variables are often related to decreases in other random variables and the assumption of negative dependence is more appropriate than independence assumption. Lehmann [16], for the first time, introduced the concept of negatively quadrant dependent random variables and considered some properties of such random variables. Newman [26] derived some limit theorems for positively and negatively dependent random variables. Bozorgnia et al. [5] obtained some limit theorems for pairwise negatively dependent random variables. Amini et al. [3] proved a strong law of large number for negatively dependent generalized Guassian random variables. Also, Amini et al. [4] derived some strong limit theorems of weighted sums for negatively dependent generalized Gaussian random variables. Li and Yang [17] investigated a class of strong limit theorem for negatively quadrant dependent random variables by using truncation methods and generalized three series theorem. Wu and Guan [38] established mean convergence theorems and weak laws of large numbers for weighted sums of dependent random variables which contained negatively quadrant dependent random variables. Ranjbar et al. [30] studied asymptotic behavior of product of two heavy-tailed dependent random variables. Shen et al. [31] established a Kolmogorov-type inequality for negatively superadditive dependent (NSD) random variables. As a direct extension of negatively dependent random variables, we introduce the concept of negative dependence for fuzzy random variables. By using definition of the variance and covariance of fuzzy random variables, introduced by Feng et al. [7], we prove some limit theorems for negatively dependent fuzzy random variables. Also, we generalize the various extended limit theorems in classical probability to fuzzy random variables. The structure of this paper is as follows. In Section 2, we recall some preliminaries of fuzzy arithmetic and fuzzy random variables. In Section 3, we introduce the concept of negative dependence for fuzzy random variables and investigate some properties of such random variables. Some weak and strong convergence theorems for negatively dependent fuzzy random variables are studied and investigated, based on the concept of variance and covariance, in Section 4. In the final section, a brief conclusion and some proposals for future research are given.

2. Preliminaries

In this section, we provide some definitions and elementary concepts of fuzzy set theory that will be used in the next sections. For more details, the reader is referred to [7, 22, 35].

2.1. Fuzzy sets and fuzzy arithmetic

Define $E = \{\tilde{u} : \mathbb{R} \to [0,1] | \tilde{u}$ satisfies (i)-(iii) $\}$, where (i)$\tilde{u}$ is normal; (ii) $\tilde{u}$ is convex fuzzy (iii) $\tilde{u}$ is upper semicontinuous. For $\tilde{u} \in E, [\tilde{u}]^r = \{x \in \mathbb{R} | \tilde{u}(x) \geq r, 0 < r \leq 1\}$ is r-level set of $\tilde{u}$. We invoke the notations $\oplus, \ominus, \odot$ and further more we have
(i) $[\tilde{u} \oplus \tilde{v}]^r = [\tilde{u}^-(r) + \tilde{v}^-(r), \tilde{u}^+(r) + \tilde{v}^+(r)]$.

(ii) $[\tilde{u} \ominus \tilde{v}]^r = [\tilde{u}^-(r) - \tilde{v}^+(r), \tilde{u}^+(r) - \tilde{v}^-(r)]$.

(iii) $[a \odot \tilde{u}]^r = [a\tilde{u}^-(r), a\tilde{u}^+(r)]$, if $a > 0$ and $[a \odot \tilde{u}]^r = [a\tilde{u}^+(r), a\tilde{u}^-(r)]$, if $a < 0$.

To prove main theorems we need to apply the order relation. Thus, we use notations $\prec, \succ, \preceq$ and $\succeq$ which mean:

$\tilde{a} \prec \tilde{b}$ if and only if $a^-(r) < b^-(r)$ and $a^+(r) < b^+(r)$ $\forall r \in [0,1]$,

$\tilde{a} \succ \tilde{b}$ if and only if $a^-(r) > b^-(r)$ and $a^+(r) > b^+(r)$ $\forall r \in [0,1]$,

$\tilde{a} \preceq \tilde{b}$ if and only if $a^-(r) \le b^-(r)$ and $a^+(r) \le b^+(r)$ $\forall r \in [0,1]$,

$\tilde{a} \succeq \tilde{b}$ if and only if $a^-(r) \ge b^-(r)$ and $a^+(r) \ge b^+(r)$ $\forall r \in [0,1]$.

It is assumed that $\tilde{1}_{\{a\}}$ is a fuzzy number in E whose membership function equals 1 at a and zero otherwise, on the other hand $\tilde{1}_{\{a\}}$ is a crisp number. Let $\tilde{u}, \tilde{v} \in E$, and set

$$d_p(\tilde{u}, \tilde{v}) = \left(\int_0^1 h^p([\tilde{u}]^r, [\tilde{v}]^r) dr \right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

$d_\infty(\tilde{u}, \tilde{v}) = sup_{0 < r \le 1} h([\tilde{u}]^r, [\tilde{v}]^r)$, where h is Hausdorff metric i.e.

$h([\tilde{u}]^r, [\tilde{v}]^r) = max\{|u^-(r) - v^-(r)|, |u^+(r) - v^+(r)|\}$. Norm $\|\tilde{u}\|_p$ of a fuzzy number $\tilde{u} \in E$ is defined by $\|\tilde{u}\|_p = d_p(\tilde{u}, \tilde{0})$, where $\tilde{0}$ is the fuzzy number in E whose membership function equals 1 at 0 and zero otherwise. The norm of $\tilde{u}$ is defined by $\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0})$. The operation $\langle ., . \rangle : E \times E \to [-\infty, \infty]$ is defined by

$$< \tilde{u}, \tilde{v} >= \int_0^1 (\tilde{u}^-(r)\tilde{v}^-(r) + \tilde{u}^+(r)\tilde{v}^+(r)) dr.$$

If the indeterminacy of the form $\infty - \infty$ arises in the Lebesgue integral, then we say that $\langle \tilde{u}, \tilde{v} \rangle$ does not exist. It is easy to see that the operation $\langle ., . \rangle$ has following properties:

(i) $\langle \tilde{u}, \tilde{u} \rangle \ge 0$ *and* $\langle \tilde{u}, \tilde{u} \rangle = 0 \Leftrightarrow u = \tilde{0}$,

(ii) $\langle \tilde{u}, \tilde{v} \rangle = \langle \tilde{v}, \tilde{u} \rangle$,

(iii) $\langle \tilde{u} + \tilde{v}, \tilde{w} \rangle = \langle \tilde{u}, \tilde{w} \rangle + \langle \tilde{v}, \tilde{w} \rangle$,

(iv) $\langle \lambda \tilde{u}, \tilde{v} \rangle = \lambda \langle \tilde{u}, \tilde{v} \rangle$,

(v) $|\langle \tilde{u}, \tilde{v} \rangle| < \sqrt{\langle \tilde{u}, \tilde{u} \rangle \langle \tilde{v}, \tilde{v} \rangle}$.

For all $\tilde{u}, \tilde{v} \in E$, if $\langle \tilde{u}, \tilde{u} \rangle < \infty$ and $\langle \tilde{v}, \tilde{v} \rangle < \infty$ then the property (v) implies that $\langle \tilde{u}, \tilde{v} \rangle < \infty$. So, we can define

$$d_*(\tilde{u}, \tilde{v}) = \sqrt{\langle \tilde{u}, \tilde{u} \rangle - 2\langle \tilde{u}, \tilde{v} \rangle + \langle \tilde{v}, \tilde{v} \rangle}.$$

In fact, d_* is a metric in $\{\tilde{u} \in E | \langle \tilde{u}, \tilde{u} \rangle < \infty\}$ i.e. for $\tilde{x}, \tilde{y}, \tilde{z}$ in E, the metric d_* satisfies the following conditions.

i) $d_*(\tilde{x}, \tilde{y}) \ge 0$

ii) $d_*(\tilde{x}, \tilde{y}) = 0$ iff $\tilde{x} = \tilde{y}$,

iii) $d_*(\tilde{x}, \tilde{z}) \le d_*(\tilde{x}, \tilde{y}) + d_*(\tilde{y}, \tilde{z})$ (subadditivity or triangle inequality).

Moreover, the norm $\|\tilde{u}\|_*$ of fuzzy number $\tilde{u} \in E$ is defined by $\|\tilde{u}\|_* = d_*(\tilde{u}, \tilde{0})$.

2.2. Fuzzy random variables

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A fuzzy random variable (briefly: f.r.v.) is a Borel measurable function $\tilde{X} : (\Omega, \mathcal{A}) \to (E, d_\infty)$. Let $\tilde{X}$ be a f.r.v. is defined on $(\Omega, \mathcal{A}, P)$ then $[\tilde{X}]^r = [\tilde{X}^-(r), \tilde{X}^+(r)], r \in (0,1]$, is a random closed interval set, and $\tilde{X}^-(r)$ and $\tilde{X}^+(r)$ are real valued random variables. A f.r.v. $\tilde{X}$ is called integrably bounded if $E\|\tilde{X}\|_\infty < \infty$ and the expectation value $E\tilde{X}$ is defined as the unique fuzzy number which satisfies the property $[E\tilde{X}]^r = E[\tilde{X}]^r, 0 < r \le 1$ [28].

Example 2.1. *Let $\tilde{X}$ be a fuzzy random variable with the following membership function*

$$\mu_{\tilde{X}}(x) = \begin{cases} \dfrac{x - \xi}{\gamma}, & \xi < x \le \xi + \gamma, \\[2ex] \dfrac{\xi + \gamma + \zeta - x}{\zeta}, & \xi + \gamma < x < \xi + \gamma + \zeta, \\[2ex] 0, & otherwise, \end{cases}$$

where ξ, γ and ζ are random variables. It is easy to see that

$$[\tilde{X}]^r = [\xi + r\gamma, \xi + \gamma + (1 - r)\zeta].$$

By invoking the equation $[E\tilde{X}]^r = E[\tilde{X}]^r, \ \forall r \in [0, 1]$, we can write $E\tilde{X}$ has the following membership function

$$\mu_{\bar{X}}(x) = \begin{cases} \dfrac{x - E\xi}{E\gamma}, & E\xi < x \le E\xi + E\gamma, \\[2ex] \dfrac{E\xi + E\gamma + E\zeta - x}{E\zeta}, & E\xi + E\gamma < x < E\xi + E\gamma + E\zeta, \\[2ex] 0, & otherwise. \end{cases}$$

Definition 2.2. *([7]) Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s in L_2 ($L_2 = \{\tilde{X}|\tilde{X}$ is f.r.v. and $E\|\tilde{X}\|_2^2 < \infty\}$). The covariance of $\tilde{X}$ and $\tilde{Y}$ is defined as*

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2} \int_0^1 (Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r)))dr.$$

Specially, the variance of $\tilde{X}$ is defined by $Var(\tilde{X}) = Cov(\tilde{X}, \tilde{X})$.

Theorem 2.3. *([7]) Let $\tilde{X}$ and $\tilde{Y}$ be f.r.v.'s in L_2 and $\tilde{u}, \tilde{v} \in E$ and $\lambda \in \mathbb{R}$, then*

$$(i) \quad Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}(E\langle \tilde{X}, \tilde{Y} \rangle - \langle E\tilde{X}, E\tilde{Y} \rangle)$$

$$(ii) \quad Var(\tilde{X}) = \frac{1}{2}Ed_*^2(\tilde{X}, E\tilde{X})$$

$(iii) \quad Cov(\lambda\tilde{X} \oplus \tilde{u}, k\tilde{Y} \oplus \tilde{v}) = \lambda k Cov(\tilde{X}, \tilde{Y})$

$(iv) \quad Var(\lambda\tilde{X} \oplus \tilde{u}) = \lambda^2 Var(\tilde{X});$

$(v) \quad Var(\tilde{X} \oplus \tilde{Y}) = Var(\tilde{X}) + Var(\tilde{Y}) + 2Cov(\tilde{X}, \tilde{Y}).$

The following Lemma which is due to Hoeffding shows the relationship between quadrant dependent and correlated real valued random variables.

Lemma 2.4. *([12]) Let X and Y be real valued random variables with joint distribution F and margins F_1 and F_2, respectively, then*

$$\begin{aligned} Cov(X, Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{F(x, y) - F_1(x)F_2(y)\}dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{P(X > x, Y > y) - P(X > x)P(Y > y)\}dxdy. \end{aligned}$$

In order to establish strong and weak convergence, we need the following definitions.

Definition 2.5. *([37]) Let $\tilde{X}$ and $\tilde{X}_n$ be f.r.v.'s are defined on the same probability space $(\Omega, \mathcal{A}, P)$. i) We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ in probability with respect to the metric d_* if, for all $\epsilon > 0$, $\lim_{n\to\infty} P(\omega : d_*(\tilde{X}_n(\omega), \tilde{X}(\omega)) > \epsilon) = 0$. ii) We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ almost surely (briefly: a.s.) with respect to the metric d_* if $P(\omega : \lim_{n\to\infty} d_*(\tilde{X}_n(\omega), \tilde{X}(\omega)) = 0) = 1$.*

Statistical convergence and limit theorems with respect to the fuzzy normed space play important roles in mathematical analysis, for more details, see [23]. Throughout this paper it is assumed that all of f.r.v.'s are defined on the probability space $(\Omega, \mathcal{A}, P)$.

3. Negatively Dependent f.r.v.'s: Definition and Some Properties

In the following, we introduce and investigate the concept of negatively dependent f.r.v.'s. It should be mentioned that, in order to prove a generalized law of large numbers for fuzzy valued random variables, Viertl [35] introduced the concept of independence for f.r.v.'s based on random sets as follows.

Definition 3.1. *[35] Two f.r.v.'s $\tilde{X}$ and $\tilde{Y}$ are said independent if for any Borel sets B_1 and B_2 and all $r \in (0, 1]$*

$$P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) = P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_2),$$

where, $P([\tilde{X}]^r \subset B) = P(\omega : [\tilde{X}]^r(\omega) \subset B)$.

By inception of Viertl's definition, we introduce the concept of negatively dependent for f.r.v.'s.

Definition 3.2. *Two f.r.v.'s $\tilde{X}$ and $\tilde{Y}$ are said negatively dependent if for any Borel sets B_1 and B_2 and all $r \in (0, 1]$*

$$P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) \leq P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_2),$$

where, $P([\tilde{X}]^r \subset B) = P(\omega : [\tilde{X}]^r(\omega) \subset B)$.

Remark 3.3. *If $\tilde{X}$ and $\tilde{Y}$ reduce to real valued random variables and $B_1 = (-\infty, x_1]$ or (x_2, ∞) and $B_2 = (-\infty, y_1]$ or $(y_2, +\infty)$, Definition 3.2 conclude the concept of negatively dependence in the case of real valued random variables. Note that, in the ordinary case, two real valued random variables X and Y are said to be negatively dependent random variables if*

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y) \quad \forall \ x \text{ and } y \ \in \mathbb{R}.$$

For more details, see [25].

The following example explains a case where we have two f.r.v.s which are not independent but negatively dependent.

Example 3.4. *If $\tilde{X}$ and $\tilde{Y}$ have following probability mass, then $\tilde{X}$ and $\tilde{Y}$ are ND f.r.v.'s,*

$$P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) = \frac{1}{2}, P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) = 0,$$

$$P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) = \frac{1}{2}, P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = 0,$$

where $\tilde{u}$ and $\tilde{v}$ are fuzzy numbers with the following membership function respectively

$$\mu_{\tilde{u}}(x) = \begin{cases} x, & 0 \leq x < 1, \\ 1, & 1 \leq x \leq 2, \\ 3 - x, & 2 < x < 3, \\ 0, & otherwise, \end{cases}$$

and

$$\mu_{\tilde{v}}(x) = \begin{cases} 2x - 2, & 1 \leq x < \frac{3}{2}, \\ 4 - 2x, & \frac{3}{2} \leq x \leq 2, \\ 0, & otherwise. \end{cases}$$

The membership functions of ũ and ṽ are presented in Figure 1.
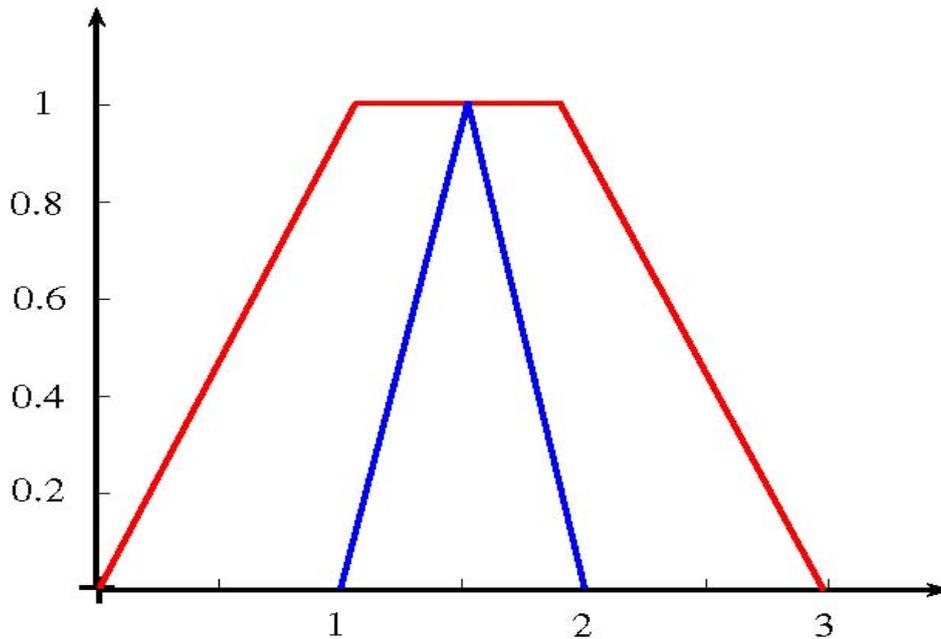
Figure 1: The membership functions of $\tilde{u}$ and $\tilde{v}$ in Example 1

Then, $\tilde{X}$ and $\tilde{Y}$ are ND f.r.v.s. Since, for $B_1 = [x, y], (x, y], [x, y), (x, y), (x, +\infty), [x, +\infty), (-\infty, y), (-\infty, y]$ where $1 \leq x \leq \frac{3}{2}$ and $\frac{3}{2} \leq y \leq 2$ also $B_2 = [z, w], (z, w], [z, w), (z, w), (z, +\infty), [z, +\infty), (-\infty, w), (-\infty, w]$ where $1 \leq z \leq \frac{3}{2}$ and $\frac{3}{2} \leq w \leq 2$, we obtain

$$
\begin{aligned}
P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) = 0 \\
&< P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_2) \\
&= P(\tilde{X} = \tilde{v})P(\tilde{Y} = \tilde{v}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.
\end{aligned}
$$

For $B_3 = [x, y], (x, y], [x, y), (x, y), (x, +\infty), [x, +\infty), (-\infty, y), (-\infty, y]$ where $0 \leq x \leq 1$ and $2 \leq y \leq 3$ also $B_4 = [z, w], (z, w], [z, w), (z, w), (z, +\infty), [z, +\infty), (-\infty, w), (-\infty, w]$ where $0 \leq z \leq 1$ and $2 \leq w \leq 3$, we obtain

$$
\begin{aligned}
P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_3) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) \\
&\quad \frac{3}{10} + \frac{2}{10} = \frac{5}{10} \\
&= P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_3) \\
&= P(\tilde{X} = \tilde{v})\{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v})\} \\
&= \frac{5}{10} \times 1.
\end{aligned}
$$

Also,

$$
\begin{aligned}
P([\tilde{X}]^r \subset B_3, [\tilde{Y}]^r \subset B_1) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) \\
&= 0 + \frac{1}{2} = \frac{1}{2} \\
&= P([\tilde{X}]^r \subset B_3)P([\tilde{Y}]^r \subset B_1) \\
&= P(\tilde{Y} = \tilde{v})\{P(\tilde{X} = \tilde{u}) + P(\tilde{X} = \tilde{v})\} \\
&= \frac{1}{2} \times 1.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
P([\tilde{X}]^r \subset B_3, [\tilde{Y}]^r \subset B_4) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) \\
&+ P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = 1 \\
&= P([\tilde{X}]^r \subset B_3)P([\tilde{Y}]^r \subset B_4) \\
&= \{P(\tilde{Y} = \tilde{v}) + P(\tilde{Y} = \tilde{u})\}\{P(\tilde{X} = \tilde{u}) + P(\tilde{X} = \tilde{v})\} \\
&= 1 \times 1.
\end{aligned}
$$

Definition 3.5. *A finite collection of f.r.v.'s $\tilde{X}_1, ..., \tilde{X}_m$ is said to be pairwise negatively dependent if for every Borel sets $B_1, ..., B_m$*

$$
P([\tilde{X}_i]^r \subset B_i, [\tilde{X}_j]^r \subset B_j]) \leq P([\tilde{X}_i]^r \subset B_i)P([\tilde{X}_j]^r \subset B_j), \quad \forall \ i \neq j.
$$

An infinite sequence of f.r.v.'s $\{\tilde{X}_n, n \geq 1\}$ is said pairwise negatively dependent if every finite sub collection of it, is pairwise negatively dependent.

Nondecreasing continuous functions play important roles in convergence of negative dependent random variables. Thus, we need the following lemma to prove the main theorems.

Lemma 3.6. *Let $\tilde{X}$ and $\tilde{Y}$ be negatively dependent f.r.v.'s and f and g be nondecreasing continuous functions. Then, $f(\tilde{X})$ and $g(\tilde{Y})$ are negatively dependent f.r.v.'s.*

Proof. Continuity and monotonicity of f imply that $[f(\tilde{X})]^r = f([\tilde{X}]^r) = [f(X^-(r)), f(X^+(r))]$ [22, 27]. So, for all Borel sets B_1 and B_2 we have

$$
\begin{aligned}
P([f(\tilde{X})]^r \subset B_1, [g(\tilde{Y})]^r \subset B_2) &= P(f([\tilde{X}]^r) \subset B_1, g([\tilde{Y}]^r) \subset B_2) \\
&= P([\tilde{X}]^r \subset f^{-1}(B_1), [\tilde{Y}]^r \subset g^{-1}(B_2)) \\
&\leq P([\tilde{X}]^r \subset f^{-1}(B_1))P([\tilde{Y}] \subset g^{-1}(B_2)) \\
&= P(f([\tilde{X}]^r) \subset B_1)P(g([\tilde{Y}]^r) \subset B_2) \\
&= P([f(\tilde{X})]^r \subset B_1)P([g(\tilde{Y})]^r \subset B_2),
\end{aligned}
$$

where f^{-1} and g^{-1} are inverse images of f and g, respectively. $\square$

Lemma 3.7. *Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s, then*

$$
\begin{aligned}
Cov(\tilde{X}, \tilde{Y}) &= \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (x, \infty), [\tilde{Y}]^r \subset (y, \infty)) \\
&- P([\tilde{X}]^r \subset (x, \infty))P([\tilde{Y}]^r \subset (y, \infty))dxdydr \\
&+ \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (-\infty, w), [\tilde{Y}]^r \subset (-\infty, z)) \\
&- P([\tilde{X}]^r \subset (-\infty, w))P([\tilde{Y}]^r \subset (-\infty, z))dwdzdr.
\end{aligned}
$$

Proof. By Definition 2.2, we have

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2} \int_0^1 \{Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r))\} dr.$$

But, by Lemma 2.4,

$$
\begin{aligned}
Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y) dx dy \\
&\quad - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\tilde{X}^-(r) > x) P(\tilde{Y}^-(r) > y) dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (x, \infty), [\tilde{Y}]^r \subset (y, \infty)) dx dy \\
&\quad - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (x, \infty)) P([\tilde{Y}]^r \subset (y, \infty)) dx dy,
\end{aligned}
$$

and

$$
\begin{aligned}
Cov(\tilde{X}^+(r), \tilde{Y}^+(r)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\tilde{X}^+(r) \le w, \tilde{Y}^+(r) \le z) dw dz \\
&\quad - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\tilde{X}^+(r) \le w) P(\tilde{Y}^+(r) \le z) dw dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (-\infty, w], [\tilde{Y}]^r \subset (-\infty, z]) dw dz \\
&\quad - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (-\infty, w]) P([\tilde{Y}]^r \subset (-\infty, z]) dw dz.
\end{aligned}
$$

These complete the proof. □

Corollary 3.8. *Let $\tilde{X}$ and $\tilde{Y}$ be two negatively dependent fuzzy random variables, then*

i) $Cov(\tilde{X}, \tilde{Y}) \le 0$,   ii) $E\langle \tilde{X}, \tilde{Y} \rangle \le \langle E\tilde{X}, E\tilde{Y} \rangle$.

Proof. The proofs are straightforward. □

Remark 3.9. *Ziaei and Deiri [40] defined negatively dependent f.r.v.'s based on α-cuts. In their method, two f.r.v.s $\tilde{X}$ and $\tilde{Y}$ are called negatively dependent f.r.v.'s if $\forall\, x$ and $y \in \mathbb{R}$ and $\forall\, r \in [0, 1]$*

$$P(\tilde{X}^+(r) \le x, \tilde{Y}^+(r) \le y) \le P(\tilde{X}^+(r) \le x) P(\tilde{Y}^+(r) \le y),$$

and

$$P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) \le P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y).$$

The following proposition displays that Definition 3.2 is stronger than the above definition.

Proposition 3.10. *Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\tilde{X}$ and $\tilde{Y}$ be negatively dependent f.r.v.'s. Then, $\tilde{X}^-(r)$ and $\tilde{Y}^-(r)$ as well as $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$ are negatively dependent real valued random variables.*

Proof. For all $x, y \in \mathbb{R}$ and $r \in (0, 1]$, we have

$$
\begin{aligned}
P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y) &= P([\tilde{X}]^r \subset (x, \infty), [\tilde{Y}]^r \subset (y, \infty)) \\
&\le P([\tilde{X}]^r \subset (x, \infty)) P([\tilde{Y}]^r \subset (y, \infty)) \\
&= P(\tilde{X}^-(r) > x) P(\tilde{Y}^-(r) > y).
\end{aligned}
$$

A similar proof can be stated for $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$. □

Counterexample

Example 3.11. *Let $\tilde{X}$ and $\tilde{Y}$ have the following probability mass*

$$P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) = 0, P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) = \frac{1}{9}, P(\tilde{X} = \tilde{u}, \tilde{Y} = -\tilde{u}) = \frac{2}{9},$$

$$P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{0}) = \frac{1}{9}, P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{0}) = \frac{1}{9}, P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{0}) = 0,$$

$$P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{u}) = \frac{2}{9}, P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{u}) = \frac{1}{9}, P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = \frac{1}{9},$$

where,

$$\mu_{\tilde{u}}(x) = \begin{cases} x, & 0 < x \le 1, \\ 2 - x, & 1 < x \le 2, \\ 0, & otherwise. \end{cases}$$

It is easy to see that $\tilde{X}$ and $\tilde{Y}$ are not ND f.r.v.'s, since

$$P([\tilde{X}]^r \subset [\frac{1}{2}, \frac{3}{2}], [\tilde{Y}]^r \subset [-\frac{3}{2}, -\frac{1}{2}]) = \frac{2}{9} > P([\tilde{Y}]^r \subset [-\frac{3}{2}, -\frac{1}{2}]) P([\tilde{X}]^r \subset [\frac{1}{2}, \frac{3}{2}]) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.$$

But, $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$ are ND random variables also $\tilde{X}^-(r)$ and $\tilde{Y}^-(r)$ are ND random variables. Since, for $x \in (-\infty, 0)$ and $y \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) = 0 \\ &< P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= P(\tilde{X} = -\tilde{u}) P(\tilde{Y} = -\tilde{u}) = \frac{3}{9} \times \frac{3}{9} = \frac{1}{9}. \end{aligned}$$

Also, for $x \in [0, 1)$ and $y \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) \\ &= 0 + \frac{1}{9} \\ &< P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{0})\} \times P(\tilde{Y} = -\tilde{u}) \\ &= (\frac{3}{9} + \frac{2}{9}) \times \frac{3}{9} = \frac{5}{27}. \end{aligned}$$

For $x \in [0, 1)$ and $y \in [0, 1)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y} \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{0}) \\ &\quad + P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{0}) \\ &= 0 + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} \\ &< P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{0})\} \times \{P(\tilde{Y} = -\tilde{u}) + P(\tilde{Y} = \tilde{0})\} \\ &= \frac{5}{9} \times \frac{2}{3} \end{aligned}$$

For $x \in [0, 1)$ and $y, \in [1, \infty)$

$$
\begin{aligned}
P(\tilde{X}^-(r) \leq x, \tilde{Y}^-(r) \leq y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{0}) \\
&+ P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{u}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) \\
&+ P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{0}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{u}) \\
&= \frac{2}{3} \\
&= P(\tilde{X}^-(r) \leq x) P(\tilde{Y}^-(r) \leq y) \\
&= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{0})\} \\
&\times \{P(\tilde{Y} = -\tilde{u}) + P(\tilde{Y} = \tilde{0}) + P(\tilde{Y} = \tilde{u})\} \\
&= \frac{2}{3}.
\end{aligned}
$$

And finally, $x \in [1, \infty)$ and $y \in [1, \infty)$, we obtain

$$
P(\tilde{X}^-(r) \leq x, \tilde{Y}^-(r) \leq y) = 1 = P(\tilde{X}^-(r) \leq x) P(\tilde{Y}^-(r) \leq y) = 1 \times 1.
$$

4. Some Limit Theorems

In this section, by using the concept of variance and covariance of f.r.v.'s, we obtain some limit theorems for negatively dependent f.r.v.'s.

Theorem 4.1. *Let $\{\tilde{X}_{n,i}, 1 \leq i \leq n\}$ be an array of pairwise negatively dependent f.r.v.'s such that $P(\|\tilde{X}_{n,i}\|_* > \lambda) \leq P(X > \lambda)$, where X is a nonnegative random variable for which $nP(X > n) \to 0$. Then*

$$
\frac{1}{n} d_*(\oplus_{i=1}^n \tilde{X}_{n,i}, \oplus_{i=1}^n \tilde{C}_{n,i}) \to 0 \text{ in probability,} \tag{1}
$$

where $\tilde{C}_{n,i} = E\tilde{X}_{n,i} I_{\{\tilde{1}_{\{\frac{-n}{\sqrt{2}}\}} \leq \tilde{X}_{n,i} \leq \tilde{1}_{\{\frac{n}{\sqrt{2}}\}}\}}$.

Proof. Set, for $i = 1, 2, ..., n, n \geq 1$,

$$
\tilde{Y}_{n,i} = \tilde{X}_{n,i} I_{\{\tilde{1}_{\{\frac{-n}{\sqrt{2}}\}} \leq \tilde{X}_{n,i} \leq \tilde{1}_{\{\frac{n}{\sqrt{2}}\}}\}} \oplus \tilde{1}_{\{\frac{n}{\sqrt{2}}\}} I_{\{\tilde{X}_{n,i} > \tilde{1}_{\{\frac{n}{\sqrt{2}}\}}\}} \oplus \tilde{1}_{\{\frac{-n}{\sqrt{2}}\}} I_{\{\tilde{X}_{n,i} < \tilde{1}_{\{\frac{-n}{\sqrt{2}}\}}\}}.
$$

By subadditivity property of the metric d_*, it is easy to see that

$$
\begin{aligned}
d_*(\oplus_{i=1}^n \tilde{X}_{n,i}, \oplus_{i=1}^n \tilde{C}_{n,i}) &\leq d_*(\oplus_{i=1}^n \tilde{X}_{n,i}, \oplus_{i=1}^n \tilde{Y}_{n,i}) \\
&+ d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n E\tilde{Y}_{n,i}) \\
&+ d_*(\oplus_{i=1}^n E\tilde{Y}_{n,i}, \oplus_{i=1}^n \tilde{C}_{n,i}).
\end{aligned}
$$

But, i)

$$
\begin{aligned}
P(\frac{1}{n} d_*(\oplus_{i=1}^n \tilde{X}_{n,i}, \oplus_{i=1}^n \tilde{Y}_{n,i}) > \epsilon) &\leq P(\cup_{i=1}^n [\{\tilde{X}_{n,i} > \tilde{1}_{\{\frac{n}{\sqrt{2}}\}}\} \cup \{\tilde{X}_{n,i} < \tilde{1}_{\{\frac{-n}{\sqrt{2}}\}}\}]) \\
&\leq P(\cup_{i=1}^n \|\tilde{X}_{n,i}\|_* > n) \\
&\leq \sum_{i=1}^n P(\|\tilde{X}_{n,i}\|_* > n) \\
&\leq nP(X > n) \to 0.
\end{aligned}
$$

ii) By Markov's inequality and Theorem 2.3, we can see that

$$
\begin{aligned}
P(\frac{1}{n}d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n E\tilde{Y}_{n,i}) > \epsilon) &\leq \frac{2}{n^2\epsilon^2}Var(\oplus_{i=1}^n \tilde{Y}_{n,i}) \\
&\leq \frac{2}{n^2\epsilon^2}\sum_{i=1}^n Var(\tilde{Y}_{n,i}) \\
&\leq \frac{2}{n^2\epsilon^2}\sum_{i=1}^n E\|\tilde{Y}_{n,i}\|_*^2 \\
&= \frac{2}{n^2\epsilon^2}\sum_{i=1}^n E\|\tilde{X}_{n,i}\|_*^2 I_{\{\|\tilde{X}_{n,i}\|_*\leq n\}} \\
&+ \frac{2}{n^2\epsilon^2}\sum_{i=1}^n n^2 P(\|\tilde{X}_{n,i}\|_* > n),
\end{aligned}
\tag{2}
$$

the last term in (2) goes to 0 as $n \to \infty$, but

$$
\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^n E\|\tilde{X}_{n,i}\|_*^2 I_{\{\|\tilde{X}_{n,i}\|_*\leq n\}} &= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n E\|\tilde{X}_{n,i}\|_*^2 I_{\{j-1<\|\tilde{X}_{n,i}\|_*\leq j\}} \\
&\leq \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n j^2(P(\|\tilde{X}_{n,i}\|_* > j-1) - P(\|\tilde{X}_{n,i}\|_* > j)) \\
&= \frac{1}{n^2}\sum_{i=1}^n\{(P(\|\tilde{X}_{n,i}\|_* > 0) - n^2 P(\|\tilde{X}_{n,i}\|_* > n)) \\
&+ \sum_{j=1}^{n-1}((j-1)^2 - j^2)P(\|\tilde{X}_{n,i}\|_* > j)\} \\
&\leq \frac{1}{n^2}\sum_{i=1}^n(1 + \sum_{j=1}^{n-1}(2j-1)P(\|\tilde{X}_{n,i}\|_* > j)) \\
&\leq \frac{1}{n} + 2\frac{1}{n}\sum_{j=1}^{n-1} jP(X > j) + \frac{1}{n}\sum_{j=1}^n P(X > j) \to 0.
\end{aligned}
$$

iii) By definitions of d_*, $\tilde{C}_{n,i}$ and $E\tilde{Y}_{n,i}$, we have

$$
\begin{aligned}
\frac{1}{n}d_*(\oplus_{i=1}^n E\tilde{Y}_{n,i}, \oplus_{i=1}^n \tilde{C}_{n,i}) &= \frac{1}{n}|\sum_{i=1}^n nP(\tilde{X}_{n,i} > \tilde{1}_{\{\frac{n}{\sqrt{2}}\}}) - \sum_{i=1}^n nP(\tilde{X}_{n,i} < \tilde{1}_{\{\frac{-n}{\sqrt{2}}\}})| \\
&\leq \frac{1}{n}\sum_{i=1}^n n\{P(\tilde{X}_{n,i} > \tilde{1}_{\{\frac{n}{\sqrt{2}}\}}) + P(\tilde{X}_{n,i} < \tilde{1}_{\{\frac{-n}{\sqrt{2}}\}})\} \\
&\leq \frac{1}{n}\sum_{i=1}^n nP(\|\tilde{X}_{n,i}\|_* > n) \leq nP(X > n) \to 0.
\end{aligned}
$$

This completes the proof. □

Lemma 4.2. *If $\tilde{X}$ is a f.r.v., then $\|E\tilde{X}\|_*^2 \leq E\|\tilde{X}\|_*^2$.*

Proof. By applying Jensen's inequality and Fubini's theorem, we obtain

$$
\begin{aligned}
\|E\tilde{X}\|_*^2 &= \int_0^1 \{(EX^-(r))^2 + (EX^+(r))^2\}dr \\
&\leq \int_0^1 \{E(X^-(r))^2 + E(X^+(r))^2\}dr \\
&= E\int_0^1 \{(X^-(r))^2 + (X^+(r))^2\}dr = E\|\tilde{X}\|_*^2,
\end{aligned}
$$

this inequality follows from the fact that $[E\tilde{X}]^r = E[\tilde{X}]^r, \forall r \in (0,1]$ [28], and convexity of the function $\varphi(x) = x^2$. □

Remark 4.3. *In Lemma 4.2, If fuzzy random variable reduces to random one, then Jensen's inequality is obtained.*

In the following theorem, we provide an extension of Theorem 2.3. in [32] to pairwise negatively dependent f.r.v.'s.

Theorem 4.4. *Let $\{\tilde{X}_n, n \geq 1\}$ be a sequence of pairwise negatively dependent f.r.v.'s. Let $\{a_n, n \geq 1\}$ be a sequence of positive numbers with $a_n \uparrow \infty$. Furthermore, suppose that $\psi(t)$ is a nonnegative and even function such that $\psi(t) > 0$ as $t > 0$ and*

$$
\frac{\psi(t)}{|t|} \uparrow \quad and \quad \frac{\psi(t)}{t^2} \downarrow \quad as \quad |t| \uparrow . \tag{3}
$$

If

$$
\sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} < \infty, \tag{4}
$$

and

$$
\sum_{i=1}^n \frac{E\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} = o(n^{-1}), \tag{5}
$$

then $a_n^{-1} \oplus_{i=1}^n \tilde{X}_i \to \tilde{0}$ a.s. with respect to the metric d_*, equivalently $\|a_n^{-1} \oplus_{i=1}^n \tilde{X}_i\|_* \to 0$ a.s.

Proof. Set, for $i = 1, 2, ..., n, n \geq 1$,

$$
\tilde{Y}_{n,i} = \tilde{X}_i I_{\{\tilde{1}_{\{\frac{-a_n}{\sqrt{2}}\}} \leq \tilde{X}_i \leq \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}\}} \oplus \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}} I_{\{\tilde{X}_i > \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}\}} \oplus \tilde{1}_{\{\frac{-a_n}{\sqrt{2}}\}} I_{\{\tilde{X}_i < \tilde{1}_{\{\frac{-a_n}{\sqrt{2}}\}}\}}
$$

$$
\tilde{Z}_{n,i} = (\tilde{X}_i \ominus \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}) I_{\{\tilde{X}_i > \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}\}} \oplus (\tilde{X}_i \oplus \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}) I_{\{\tilde{X}_i < \tilde{1}_{\{\frac{-a_n}{\sqrt{2}}\}}\}},
$$

where $\tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}$ is a fuzzy number in E whose membership function equals 1 at $\frac{a_n}{\sqrt{2}}$ and zero otherwise. Since $\tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}$ is a crisp number then $\tilde{X}_i = \tilde{Y}_{n,i} \oplus \tilde{Z}_{n,i}$.

By subadditivity property of the metric d_* and the norm $\|.\|_*$, we obtain

$$
\| \oplus_{i=1}^n (\tilde{Y}_{n,i} \oplus \tilde{Z}_{n,i})\|_* \leq \| \oplus_{i=1}^n \tilde{Z}_{n,i}\|_* + d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n E\tilde{Y}_{n,i}) + \| \oplus_{i=1}^n E\tilde{Y}_{n,i}\|_*.
$$

Thus, it is sufficient to show that

$$
i) \quad \sum_{i=1}^{\infty} P(a_n^{-1}\| \oplus_{i=1}^n \tilde{Z}_{n,i}\|_* > \epsilon) < \infty, \quad ii) \sum_{i=1}^{\infty} P(a_n^{-1} d_*(\oplus_{i=1}^n \tilde{Y}_{n,i}, \oplus_{i=1}^n E\tilde{Y}_{n,i}) > \epsilon) < \infty,
$$

$$
iii) \quad a_n^{-1}\| \oplus_{i=1}^n E\tilde{Y}_{n,i}\|_* \to 0.
$$

i) Since $\psi \uparrow$, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(a_n^{-1} \| \oplus_{i=1}^{n} \tilde{Z}_{n,i} \|_* > \epsilon) &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(\|\tilde{X}_i\|_* > a_n) \\
&\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} < \infty,
\end{aligned}
$$

ii) By Lemma 3.6, it is easily seen that $\{\tilde{Y}_{n,i}, 1 \leq i \leq n\}$ are still pairwise negatively dependent for any fixed $n \geq 1$. It follows from Markov's inequality that

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(a_n^{-1} d_*(\oplus_{i=1}^{n} \tilde{Y}_{n,i}, \oplus_{i=1}^{n} E\tilde{Y}_{n,i}) > \epsilon) &\leq 2 \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{Var(\tilde{Y}_{n,i})}{\epsilon^2 a_n^2} \\
&\leq 2 \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E < \tilde{Y}_{n,i}, \tilde{Y}_{n,i} >}{\epsilon^2 a_n^2} \\
&= 2 \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E\|\tilde{Y}_{n,i}\|_*^2}{\epsilon^2 a_n^2} \\
&\leq \frac{2}{\epsilon^2} \sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-2} E\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}} \\
&\quad + \frac{2}{\epsilon^2} \sum_{n=1}^{\infty} a_n^{-2} \sum_{i=1}^{n} a_n^2 P(\|\tilde{X}_i\|_* > a_n).
\end{aligned}
$$

Moreover, by the relations (3) and (4) we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-2} E\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}} &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} I_{\{\|\tilde{X}_i\|_* \leq a_n\}} \\
&\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} < \infty,
\end{aligned}
\tag{6}
$$

and

$$
\sum_{n=1}^{\infty} a_n^{-2} \sum_{i=1}^{n} a_n^2 P(\|\tilde{X}_i\|_* > a_n) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(\|\tilde{X}_i\|_* > a_n) < \infty.
\tag{7}
$$

The relations (6) and (7) imply that

$$
\sum_{n=1}^{\infty} P(a_n^{-1} d_*(\oplus_{i=1}^{n} \tilde{Y}_{n,i}, \oplus_{i=1}^{n} E\tilde{Y}_{n,i}) > \epsilon) < \infty.
$$

iii) To prove $a_n^{-1}\|\oplus_{i=1}^{n} E\tilde{Y}_{n,i}\|_* \to 0$, as $n \to \infty$ we show that;

$$
\begin{aligned}
a_n^{-1}\|\oplus_{i=1}^{n} E\tilde{Y}_{n,i}\|_* \quad &\leq \quad \sum_{i=1}^{n} a_n^{-1}\|E\tilde{Y}_{n,i}\|_* \\
&\leq \quad \sum_{i=1}^{n} a_n^{-1}\|E\tilde{X}_i I_{\{\|\tilde{X}_i\|_* \leq a_n\}}\|_* \\
&+ \quad \sum_{i=1}^{n} P(\tilde{X}_i > \tilde{1}_{\{\frac{a_n}{\sqrt{2}}\}}) + \sum_{i=1}^{n} P(\tilde{X}_i < \tilde{1}_{\{\frac{-a_n}{\sqrt{2}}\}}) \\
&\leq \quad \sum_{i=1}^{n} a_n^{-1} E^{\frac{1}{2}}\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}} \\
&+ \quad \sum_{i=1}^{n} P(\|\tilde{X}_i\|_* > a_n),
\end{aligned}
$$

in which, the last inequality follows from Lemma 4.2.

By the relation (7), it is easy to see that $\sum_{i=1}^{n} P(\|\tilde{X}_i\|_* > a_n) \to 0$ as $n \to \infty$.

To prove $\sum_{i=1}^{n} a_n^{-1} E^{\frac{1}{2}}\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}} \to 0$ as $n \to \infty$, we show that

$$
[\sum_{i=1}^{n} a_n^{-1} E^{\frac{1}{2}}\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}}]^2 \to 0.
$$

As an application of Jensen's inequality, we have

$$
(\frac{1}{n}\sum_{i=1}^{n} x_i)^2 \leq \frac{1}{n}\sum_{i=1}^{n} x_i^2, \qquad \forall x_i \in \mathbb{R}. \tag{8}
$$

Now, using the relations (3), (5), and (8), we can write

$$
\begin{aligned}
[\sum_{i=1}^{n} a_n^{-1} E^{\frac{1}{2}}\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}}]^2 \quad &\leq \quad n\sum_{i=1}^{n} a_n^{-2} E\|\tilde{X}_i\|_*^2 I_{\{\|\tilde{X}_i\|_* \leq a_n\}} \\
&\leq \quad n\sum_{i=1}^{n} E\frac{\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} \to 0.
\end{aligned}
$$

This completes the proof. $\square$

Example 4.5. *Let $\tilde{u}$ be a fuzzy number with $\|\tilde{u}\|_* = 1$ for instance a fuzzy number with the following membership function*

$$
\tilde{u}(x) = 1 - \frac{\sqrt{6}}{3}|x|, \quad -\frac{\sqrt{6}}{2} \leq x \leq \frac{\sqrt{6}}{2},
$$

,$\psi(x) = |x|^p$, $1 \leq p \leq 2$ and $a_n = n^\beta$ and $\beta p > 3$. Let $\{\tilde{X}_n\}$ be a sequence of negatively dependent f.r.v.'s such that $P(\tilde{X}_n = n\tilde{u}) = \frac{1}{n}$, $P(\tilde{X}_n = \tilde{0}) = 1 - \frac{1}{n}$. Then, $\sum_{n=1}^{\infty}\sum_{i=1}^{n} E\frac{\psi(\|\tilde{X}_i\|_*)}{\psi(a_n)} < \infty$ and, therefore, by Theorem 4.4, $\oplus_{i=1}^{n}\{a_n^{-1} \odot \tilde{X}_i\}$ converges to $\tilde{0}$ a.s. with respect to the metric d_*.

5. Conclusion

Strong and weak convergence theorems were obtained under some relaxed conditions based on the concept of variance and covariance. Therefore, we can extend the classical probabilistic results to f.r.v.'s based on the concept of variance and covariance, instead of using the Banach techniques. The study of linearly dependent and associated f.r.v.'s, specially weak and strong laws of large numbers, for such random variables are potential works for future research.

References

[1] H. Ahmadzade, M. Amini, S. M. Taheri, A. Bozorgnia, *Some limit theorems for independent fuzzy random variables,* Thai J. Math. **12**, (2014), 537-548.

[2] H. Ahmadzade, M. Amini, S. M. Taheri, A. Bozorgnia, *Some moment inequalities for fuzzy martingales and their applications.* J. Uncertainty Anal. Appl. **2**, (2014), 1-14.

[3] M. Amini, H. A. Azarnoosh, A. Bozorgnia, *The strong law of large numbers for ND generalized Gaussian random variables,* J. Stochastic Anal. Appl. **22**, (2004), 893-901.

[4] M. Amini, H. Zarei, A. Bozorgnia, *Some strong limit theorems of weighted sums for negatively dependent generalized Gaussian random variables,* Stat. Probab. Lett. **77**, (2007), 1106-1110.

[5] A. Bozorgnia, R. F. Patterson, R. L. Taylor, *Limit theorems for dependent random variables.* World Congress on Nonlinear Analysts, Walter de Gruyter and Co. Hawthorne, NJ, USA, 92, (1996), 1639-1650.

[6] Y. Feng, *An approach to generalize laws of large numbers for fuzzy random variables,* Fuzzy Set. Syst., **128**, (2002), 237-245.

[7] Y. Feng, L. Hu, H. Shu, *The variance and covariance of fuzzy random variables and their applications,* Fuzzy Set. Syst. **120**, (2001), 487-497.

[8] K. Fu , L-X. Zhang, *Strong limit theorems for random sets and fuzzy random sets with slowly varying weights,* Inform. Sci., **178**, (2008), 2648-2660.

[9] L. Guan, S. Li, *laws of large numbers for weighted sums of fuzzy set-valued random variables,* Int. J. Uncertain. Fuzz. **12**, 2004, 811-825.

[10] D. Hong, *A convergence of fuzzy random variables,* Kybernetika, **39**, (2003), 275-280.

[11] D. Hong, K. Kim, *Weak law of large numbers for i.i.d. fuzzy random variables,* Kybernetika, **43**, (2007), 87-96.

[12] W. Hoeffding, *Masstabinvariante Korrelationstheorie,* (1940). Schriften des Matematischen Instituts und des Instituts fur Angewandte Mathematik der Universitat.

[13] S.Y. Joo, Y. K. Kim, J. S. Kwon, *On Chung's type law of large numbers for fuzzy random variables,* Stat. Probab. Lett. **74**, (2005), 67-75.

[14] E. P. Klement, M. L. Puri, D. A. Ralescu, *Limit theorems for fuzzy random variables,* Proc. Roy. Soc. London, Ser. A, **407**, (1986), 171-182.

[15] H. Kwakernaak, *Fuzzy random variables I.* Inform. Sci. **15**, (1978) 1-29.

[16] E. L. Lehmann, *Some concepts of dependence.* Ann. Math. Statist. **37**, (1966), 1137-1153.

[17] R. Li, W. Yang, *Strong convergence of pairwise NQD random sequences,* J. Math. Anal. Appl. **344**, (2008), 741-747.

[18] S. Li and J. Zhang, *A general method for convergence Theorems of fuzzy set-valued random variables and its aplications to martingales and uniform amarts,* Int. J. Uncertain. Fuzz. **13**, (2005), 243-253.

[19] M. Miyakoshi, M. Shimbo, *A strong law of large numbers for fuzzy random variables.* Fuzzy Set. Syst. **12**, (1984), 133-142.

[20] E. Miranda, I. Couso, P. Gil, *Random sets as imprecise random variables,* J. Math. Anal. Appl. **307**, (2005), 32-47.

[21] S.A. Mohiuddinea, H. S. evlib, M. Cancanc, *Statistical convergence of double sequences in fuzzy normed spaces,* Filomat, **26**, (2012), 673-681.

[22] R. E. Moore, R. B. Kearfott, M. J. Cloud, *Introduction to Interval Analysis,* Siam, 2009.

[23] M. Mursaleen, S.A. Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space,* J. Comput. Appl. Math. **223**, (2009), 142-149.

[24] W. Nather, A. Wnsche,*On the conditional variance of fuzzy random variables,* Metrika, **65**, (2007), 109-122.

[25] R. B. Nelsen, *An Introduction to Copulas,* Springer, New York, 2007.

[26] C. M. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables,* Lecture Notes-Monograph Series, Inequalities in Statistics and Probability, **5**, (1984), 127-140.

[27] H. T. Nguyen, *A note on the extension principle for fuzzy sets,* J. Math. Anal. Appl. **64**, (1978), 369-380.

[28] M. L. Puri, D. A. Ralescu, *Fuzzy random variables.* J. Math. Anal. Appl. **114**, (1986), 402-422.

[29] D. A. Ralescu, *Fuzzy random variables revisited.* Mathematics of Fuzzy Sets, Handbook of Fuzzy Sets, Series **3**, Kluwer, Boston, (1999), 701-710.

[30] V. Ranjbar, M. Amini, J. Geluk, A. Bozorgnia, *Asymptotic behavior of product of two heavy-tailed dependent random variables,* Acta Math. Sin. **29**, (2013), 355-364.

[31] Y. Shen, X. J. Wang, W. Z. Yang, S. H. Hu, *Almost sure convergence theorem and strong stability for weighted sums of NSD random variables,* Acta Math. Sin. **29**, (2013), 743-756.

[32] G. Shixin, C. Pingyan, *Some limit theorems for sequences of pairwise NQD random variables,* Acta Math. Sci. **28B**, (2008), 269-281.

[33] M. Stojakovic, *Imprecise set and fuzzy valued probability,* J. Comput. Appl. Math., **235**, (2011), 4524-4531.

[34] R. L. Taylor, L. Seymour, Y. Chen, *Weak laws of large numbers for fuzzy random sets.* Nonlinear Anal.**47**, (2001), 1245-1250.

[35] R. Viertl. *Statistical Methods for Fuzzy Data,* J. Wiley, Chichester, 2011.

[36] D. Wang, *A note on consistency and unbiasedness of point estimation with fuzzy data,* Metrika, **60**, (2004), 93-104.

[37] H. C. Wu, *The laws of large numbers for fuzzy random variables,* Fuzzy Set. Syst., **116**, (2000), 245-262.

[38] Y. Wu, M. Guan, *Mean convergence theorems and weak laws of large numbers for weighted sums of dependent random variables,* J. Math. Anal. Appl. 377, (2011), 613-623.

[39] L. Yang, B. Liu, *On inequalities and critical values of fuzzy random variables,* Int. J. Uncertain. Fuzz. **13**, (2005), 163-175.

[40] A. R. Ziaei, E. Deiri, *Law of large numbers for weighted sum of fuzzy random variables with negatively quadrant dependent,* Australian Journal of Basic and Applied Sciences, **5**, (2011), 634-640.