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Existence of Positive Periodic Solutions in Shifts $\delta_\pm$ for a Nonlinear First Order Functional Dynamic Equation on Time Scales

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Abstract. Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_\mathbb{T}$. In this paper we consider the nonlinear functional dynamic equation of the form

$$x^\nabla(t) = a(t)x(t) - \lambda b(t) f(x(h(t))), \quad t \in \mathbb{T}.$$

By using the Krasnoselskiĭ, Avery-Henderson and Leggett-Williams fixed point theorems, we present different sufficient conditions for the nonexistence and existence of at least one, two or three positive periodic solutions in shifts $\delta_\pm$ of the above problem on time scales. We extend and unify periodic differential, difference, h-difference and q-difference equations and more by a new periodicity concept on time scales.

1. Introduction

Due to their importance in applications such as population dynamics, industrial robotics, the production of blood cells, and so on, many authors are studying the existence, uniqueness and positivity of solutions for first order functional differential equations of the form

$$y'(t) = -a(t)y(t) + \lambda f(t, y(h(t)))$$

where a, h and according the first variable the function f are periodic with positive T, see [6, 11] and the references therein. If $h(t) = t - \tau(t)$ with $\tau \in C(\mathbb{R}, \mathbb{R}^+)$, $0 \leq \tau(t) \leq t$ then the above equation takes the form functional differential equations with delay terms and such equations, directly or after some transformation, appear in many mathematical ecological and population models, such as:

1. Lasota-Wazewska model [14–17, 25]

$$y'(t) = -a(t)y(t) + p(t)e^{-\gamma(t)y(t-\tau(t))},$$

2. Nicholsan's blowflies model [16, 18–20]

$$y'(t) = -a(t)y(t) + b(t)y(t - \tau(t))e^{-\gamma(t)y(t-\tau(t))},$$

3. model for blood cell production [11, 16, 20, 21]

$$y'(t) = -a(t)y(t) + b(t)\frac{y(t-\tau(t))}{1+y^n(t-\tau(t))}.$$

Since the periodic variation of the environment plays an important role in many biological and ecological systems, many researchers have been concerned with the study of the existence of periodic solutions of the above models. The monograph by Kuang [19] gives a nice discussion of such models.

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Chow [22], Freedman and Wu [23], Hadeler and Tomiuk [24], Wang [10], Weng and Sun [13] and many others studied the existence of at least one and two positive periodic solutions of nonlinear first order differential equations by several way. On the other hand, it has been observed that very few papers exist in the literature on the nonexistence and multiplicity existence of nonnegative periodic solutions for the first order differential equations. For example, see [6, 10, 11, 26, 31].

Because of the study of continuous and discrete systems are very important role in implementation and application, the theory of dynamic equations on time scales which unifies differential, difference, h-difference, q-differences equations and more has been received much attention, see [1, 2, 12, 27]. This theory has become important mathematical branch [4, 5] since it was initiated by Stefan Hilger in 1988 [7]. There is only a few results concerning periodic solutions of dynamic equations on time scales such as in [2, 9]. Although the former definition of periodicity using in these papers obliges the time scales to be additive and unbounded above and below, there are many time scales that do not satisfy this condition such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$. M. Adıvar introduced a new periodicity concept in [3] by using shift operators $\delta_{\pm}$. With the new periodicity concept, the time scale needn't be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. The construction a new equation by using shift operators $\delta_{\pm}$ on time scales not only covers previously handled equations but also enables to generalize the results to any time scales having a general rule as well as $\overline{q^{\mathbb{Z}}}$. There is only a few existence results related with the new periodicity, see [1]. Also, Wang and Agarwal gave the concept of changing-periodic time scales in [32].

Let $\mathbb{T}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$. We are concerned with the nonexistence, existence and multiplicity of periodic solutions in shifts $\delta_{\pm}$ for nonlinear dynamic equation

$$x^{\nabla}(t) = a(t)x(t) - \lambda b(t) f(x(h(t))), \quad t \in \mathbb{T} \tag{1}$$

where $a \in C(\mathbb{T}, (-\infty, 0))$ is ∇-periodic in shifts $\delta_{\pm}$ with period T and $a \in \mathcal{R}_{\nu}$, $\lambda > 0$ is a positive parameter, $b \in C(\mathbb{T}, [0, \infty))$ is ∇-periodic in shifts with period T, $\int_{t_0}^{\delta_+^T(t_0)} b(s)\nabla s > 0$, $h \in C(\mathbb{T}, [0, \infty))$ is periodic in shifts with period T, $f : C((0, \infty), (0, \infty))$ and $T \in [P, \infty)_{\mathbb{T}}$.

Hereafter, we use the notation $[a, b]_{\mathbb{T}}$ to indicate the time scale interval $[a, b] \cap \mathbb{T}$. The intervals $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ and $(a, b)_{\mathbb{T}}$ are similarly defined.

In this study, we shall show that the number of positive periodic solutions in shifts $\delta_{\pm}$ of (1.1) can be determined by the asymptotic behaviors of the quotient of $\frac{f(x)}{x}$ at zero and infinity. We shall organize this paper as follows. In section 2, we will state some facts about exponential function on time scales, the new periodicity concept for time scales and some important theorems which will be needed to show the existence of at least single, twin or triple periodic solutions in shifts $\delta_{\pm}$. Besides these, in section 2, we will give some lemmas about the exponential function and the graininess function with shift operators using nabla periodicity and nabla differentiability. Finally, we state our main results and give their proofs in section 3 by using Krasnosel'skiĭ's, Avery-Henderson and Leggett-Williams fixed point theorems.

2. Preliminaries

In this section, we mention some definitions, lemmas and theorems from calculus on time scales which can be found in [4–6]. Next, we state some definitions, lemmas and theorems about the shift operators and the new periodicity concept for time scales which can be found in [3].

Definition 2.1. [5] *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 - \nu(t)p(t) \neq 0$ for all $t \in \mathbb{T}_{\kappa}$, where $\nu(t) = t - \rho(t)$. The set of all regressive ld-continuous and ν-regressive functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}_{\nu}$ while the set $\mathcal{R}_{\nu}^{+}$ is given by $\mathcal{R}_{\nu}^{+} = \{\varphi \in \mathcal{R}_{\nu} : 1 - \nu(t)\varphi(t) > 0$ for all $t \in \mathbb{T}\}$.*

If $p, q \in \mathcal{R}_{\nu}$, then we define circle plus addition by

$$(p \oplus_{\nu} q)(t) = p(t) + q(t) - p(t)q(t)\nu(t) \quad \text{for all } t \in \mathbb{T}_{\kappa}.$$

For $p \in \mathcal{R}_{\nu}$, define circle minus p by

$$\ominus_{\nu} p = -\frac{p}{1 - p\nu},$$

and the generalized square of p by

$$p^{②} = (-p)(\ominus_\nu p) = \frac{p^2}{1 - p\nu}.$$

If $\varphi \in \mathcal{R}_\nu$, then the nabla exponential function on $\mathbb{T}$ is defined by

$$\hat{e}_\varphi(t,s) = \exp\left(\int_s^t \hat{\xi}_{\nu(r)}(\varphi(r))\nabla r\right) \tag{2}$$

where $\hat{\xi}_{\nu(r)}$ is the ν-cylinder transformation given by

$$\hat{\xi}_{\nu(r)}(\varphi(r)) := \begin{cases} -\frac{1}{\nu(r)}Log(1 - \varphi(r)\nu(r)), & \text{if } \nu(r) > 0; \\ p(r), & \text{if } \nu(r) = 0. \end{cases} \tag{3}$$

Also, the nabla exponential function $y(t) = \hat{e}_p(t,s)$ is the solution to the initial value problem $y^\nabla = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma ([5], Theorem 3.15).

Lemma 2.2. *Let $p, q \in \mathcal{R}_\nu$ and $s, t, r \in \mathbb{T}$. Then*

i. $\hat{e}_0(t,s) \equiv 1$ *and* $\hat{e}_p(t,t) \equiv 1$;
ii. $\hat{e}_p(\rho(t),s) = (1 - \nu(t)p(t))\hat{e}_p(t,s)$;
iii. $\frac{1}{\hat{e}_p(t,s)} = \hat{e}_{\ominus_\nu}(t,s)$ *where,* $\ominus_\nu p(t) = -\frac{p(t)}{1-\nu(t)p(t)}$;
iv. $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus_\nu p}(s,t)$;
v. $\hat{e}_p(t,s)\hat{e}_p(s,r) = \hat{e}_p(t,r)$;
vi. $\hat{e}_p(t,s)\hat{e}_q(t,s) = e_{p\oplus_\nu q}(t,s)$;
vii. $\frac{\hat{e}_p(t,s)}{\hat{e}_q(t,s)} = \hat{e}_{p\ominus_\nu q}(t,s)$;
viii. $\left(\frac{1}{\hat{e}_p(t,s)}\right)^\nabla = \frac{-p(t)}{\hat{e}_p^\rho(t,s)}$.

The following definitions, lemmas, corollaries and examples are about the shift operators and new periodicity concept for time scales which can be found in [3].

Definition 2.3. [3] *Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm : [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:*

(P.1) The function $\delta_\pm$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0,t),(T_0,u) \in \mathcal{D}_\pm := \{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s,t) \in \mathbb{T}^*\},$$

then

$$T_0 \le t < u \text{ implies } \delta_\pm(T_0,t) < \delta_\pm(T_0,u),$$

(P.2) If $(T_1,u),(T_2,u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$, and if $(T_1,u),(T_2,u) \in D_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$,

(P.3) If $t \in [t_0,\infty)_\mathbb{T}$, then $(t,t_0) \in D_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in D_+$ and $\delta_+(t_0,t) = t$ holds,

(P.4) If $(s,t) \in D_\pm$, then $(s,\delta_\pm(s,t)) \in D_\mp$ and $\delta_\mp(s,\delta_\pm(s,t)) = t$, respectively,

(P.5) If $(s,t) \in D_\pm$ and $(u,\delta_\pm(s,t)) \in D_\mp$, then $(s,\delta_\mp(u,t)) \in D_\pm$ and $\delta_\mp(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\mp(u,t))$, respectively.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be backward and forward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0,\infty)_\mathbb{T}$ in $\delta_\pm(s,t)$ is called the shift size. The value $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathcal{D}_\pm$ are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 2.4. [3] *(Periodicity in shifts) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shift $\delta_\pm$ if there exists a $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p,t) \in D_\pm,\ \forall t \in \mathbb{T}^*\} \neq t_0$$

then P is called the period of the time scale $\mathbb{T}$.

Example 2.5. [3] *The following time scales are periodic in the sense of shift operators given in Definition 2.4.*

i. $\mathbb{T}_1 = \{\pm n^2 : n \in \mathbb{Z}\}$, $\delta_\pm(P,t) = \begin{cases} (\sqrt{t} \pm \sqrt{P})^2, & t > 0; \\ \pm P, & t = 0; \\ -(\sqrt{-t} \pm \sqrt{P})^2, & t < 0; \end{cases}$, $P = 1$, $t_0 = 0$,

ii. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1} t$, $P = q$, $t_0 = 1$,

iii. $\mathbb{T}_3 = \overline{\cup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}]}$, $\delta_\pm(P,t) = P^{\pm 1} t$, $P = 4$, $t_0 = 1$,

iv. $\mathbb{T}_4 = \{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\} \cup \{0, 1\}$,

$$\delta_\pm(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1 + q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \quad P = \frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ in Example 2.5 is bounded above and below and $\mathbb{T}_4^* = \{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\}$.

Remark 2.6. [3] *Let $\mathbb{T}$ be a time scale that is periodic in shifts with the period P. Thus, by P.4 of Definition 2.3 the mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ defined by $\delta_+^P(t) = \delta_+(P,t)$ is surjective. On the other hand, by P.1 of Definition 2.3 shift operators $\delta_\pm$ are strictly increasing in their second arguments. That is, the mapping $\delta_+^P(t) = \delta_+(P,t)$ is injective. Hence, δ_+^P is an invertible mapping with the inverse $(\delta_+^P)^{-1} = \delta_-^P$ defined by $\delta_-^P(t) := \delta_-(P,t)$.*

We assume that $\mathbb{T}$ is a periodic time scale in shift $\delta_\pm$ with period P. The operators $\delta_\pm^P : \mathbb{T}^* \to \mathbb{T}^*$ are commutative with the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ given by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. That is, $(\delta_\pm^P \circ \rho)(t) = (\rho \circ \delta_\pm^P)(t)$ for all $t \in \mathbb{T}^*$.

Corollary 2.7. [3] *$\delta_+(P, \rho(t)) = \rho(\delta_+(P,t))$ and $\delta_-(P, \rho(t)) = \rho(\delta_-(P,t))$ for all $t \in \mathbb{T}^*$.*

Definition 2.8. [3] *(Periodic function in shift $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in D_\pm \text{ and } f(\delta_\pm^T(t)) = f(t) \text{ for all } t \in \mathbb{T}^* \tag{4}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (4) holds is called the period of f.

Definition 2.9. *(∇-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is ∇-periodic in shifts $\delta_\pm$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in D_\pm \text{ for all } t \in \mathbb{T}^*, \tag{5}$$

$$\text{the shifts } \delta_\pm^T \text{ are } \nabla\text{-differentiable with ld-continuous derivatives} \tag{6}$$

and

$$f(\delta_\pm^T(t))\delta_\pm^{\nabla T} = f(t) \text{ for all } t \in \mathbb{T}^*, \tag{7}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (5-7) hold is called the period of f.

Notice that Definition 2.8 and Definition 2.9 give the classic periodicity definition on time scales whenever $\delta_{\pm}^{T} := t \pm T$ are the shifts satisfying the assumptions of Definition 2.8 and Definition 2.9.

Theorem 2.10. [29] *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu\mathbb{T}$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an ld-continuous function and ν is ∇-differentiable with ld-continuous derivative, then for $a, b \in \mathbb{T}$*

$$\int_{a}^{b} g(s)\nu^{\nabla}(s)\nabla s = \int_{\nu(b)}^{\nu(a)} (f \circ \nu^{-1})(s)\widetilde{\nabla} s. \tag{8}$$

Now, we give a theorem which is the substitution rule on periodic time scales in shifts $\delta_{\pm}$

Theorem 2.11. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}^*}$ and f a ∇-periodic function in shifts $\delta_{\pm}$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{ld}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\nabla s = \int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} f(s)\nabla s. \tag{9}$$

Proof. We will prove this theorem similarly with Theorem 2 in [3]. Substituting $\nu(s) = \delta_{+}^{T}(t)$ and $g(s) = f(\delta_{+}^{T}(t))$ in (8) and taking (7) into account we have

$$\begin{aligned} \int_{\delta_{+}^{T}(t_0)}^{\delta_{+}^{T}(t)} f(s)\nabla s &= \int_{\nu(t_0)}^{\nu(t)} g(\nu^{-1}(s))\nabla s \\ &= \int_{t_0}^{t} g(s)\nu^{\nabla}(s)\nabla s \\ &= \int_{t_0}^{t} f(\delta_{+}^{T}(t))\delta_{+}^{\nabla T}(t)(s)\nabla s \\ &= \int_{t_0}^{t} f(s)\nabla s. \end{aligned}$$

The equality

$$\int_{\delta_{-}^{T}(t_0)}^{\delta_{-}^{T}(t)} f(s)\nabla s = \int_{t_0}^{t} f(s)\nabla s$$

can be obtained similarly. The proof is complete. □

We give some interesting properties of the nabla exponential functions $\hat{e}_p(t, t_0)$ and shift operators on time scales.

Lemma 2.12. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. Suppose that the shifts $\delta_{\pm}^{T}$ are ∇-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Then the backwards graininess function $\nu : \mathbb{T}_{\kappa} \to [0, \infty)$ satisfies*

$$\nu(\delta_{\pm}^{T}(t)) = \delta_{\pm}^{\nabla T}(t)\nu(t).$$

Proof. Since $\delta_{\pm}^{T}$ are ∇-differentiable at t, from the calculus on time scales we know

$$\nu(t)\delta_{+}^{\nabla T}(t) = \delta_{+}^{T}(t) - \delta_{+}^{T}(\rho(t)).$$

Then by using Corollary 3.5, we have

$$\begin{aligned} \nu(t)\delta_{+}^{\nabla T}(t) &= \delta_{+}^{T}(t)) - \rho(\delta_{+}^{T}(t)) \\ &= \nu(\delta_{+}^{T}(t)). \end{aligned}$$

Similarly, $\nu(t)\delta_{-}^{\nabla T}(t) = \nu(\delta_{-}^{T}(t))$ can be obtained. Thus, the proof is complete. □

Lemma 2.13. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^T$ are ∇-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}_\nu$ is ∇-periodic in shifts $\delta_\pm$ with the period T. Then*
i. $\hat{e}_p(\delta_\pm^T(t), \delta_\pm^T(t_0)) = \hat{e}_p(t, t_0)$ for $t, t_0 \in \mathbb{T}^*$,
ii. $\hat{e}_p(\delta_\pm^T(t), \rho(\delta_\pm^T(s))) = \hat{e}_p(t, \rho(s)) = \frac{\hat{e}_p(t,s)}{1-\nu(t)p(t)}$ for $t, s \in \mathbb{T}^*$.

Proof. (i) Assume that $\nu(\tau) \neq 0$. Set $f(\tau) = \dfrac{1}{\nu(\tau)} Log(1 - p(\tau)\nu(\tau))$. Using Lemma 2.12 and ∇-periodicity of p in shifts $\delta_\pm$ we get

$$\begin{aligned} f(\delta_+^T(\tau))\delta_+^{\nabla T}(\tau) &= \frac{\delta_+^{\nabla T}(\tau)}{\nu(\delta_+^T(\tau))} Log(1 - p(\delta_+^T(\tau))\nu(\delta_+^T(\tau))) \\ &= \frac{\delta_+^{\nabla T}(\tau)}{\nu(\delta_+^T(\tau))} Log(1 - p(\delta_+^T(\tau))\delta_+^{\nabla T} \frac{1}{\delta_+^{\nabla T}} \nu(\delta_+^T(\tau))) \\ &= \frac{1}{\nu(\tau)} Log(1 - p(\tau)\nu(\tau)) \\ &= f(\tau). \end{aligned}$$

Thus, f is ∇-periodic in shifts $\delta_\pm$ with the period T. By using Theorem 2.11 we have

$$\begin{aligned} \hat{e}_p(\delta_+^T(t), \delta_+^T(t_0)) &= \begin{cases} \exp\left(-\displaystyle\int_{\delta_+^T(t_0)}^{\delta_+^T(t)} \frac{1}{\nu(\tau)} Log(1 - p(\tau)\nu(\tau))\nabla\tau\right), & \text{if } \nu(\tau) \neq 0; \\ \exp\left(\displaystyle\int_{\delta_+^T(t_0)}^{\delta_+^T(t)} p(\tau)\nabla\tau\right), & \text{if } \nu(\tau) = 0, \end{cases} \\ &= \begin{cases} \exp\left(-\displaystyle\int_{t_0}^{t} \frac{1}{\nu(\tau)} Log(1 - p(\tau)\nu(\tau))\nabla\tau\right), & \text{if } \nu(\tau) \neq 0; \\ \exp\left(\displaystyle\int_{t_0}^{t} p(\tau)\nabla\tau\right), & \text{if } \nu(\tau) = 0, \end{cases} \\ &= \hat{e}_p(t, t_0). \end{aligned}$$

Similarly, we can obtain $\hat{e}_p(\delta_-^T(t), \delta_-^T(t_0)) = \hat{e}_p(t, t_0)$.
(ii) From Corollary 3.5, we know $\rho(\delta_+^T(s)) = \delta_+^T(\rho(s))$. By Lemma 2.12 and Lemma 2.2 we obtain

$$\hat{e}_p(\delta_+^T(t), \rho(\delta_+^T(s))) = \hat{e}_p(\delta_+^T(t), \delta_+^T(\rho(s))) = \hat{e}_p(t, \rho(s)) = \frac{\hat{e}_p(t, s)}{1 - \nu(t)p(t)}.$$

Similarly, we can obtain $\hat{e}_p(\delta_-^T(t), \rho(\delta_-^T(s))) = \hat{e}_p(\delta_-^T(t), \delta_-^T(\rho(s))) = \hat{e}_p(t, \rho(s))$. The proof is complete. □ Let define us the set

$$P_T := \{x \in C(\mathbb{T}, \mathbb{R}) : x(\delta_+^T(t)) = x(t)\}$$

where $C(\mathbb{T}, \mathbb{R})$ is the space of all real valued continuous functions. Endowed with the norm

$$\|x\| = \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} |x(t)|,$$

then P_T is a Banach space.

Lemma 2.14. *$x(t) \in P_T$ is a solution of (1) if and only if*

$$x(t) = \lambda \int_t^{\delta_+^T(t)} G(t, s)b(s)f(x(h(s)))\nabla s$$

where $G(t, s) = \dfrac{\hat{e}_{a(s)}(t, \rho(s))}{\hat{e}_{a(t)}(t, \delta_+^T(t)) - 1}$ is the Green's function of the problem (1).

Proof. Let $x(t) \in P_T$ be a solution of (1). We can reform the equation (1) as

$$\begin{aligned} x^{\nabla}(t) &= a(t)[x(\rho(t)) + \nu(t)x^{\nabla}(t)] + \lambda b(t)f(x(h(t))) \\ (1-\nu(t)a(t))x^{\nabla}(t) &= a(t)x(\rho(t)) + \lambda b(t)f(x(h(t))) \\ x^{\nabla}(t) &= \frac{a(t)}{1-\nu(t)a(t)}x(\rho(t)) + \lambda\frac{b(t)f(x(h(t)))}{1-\nu(t)a(t)} \\ x^{\nabla}(t) + \ominus_{\nu}(a(t))x(\rho(t)) &= \lambda \hat{e}_{a(t)}(t,\rho(t))b(t)f(x(h(t))) \end{aligned}$$

Multiply both sides of the above equation by $\hat{e}_{\ominus_{\nu}a(t)}(t,t_0)$ and then integrate from t to $\delta_+^T(t)$, we obtain

$$\int_t^{\delta_+^T(t)} [x(s)\hat{e}_{\ominus_{\nu}a(s)}(s,t_0)]^{\nabla}\nabla s = \lambda \int_t^{\delta_+^T(t)} \hat{e}_{\ominus_{\nu}a(s)}(s,t_0)\hat{e}_{a(s)}(s,\rho(s))b(s)f(x(h(s)))\nabla s.$$

We arrive at

$$[\hat{e}_{\ominus_{\nu}a(t)}(\delta_+^T(t),t_0) - \hat{e}_{\ominus_{\nu}a(t)}(t,t_0)]x(t) = \lambda \int_t^{\delta_+^T(t)} \hat{e}_{a(s)}(t_0,s)\hat{e}_{a(s)}(s,\rho(s))b(s)f(x(h(s)))\nabla s.$$

Dividing both sides of the above equation by $\hat{e}_{a(t)}(t,t_0)$ and using Lemma 2.2 we have

$$x(t)[\hat{e}_{\ominus_{\nu}a(t)}(\delta_+^T(t),t) - 1] = \lambda \int_t^{\delta_+^T(t)} \hat{e}_{a(s)}(t,\rho(s))b(s)f(x(h(s)))\nabla s.$$

We get

$$x(t) = \lambda \int_t^{\delta_+^T(t)} \frac{\hat{e}_{a(s)}(t,\rho(s))}{\hat{e}_{a(t)}(t,\delta_+^T(t)) - 1} b(s)f(x(h(s)))\Delta s$$

The proof is complete. □

If we define $\theta := e_{a(t_0)}(t_0,\delta_+^T(t_0))$ we have

$$0 < \alpha := \frac{a^{②}(t_0)}{a^2(t_0)(\theta-1)} \leq G(t,s) \leq \frac{\theta}{\theta-1} := \beta \text{ for } s \in [t,\delta_+^T(t)]_{\mathbb{T}}, \tag{10}$$

and

$$G(\delta_{\pm}^T(t),\delta_{\pm}^T(s)) = G(t,s) \text{ for } t \in \mathbb{T}^*, s \in [t,\delta_+^T(t)]_{\mathbb{T}}. \tag{11}$$

Lemma 2.15. *Let $G(t,s)$ be the Green's function of the problem (1). Then*

$$\int_t^{\delta_+^T(t)} a(s)G(t,s)\nabla s = -1$$

Proof. We consider the following problem

$$x^{\nabla}(t) = a(t)x(t) - a(t), \quad t \in \mathbb{T} \tag{12}$$

for $\lambda = 1$ and $b(t)f(x(h(t))) = -a(t)$ in the problem (1). By Lemma 2.14, $x(t) = -\int_t^{\delta_+^T(t)} a(s)G(t,s)\nabla s$ is the solution of the problem (12). Also, $x(t) = 1$ is the solution of this problem. Now, we show that $x(t) = 1$ is the unique solution. If the problem (12) has two solutions such that $y_1(t), y_2(t)$, then we get

$$\begin{aligned} y_1^{\nabla}(t) &= a(t)y_1(t) - a(t) \\ y_2^{\nabla}(t) &= a(t)y_2(t) - a(t). \end{aligned}$$

So we have the first order problem $(y_1 - y_2)^{\nabla}(t) = a(t)(y_1(t) - y_2(t))$, again by Lemma 2.14 we obtain $(y_1 - y_2)(t) = \int_t^{\delta_+^T(t)} 0G(t,s)\nabla s = 0$, since $b(t)f(x(h(t))) = 0$. This implies $y_1(t) = y_2(t)$ for $t \in \mathbb{T}$. Thus, the problem (12) has only unique solution $x(t) = 1$. Consequently,

$$\int_t^{\delta_+^T(t)} a(s)G(t,s)\nabla s = -1$$

Define K be a cone in P_T by

$$K := \{x \in P_T : x(t) \geq \gamma \|x\|, \forall t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}\}$$

where $\gamma := \dfrac{a^{②}(t_0)}{\theta a^2(t_0)}$ and an operator $A : K \to P_T$ by

$$Ax(t) = \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s.$$

Also define the numbers

$$B := \int_{t_0}^{\delta_+^T(t_0)} b(s)\nabla s, \quad C := \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} a(t). \tag{13}$$

Lemma 2.16. *$A(K) \subset K$ and $A : K \to K$ is compact and continuous.*

Proof. By using (11), b is ∇-periodic, h is periodic in shifts and Theorem 2.10 and substituting $\nu(s) = \delta_+^T(t)$ in (8) for $x \in K$, we have

$$\begin{aligned}
(Ax)(\delta_+^T(t)) &= \lambda \int_{\delta_+^T(t)}^{\delta_+^T(\delta_+^T(t))} G(\delta_+^T(t), s)b(s)f(x(h(s)))\nabla s \\
&= \lambda \int_t^{\delta_+^T(t)} G(\delta_+^T(t), \delta_+^T(s))b(\delta_+^T(s))\delta_+^{\nabla T}(s)f(x(h(\delta_+^T(s))))\nabla s \\
&= \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&= Ax(t).
\end{aligned}$$

We show that, for $x \in K$,

$$\begin{aligned}
Ax(t) &\geq \lambda \int_t^{\delta_+^T(t)} \frac{a^{②}(t_0)}{a^2(t_0)(\theta - 1)} b(s)f(x(h(s)))\nabla s \\
&= \frac{a^{②}(t_0)}{\theta a^2(t_0)} \lambda \int_{t_0}^{\delta_+^T(t_0)} \frac{\theta}{\theta - 1} b(s)f(x(h(s)))\nabla s \\
&\geq \gamma \|Ax\|.
\end{aligned}$$

Therefore, $A(K) \subset K$. We will prove that A is continuous and compact. Firstly, we will consider the continuity of A. Let $x_n \in K$ and $\|x_n - x\| \to 0$ as $n \to \infty$, the $x \in K$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. Because of continuity of f, for any $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$ and $\epsilon^* > 0$, we have

$$\begin{aligned}
\|Ax_n - Ax\| &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} |Ax_n - Ax| \\
&\leq \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)|f(x_n(h(s))) - f(x(h(s)))|\nabla s \\
&\leq \lambda \epsilon^* \beta \int_{t_0}^{\delta_+^T(t_0)} b(s)\nabla s = \epsilon
\end{aligned}$$

where $\epsilon^* = \dfrac{\epsilon}{\lambda\beta B}$. Thus A is continuous on K.

Next, we prove that A is a compact operator. It is equal to proving that A maps bounded sets in to relatively compact sets.

Let $S \subset K$ be an arbitrary bounded set in K, then there exists a number $R > 0$ such that $\|x\| < R$ for any $x \in K$. We prove $\overline{AS}$ is compact. In fact, for any $\{x_n\}_{n\in\mathbb{N}} \in K$ and $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned}
\|Ax_n\| &\leq \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)|f(x_n(h(s)))|\nabla s \\
&\leq \lambda\beta \int_{t_0}^{\delta_+^T(t_0)} b(s)|f(x_n(h(s)))|\nabla s := D.
\end{aligned}$$

Now, we find the ∇-derivative of the operator $Ax(t)$

$$(Ax)^{\nabla}(t) = \lambda\left(\frac{1}{\hat{e}_{a(t)}(t,\delta_+^T(t))-1}\right)^{\nabla} \int_t^{\delta_+^T(t)} \hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s \tag{14}$$

$$+ \lambda\frac{1}{\hat{e}_{a(t)}(\rho(t),\delta_+^T(\rho(t)))-1}\left(\int_t^{\delta_+^T(t)} \hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s\right)^{\nabla} \tag{15}$$

Firstly, we consider the ∇-derivative in (14)

$$\begin{aligned}
\left(\frac{1}{\hat{e}_a(t,\delta_+^T(t))-1}\right)^{\nabla} &= -\frac{\hat{e}_a^{\nabla}(t,\delta_+^T(t))}{(\hat{e}_a(t,\delta_+^T(t))-1)(\hat{e}_a(\rho(t),\delta_+^T(\rho(t)))-1)} \\
&= -\frac{a(t)[\hat{e}_a(t,\delta_+^T(t))-\hat{e}_a(\rho(t),\delta_+^T(\rho(t)))]}{(\hat{e}_a(t,\delta_+^T(t))-1)(\hat{e}_a(\rho(t),\delta_+^T(\rho(t)))-1)} \\
&= a(t)\left[\frac{1}{\hat{e}_a(t,\delta_+^T(t))-1} - \frac{1}{\hat{e}_a(\rho(t),\rho(\delta_+^T(t)))-1}\right]
\end{aligned}$$

Secondly, we consider the ∇-derivative in (15)

$$\begin{aligned}
&\left(\int_t^{\delta_+^T(t)} \hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s\right)^{\nabla} \\
&= \left(\int_t^{\tau} \hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s + \int_{\tau}^{\delta_+^T(t)} \hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s\right)^{\nabla} \\
&= \left(\int_s^{\tau} \hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s\right)^{\nabla} + \left(\int_{\delta_-^T(\tau)}^{t} \hat{e}_{a(t)}(t,\rho(\delta_+^T(s))b(s)f(x(h(s)))\nabla s\right)^{\nabla} \\
&= -\hat{e}_{a(t)}(\rho(t),\rho(t))b(t)f(x(h(t))) + \int_t^{\tau} a(t)\hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s \\
&+ \hat{e}_{a(t)}(\rho(t),\rho(\delta_+^T(t)))b(t)f(x(h(t))) + \int_{\delta_-^T(\tau)}^{t} a(t)\hat{e}_{a(t)}(t,\rho(\delta_+^T(s))b(s)f(x(h(s)))\nabla s \\
&= b(t)f(x(h(t)))[\hat{e}_{a(t)}(\rho(t),\rho(\delta_+^T(t)))-1] - a(t)\int_t^{\delta_+^T(t)} \hat{e}_{a(t)}(t,\rho(s)b(s)f(x(h(s)))\nabla s.
\end{aligned}$$

Thus, we obtain the $(Ax)^{\nabla}(t)$ by using above equations

$$\begin{aligned}(Ax)^{\nabla}(t) &= \lambda a(t)\left[\frac{1}{\hat{e}_a(t,\delta_+^T(t))-1}-\frac{1}{\hat{e}_a(\rho(t),\rho(\delta_+^T(t)))-1}\right]\int_t^{\delta_+^T(t)}\hat{e}_{a(t)}(t,\rho(s))b(s)f(x(h(s)))\nabla s\\ &+ \lambda\frac{1}{\hat{e}_{a(t)}(\rho(t),\rho(\delta_+^T(t)))-1}\Big(b(t)f(x(h(t)))[\hat{e}_{a(t)}(\rho(t),\rho(\delta_+^T(t)))-1]\\ &- a(t)\int_t^{\delta_+^T(t)}\hat{e}_{a(t)}(t,\rho(s)b(s)f(x(h(s)))\nabla s\Big)\\ &= a(t)Ax(t)+\lambda b(t)f(x(h(t))).\end{aligned}$$

Consequently, we have

$$\begin{aligned}\|(Ax_n)^{\nabla}\| &\le \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|a(t)Ax_n(t)+\lambda b(t)f(x_n(h(t)))|\\ &\le \|Ax_n\|\max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|a(t)|+\lambda\max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|b(t)\|f(t,x_n(h(t)))|\\ &\le CD+\lambda\max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|b(t)\|f(x_n(h(t)))|:=L\end{aligned}$$

which imply that $\{Ax_n\}_{n\in\mathbb{N}}$ and $\{A^{\nabla}x_n\}_{n\in\mathbb{N}}$ are uniformly bounded on $[t_0,\delta_+^T(t_0)]_{\mathbb{T}}$. There exists a subsequence of $\{Ax_n\}_{n\in\mathbb{N}}$ converges uniformly on $[t_0,\delta_+^T(t_0)]_{\mathbb{T}}$, namely, $\overline{A(S)}$ is compact. The proof is complete. □

Lemma 2.17. *Existence of positive periodic solutions in shifts $\delta_\pm$ of (1) is equivalent to the existence of fixed point problem of A_λ in K.*

The proof of the Lemma 2.17 is straight forward and hence omitted.

Let X be a Banach space and K be a cone in X. A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to [0,\infty)$ is continuous and

$$\psi(\mu x+(1-\mu)y)\ge\mu\psi(x)+(1-\mu)\psi(y),\ x,y\in K,\ \mu\in[0,1].$$

Let $a,b,c>0$ be constants with K and X as above. Define the convex sets K_a, $K(\phi,r)$ and $K(\psi,b,c)$ by

$$K_a=\{x\in K:\|x\|<a\},\quad K(\phi,r)=\{x\in K:\phi(x)<r\}\ \text{ and }\ K(\psi,b,c)=\{x\in K:\psi(x)\ge b,\ \|x\|\le c\}.$$

In order to follow the main results of this paper easily, now we state the following fixed point theorems in a cone.

Theorem 2.18. *(Guo-Krasnoselskiĭ Fixed Point Theorem [8]) Let X be a Banach space, $K\subset X$ be a cone, and suppose that Ω_1 and Ω_2 are open, bounded subsets of X with $0\in\Omega_1$ and $\overline{\Omega_1}\subset\Omega_2$. Suppose further that $A : K\cap(\overline{\Omega_2}\setminus\Omega_1)\to K$ is a completely continuous operator such that either*

(i) $\|Au\|\le\|u\|$ *for* $u\in K\cap\partial\Omega_1$, $\|Au\|\ge\|u\|$ *for* $u\in K\cap\partial\Omega_2$, *or*

(ii) $\|Au\|\ge\|u\|$ *for* $u\in K\cap\partial\Omega_1$, $\|Au\|\le\|u\|$ *for* $u\in K\cap\partial\Omega_2$

holds. Then A has a fixed point in $K\cap(\overline{\Omega_2}\setminus\Omega_1)$.

Theorem 2.19. *(Avery-Henderson Fixed Point Theorem [30]) Let K be a cone in a real Banach space E. If μ and ϕ are increasing, nonnegative, continuous functionals on K, let θ be a nonnegative continuous functional on K with $\theta(0)=0$ such that for some positive constants r and M,*

$$\phi(u)\le\theta(u)\le\mu(u)\ \text{ and }\ \|u\|\le M\phi(u)$$

for all $u\in\overline{K(\phi,r)}$. Suppose that there exist positive numbers $p<q<r$ such that

$$\theta(\lambda u)\le\lambda\theta(u)\ \text{ for all } 0\le\lambda\le1\ \text{ and }\ u\in\partial K(\theta,q).$$

If $A:\overline{K(\phi,r)}\to K$ is a completely continuous operator satisfying

(i) *$\phi(Au) > r$ for all $u \in \partial K(\phi, r)$,*

(ii) *$\theta(Au) < q$ for all $u \in \partial K(\theta, q)$,*

(iii) *$K(\mu, q) \neq \emptyset$ and $\mu(Au) > p$ for all $u \in \partial K(\mu, p)$,*

then A has at least two fixed points u_1 and u_2 such that

$$p < \mu(u_1) \ \text{with} \ \theta(u_1) < q \ \text{and} \ q < \theta(u_2) \ \text{with} \ \phi(u_2) < r.$$

Theorem 2.20. *(Leggett-Williams Fixed Point Theorem [28]) Let K be a cone in Banach space X. Suppose $A : \overline{K_r} \to \overline{K_r}$ be a completely continuous operator and ψ be concave nonnegative continuous functional on K with $\psi(u) \leq u$ for all $u \in \overline{K_r}$. If there exist $0 < p < q < l < r$ such that the following conditions holds:*

(i) *$\{u \in K(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in K(\psi, q, l)$;*

(ii) *$\|Au\| < p$ for all $\|u\| \leq p$;*

(iii) *$\psi(Au) > q$ for all $u \in K(\psi, q, r)$ with $\|Au\| > l$.*

Then A has at least three positive solutions u_1, u_2 and u_3 in $\overline{K_r}$ satisfying

$$\|u_1\| < p, \ \ \psi(u_2) > q, \ \ p < \|u_3\| \ \text{with} \ \psi(u_3) < q.$$

3. Main Results

In this section, we use the notation i_0 =number of zeros in the set f_0, f_∞ and i_∞ =number of infinities in the set f_0, f_∞ as in the reference [10], where $f_0 = \lim_{u \to 0^+} \dfrac{f(u)}{u}$ and $f_\infty = \lim_{u \to \infty} \dfrac{f(u)}{u}$. It is clear that $i_0, i_\infty = 0, 1$ or 2. Then we shall show that the problem (1) has i_0 or i_∞ positive periodic solution(s) in shifts $\delta_\pm$.

Theorem 3.1. *The problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$ in the case $i_0 = 1$ and $i_\infty = 1$.*

Proof. First, we consider the case $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, then there exists $R_1 > 0$ such that $f(x) \leq \epsilon x$ for $0 < x < R_1$, where ϵ satisfies

$$\epsilon \leq (\lambda \beta B)^{-1}.$$

If $x \in K$ with $\|x\| = R_1$, then

$$\begin{aligned} \|Ax\| &\leq \lambda\beta \int_{t_0}^{\delta_+^T(t_0)} b(s) f(x(h(s)))\nabla s \\ &\leq \lambda\beta \int_{t_0}^{\delta_+^T(t_0)} b(s)\epsilon x(h(s))\nabla s \\ &\leq \lambda\beta\epsilon\|x\| \int_{t_0}^{\delta_+^T(t_0)} b(s)\nabla s \\ &= \lambda\beta\epsilon B\|x\| \leq \|x\|. \end{aligned}$$

It follows that if $\Omega_{R_1} = \{x \in P_T : \|x\| < R_1\}$, then $\|Ax\| \leq \|x\|$ for $x \in K \cap \partial\Omega_{R_1}$.
Since $f_\infty = \infty$, there exists $R_2' > 0$ such that $f(x) \geq \eta x$ for $x \geq R_2'$ where $\eta > 0$ is chosen such that

$$\eta \geq (\lambda \alpha \gamma B)^{-1}.$$

Set $R_2 = \max\{2R_1, \dfrac{R_2'}{\gamma}\}$ and $\Omega_{R_2} = \{x \in P_T : \|x\| < R_2\}$.
If $x \in K$ with $\|x\| = R_2$ then $x(t) \geq \gamma\|x\| \geq R_2$. So that

$$\begin{aligned} Ax(t) &\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)f(x(h(s)))\nabla s \\ &\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)\eta x(h(s))\nabla s \\ &\geq \lambda\alpha\eta \int_{t_0}^{\delta_+^T(t_0)} b(s)\gamma\|x\|\nabla s \\ &= \lambda\alpha\eta\gamma B\|x\| \\ &\geq \|x\|. \end{aligned}$$

In other words if $x \in K \cap \partial\Omega_{R_2}$, then $\|Ax\| \geq \|x\|$. Thus by the Krasnoselskiĭ's Fixed Point Theorem, it follows that A has a fixed point x in $x \in K \cap (\overline{\Omega_{R_2}} \backslash \Omega_{R_1})$ with $R_1 \leq x \leq R_2$.
Now we consider the case $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, there exists $R_3 > 0$ such that $f(x) \geq mx$ for $0 < x \leq R_3$, where m is such that

$$m \geq (\lambda\alpha\gamma B)^{-1}.$$

If $x \in K$ with $\|x\| = R_3$, then we have

$$\begin{aligned} Ax(t) &\geq \lambda \int_{t_0}^{\delta_+^T(t_0)} \alpha b(s)f(x(h(s)))\nabla s \\ &\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)mx(h(s))\nabla s \\ &\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)m\gamma\|x\|\nabla s \\ &\geq \lambda\alpha m\gamma B\|x\| \\ &\geq \|x\|. \end{aligned}$$

Thus we let $\Omega_{R_3} = \{x \in P_T : \|x\| < R_3\}$, so that $\|Ax\| \geq \|x\|$, for $x \in K \cap \partial\Omega_{R_3}$.
Next consider $f_\infty = 0$. By definition there exists $R_4' > 0$ such that $f(x) \leq \delta x$ for $x \geq R_4'$, where $\delta > 0$ satisfies

$$\delta \leq (\lambda\beta B)^{-1}. \tag{16}$$

Suppose, f is bounded. Then $f(x) \leq C$ for all $x \geq 0$ for some constant $C > 0$. Pick

$$R_4 = \max\{2R_3, \lambda\beta CB\}.$$

If $x \in K$ with $\|x\| = R_4$, then

$$\begin{aligned} \|Ax(t)\| &\leq \lambda\beta \int_{t_0}^{\delta_+^T(t_0)} b(s)f(x(h(s)))\nabla s \\ &\leq \lambda\beta CB \\ &\leq R_4 = \|x\|. \end{aligned}$$

Now suppose that f is unbounded. From $f \in C((0,\infty),(0,\infty))$ it is easy to know that there exists $R_4 \geq \max\{2R_3, R_4'\}$ such that $f(x) \leq f(R_4)$ for $0 < x \leq R_4$. If $x \in K$ with $\|x\| = R_4$, then by using (16) we have

$$\begin{aligned} Ax(t) &\leq \lambda \int_{t_0}^{\delta_+^T(t_0)} \beta b(s) f(x(h(s)))\nabla s \\ &\leq \lambda\beta \int_{t_0}^{\delta_+^T(t_0)} b(s) f(R_4)\nabla s \\ &\leq \lambda\beta\delta R_4 \int_{t_0}^{\delta_+^T(t_0)} b(s)\nabla s \\ &= \lambda\beta\delta R_4 B \\ &\leq R_4 = \|x\|. \end{aligned}$$

Consequently, in either case we take $\Omega_{R_4} = \{x \in P_T : \|x\| < R_4\}$, so that for $x \in K \cap \partial\Omega_{R_4}$, then $\|Ax\| \leq \|x\|$. Thus by the Krasnoselskiĭ Fixed Point Theorem, it follows that A has a fixed point x in $K \cap (\overline{\Omega_{R_4}} \backslash \Omega_{R_3})$ with $R_3 \leq x \leq R_4$. The proof is complete. □

Now, we shall discuss the existence for the positive T-periodic solutions in shifts $\delta_\pm$ for the problem (1) under the condition $i_0 = 0$ and $i_\infty = 0$.

Theorem 3.2. *Suppose that the following conditions hold:*
(A1) *There exists constant $r' > 0$ such that $f(x) \leq r'\Lambda_1$ for $0 < x < r'$, where $\Lambda_1 = (\lambda\beta B)^{-1}$,*
(A2) *There exists constant $s' > 0$ such that $f(x) \geq s'\Lambda_2$ for $\gamma s' < x < s'$, where $\Lambda_2 = (\lambda\alpha B)^{-1}$, with $r' < s'$.*
Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$ such that $r' \leq \|x\| \leq s'$.

Proof. Without lost of generality, we assume $r' < s'$. Let $\Omega_{r'} = \{x \in P_T : \|x\| < r'\}$, for any $x \in K \cap \partial\Omega_{r'}$.
In view of (A1), we have

$$\begin{aligned} Ax(t) &\leq \lambda \int_{t_0}^{\delta_+^T(t_0)} \beta b(s) f(x(h(s)))\nabla s \\ &\leq \lambda\beta \int_{t_0}^{\delta_+^T(t_0)} b(s) r'\Lambda_1 \nabla s \\ &= \lambda\beta r'\Lambda_1 B = r' = \|x\| \end{aligned}$$

which yields $\|Ax\| \leq \|x\|$ for $x \in K \cap \partial\Omega_{r'}$.
Now, set $\Omega_{s'} = \{x \in P_T : \|x\| < s'\}$, for any $x \in K \cap \partial\Omega_{s'}$, we have $\gamma s' < x(t) < s'$, for $t \in [t_0, \delta_+^T(t_0)]$. Hence, by (A2) we can get

$$\begin{aligned} Ax(t) &\geq \lambda \int_{t_0}^{\delta_+^T(t_0)} \alpha b(s) f(x(h(s)))\nabla s \\ &\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s) s'\Lambda_2 \nabla s \\ &= \lambda\alpha s'\Lambda_2 B \\ &= s' = \|x\|. \end{aligned}$$

Thus, we have $\|Ax\| \geq \|x\|$ for $x \in K \cap \partial\Omega_{s'}$. Krasnoselskiĭ Fixed Point Theorem guarantees that A has a fixed point x in $K \cap (\overline{\Omega_{s'}} \backslash \Omega_{r'})$. Moreover, it is a positive periodic solution in shifts $\delta_\pm$ of the problem (1) such that $r' \leq \|x\| \leq s'$ The proof is complete. □

Now, we will give the results of the existence for the positive T-periodic solution in shifts of the problem (1) under $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$.

Theorem 3.3. *Suppose that $f_0 \in [0, \Lambda_1)$ and $f_\infty \in (\frac{1}{\gamma}\Lambda_2, \infty)$ hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Proof. It is easy to see that under the assumptions, the conditions (A1) and (A2) in Theorem 3.2 are satisfied. So the proof is omitted. □

Theorem 3.4. *Suppose that $f_0 \in (\frac{1}{\gamma}\Lambda_2, \infty)$ and $f_\infty \in [0, \Lambda_1)$ hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Proof. First, in the view of $f_0 \in (\frac{1}{\gamma}\Lambda_2, \infty)$ for $\epsilon = f_0 - \frac{1}{\gamma}\Lambda_2 > 0$, there exists a sufficiently small $s' > 0$ such that

$$\frac{f(x)}{x} \geq f_0 - \epsilon = \frac{1}{\gamma}\Lambda_2 \; for \; x \in (0, s')$$

Thus, if $x \in [\gamma s', s']$, then one has

$$f(x) \geq \frac{1}{\gamma}\Lambda_2 x \geq \frac{1}{\gamma}\Lambda_2 \gamma s' = \Lambda_2 s',$$

which yieds the condition (A2) in Theorem 3.2.
Next, by $f_\infty \in [0, \Lambda_1)$, for $\epsilon = \Lambda_1 - f_\infty > 0$ there exists a sufficiently large $r'' > s'$ such that

$$\frac{f(x)}{x} \leq f_\infty + \epsilon = \Lambda_1 \text{ for } x \in [r'', f_\infty).$$

We consider two cases:
Case 1: Suppose that f is bounded, we say $f(x) \leq M$ for $x \in [0, \infty)$. In this case, take sufficiently large $r' > 0$ such that $r' \geq \max\{\frac{M}{\Lambda_1}, r''\}$ then we get

$$f(x) \leq M \leq \Lambda_1 r' \text{ for } x \in [0, r'].$$

Then from the above inequality, the condition (A1) of Theorem 3.2 is satisfied.
Case2: Suppose that f is unbounded. Then from $f \in C([0, \infty), [0, \infty))$, we know that there is $r' > r''$ such that

$$f(x) \leq f(r') \text{ for } x \in [0, r'].$$

Since $r' > r''$, we have

$$f(x) \leq f(r') \leq \Lambda_1 r' \text{ for } x \in [0, r'].$$

Thus, the condition (A2) of Theorem 3.2 is satisfied. Hence, from Theorem 3.2, the conclusion of this theorem holds. The proof is complete. □

Also we get the following two results.

Corollary 3.5. *Suppose that $f_0 = 0$ and condition (A2) in Theorem 3.2 hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Corollary 3.6. *Suppose that $f_\infty = 0$ and condition (A2) in Theorem 3.2 hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Theorem 3.7. *Suppose that $f_0 \in (0, \Lambda_1)$ and $f_\infty = \infty$ hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Proof. Since $f_\infty = \infty$, similar to the first part of Theorem 3.1 we get

$$\|Ax\| \geq \|x\| \text{ for } x \in K \cap \partial\Omega_{R_2}.$$

By $f_0 \in (0, \Lambda_1)$, for $\epsilon = \Lambda_1 - f_0 > 0$, there exists a sufficiently small $r' \in (0, R_2)$ such that

$$f(x) \leq (f_0 + \epsilon)x = \Lambda_1 x \leq \Lambda_1 . r' \; for \; x \in [0, r'].$$

Similar to the proof of Theorem 3.2, we obtain

$$\|Ax\| \leq \|x\| \text{ for } x \in K \cap \partial\Omega_{r'}.$$

Thus, we have that A has a fixed point x in $K \cap (\overline{\Omega_{R_2}} \backslash \Omega_{r'})$. The proof is complete. □

Theorem 3.8. *Suppose that $f_0 = \infty$ and $f_\infty \in (0, \Lambda_1)$ hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Proof. First, in view of $f_0 = \infty$, similar to the second part of Theorem 3.1, we have

$$\|Ax\| \geq \|x\| \text{ for } x \in K \cap \partial\Omega_{R_3}.$$

Next, by $f_\infty \in (0, \Lambda_1)$, similar to the second part of the proof of Theorem 3.4 and Theorem 3.2, we have

$$\|Ax\| \leq \|x\| \text{ for } x \in K \cap \partial\Omega_{r'},$$

where $r' > R_3$. Thus, the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$ and the proof is complete. □

Also, we get the following corollaries easily.

Corollary 3.9. *Suppose that $f_\infty = \infty$ and condition (A1) in Theorem 3.2 hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Corollary 3.10. *Suppose that $f_0 = \infty$ and condition (A1) in Theorem 3.2 hold. Then the problem (1) has at least one positive T-periodic solution in shifts $\delta_\pm$.*

Now, we will give the existence results of multiple positive T-periodic solutions in shifts $\delta_\pm$ for the problem (1) under the conditions $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$. The following two theorems are easily proved combining the proof of Theorem 3.1 and Theorem 3.2.

Theorem 3.11. *Suppose that $i_0 = 0$ and $i_\infty = 2$ condition (A1) of Theorem 3.2 hold. Then the problem (1) has at least two positive T-periodic solutions x_1, x_2 in shifts $\delta_\pm$ such that $0 < \|x_1\| < r' < \|x_2\|$.*

Theorem 3.12. *Suppose that $i_0 = 2$ and $i_\infty = 0$ condition (A2) of Theorem 3.2 hold. Then the problem (1) has at least two positive T-periodic solutions x_1, x_2 in shifts $\delta_\pm$ such that $0 < \|x_1\| < s' < \|x_2\|$.*

Now, we will give the existence of two positive T-periodic solutions in shifts $\delta_\pm$ under the more general case. To get this result, we define the nonnegative increasing continuous functionals θ, φ and Φ by

$$\theta(x) = \min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} x(t), \quad \varphi(x) = \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} x(t), \quad \Phi(x) = \max_{t\in\mathbb{T}} x(t).$$

For each $x \in K$, $\theta(x) \leq \varphi(x) = \Phi(x)$. Also we define, for each $d > 0$, the set $K(\psi, d) = \{x \in K : \psi(x) < d\}$, for a nonnegative continuous functional ψ on a cone K of areal Banach space P_T.
In additional, for each $x \in K$, $\theta(x) = x(\xi) \geq \gamma\|x\|$ where $\xi \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. Thus $\|x\| \leq \dfrac{1}{\gamma}\theta(x)$ for all $x \in K$. Finally, we also note that $\varphi(\lambda x) = \lambda\varphi(x)$, $0 \leq \lambda \leq 1$ and $x \in \partial K(\varphi, b')$.

Theorem 3.13. *Suppose that there exist positive numbers $0 < a' < b' < c'$ such that $0 < a' < \frac{\alpha}{\beta}b' < \gamma\frac{\alpha}{\beta}c'$. Assume $f(x)$ satisfies the following conditions:*

(i) $f(x) > \frac{c'}{\lambda\alpha B}$*, for* $x \in [c', \frac{c'}{\gamma}]$*,*

(ii) $f(x) < \frac{b'}{\lambda\beta B}$*, for* $x \in [0, \frac{b'}{\gamma}]$*,*

(iii) $f(x) > \frac{a'}{\lambda\alpha B}$*, for* $x \in [\gamma a', \frac{a'}{\gamma}]$*.*

Then the problem (1) has at least two positive T-periodic solutions x_1, x_2 in shifts $\delta_\pm$ such that

$$a' < \max_{t\in\mathbb{T}} x_1(t) \text{ with } \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} x_1(t) < b' \text{ and} b' < \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} x_2(t) \text{ with } \min_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} x_2(t) < c'.$$

Proof. By the definition of the operator A and its properties, it suffices to show that the conditions of Theorem 2.19 hold with respect to A. We first show that if $x \in \partial K(\theta, c')$, then $\theta(Ax) > c'$. Indeed, if $x \in \partial K(\theta, c')$, then $\theta(x) = \min_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} x(\xi) < c'$, where $\xi \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$, one gets $x(t) \geq c'$ for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. If we recall that $\|x\| \leq \frac{1}{\gamma}\theta(x) \leq \frac{1}{\gamma}c'$, for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. As a consequence of (i), $f(x(t)) > \frac{c'}{\lambda\alpha B}$ for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$.
Also, $Ax \in K$, so we get

$$\begin{aligned}
\theta(Ax) &= \min_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} Ax(t) \\
&= \min_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)\frac{c'}{\lambda\alpha B}\nabla s \\
&\geq \lambda\alpha \frac{c'}{\lambda\alpha B}B = c'.
\end{aligned}$$

Next, we verify $\varphi(Ax) < b'$ for $x \in \partial K(\varphi, b')$. Let us choose $x \in \partial K(\varphi, b')$, then $\varphi(x) = \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} x(t) = x(\eta) = b'$, where $\eta \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. This implies $0 \leq x(t) \leq b'$, for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$ and since $x \in K$ we also have

$$0 \leq x(t) \leq \|x\| \leq \frac{1}{\gamma}x(t) \leq \frac{1}{\gamma}x(\eta) = \frac{1}{\gamma}b' \text{ for } t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}.$$

Using (ii), $f(x(t)) \leq \frac{b'}{\lambda\beta B}$, for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$.
$Ax \in K$, and so

$$\begin{aligned}
\varphi(Ax) &= \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} Ax(t) \\
&= \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)\frac{b'}{\lambda\beta B}\nabla s \\
&\geq \lambda\beta \frac{b'}{\lambda\beta B}B = b'.
\end{aligned}$$

Finally, we prove that $K(\Phi, a') \neq \emptyset$ and $\Phi(Ax) > a'$, for all $x \in \partial K(\Phi, a')$.
In fact, the constant function $\frac{a'}{2} \in K(\Phi, a')$. Moreover, for $x \in \partial K(\Phi, a')$, we have $\Phi(x) = \max_{t \in \mathbb{T}} x(t) = \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} x(t) = x(\eta) = a'$.
This implies $\gamma a' = \gamma\|x\| \leq x(t) \leq \frac{1}{\gamma}x(t) \leq \frac{1}{\gamma}x(\eta) = \frac{1}{\gamma}a'$, for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. Using assumption (iii), $f(x(t)) > \frac{a'}{\lambda\alpha B}$, for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$. As before $Ax \in K$, and so

$$\begin{aligned}
\Phi(Ax) &= \max_{t \in \mathbb{T}} Ax(t) \\
&= \max_{t \in \mathbb{T}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&\geq \lambda\alpha \int_{t_0}^{\delta_+^T(t_0)} b(s)\frac{a'}{\lambda\alpha B}\nabla s \\
&\geq \lambda\alpha \frac{a'}{\lambda\alpha B}B = a'.
\end{aligned}$$

Thus, by Theorem 2.19 there exists two fixed points of A which are at least two positive T-periodic solutions x_1, x_2 in shifts $\delta_\pm$, belonging to $\overline{K(\theta, c')}$, of the problem (1) such that

$$a' < \Phi(x_1) \text{ with } \varphi(x_1) < b' \text{ and } b' < \varphi(x_2) \text{ with } \theta(x_2) < c'.$$

The proof is complete. □

Now, we will give the nonexistence results for the problem (1).

Theorem 3.14. *If $i_0 = 0$ or $i_\infty = 0$ then the problem (1) has no positive T-periodic solution in shifts $\delta_\pm$ for sufficiently large or small $\lambda > 0$, respectively.*

Proof. If $i_0 = 0$, then $f_0 > 0$ and $f_\infty > 0$. It follows that there exist positive numbers δ_1, δ_2, r_1 and r_2 such that $r_1 > r_2$ and

$$f(x) \geq \delta_1 x \ \text{ for } \ x \in [0, r_1],$$

$$f(x) \geq \delta_2 x \ \text{ for } \ x \in [r_2, \infty].$$

Let $\delta = \min\{\delta_1, \delta_2, \min\limits_{r_1 \leq x \leq r_1} \dfrac{f(x)}{x}\} > 0$. Thus, we have

$$f(x) \geq \delta x \ \text{ for } \ x \in [0, \infty].$$

Assume $y(t)$ is a positive periodic solution in shift $\delta_\pm$ of the problem (1). We will show that this leads to a contraction for $\lambda > \lambda_0$ where $\lambda_0 = \dfrac{1}{\alpha \delta B}$.
Since $Ay(t) = y(t)$ for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$ we have

$$\begin{aligned} \|y\| = \|Ay\| &\geq \lambda \int_{t_0}^{\delta_+^T(t_0)} \alpha b(s) f(y(h(s))) \nabla s \\ &\geq \lambda \alpha \delta \|y\| \int_{t_0}^{\delta_+^T(t_0)} b(s) \nabla s \\ &> \|y\|, \end{aligned}$$

which is a contradiction.
If $i_\infty = 0$, then $f_0 < \infty$ and $f_\infty < \infty$. It follows that there exist positive numbers $\epsilon_1, \epsilon_2, r_1$ and r_2 such that $r_1 < r_2$,

$$f(x) \leq \epsilon_1 x \ \text{ for } \ x \in [0, r_1],$$

$$f(x) \leq \epsilon_2 x \ \text{ for } \ x \in [r_2, \infty).$$

Let $\epsilon = \max\{\epsilon_1, \epsilon_2, \max\limits_{x \in [r_1, r_2]} \dfrac{f(x)}{x}\} > 0$. Thus, we have

$$f(x) \leq \epsilon x \ \text{ for } \ x \in [0, \infty).$$

Assume $y(t)$ is a positive periodic solution in shift $\delta_\pm$ of the problem (1). We show that this leads to a contraction for $0 < \lambda < \lambda_1$, where $\lambda_1 = \frac{1}{\beta \epsilon B}$. Since $Ay(t) = y(t)$ for $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$ we have

$$\begin{aligned} \|y\| = \|Ay\| &\leq \lambda \int_{t_0}^{\delta_+^T(t_0)} \beta b(s) f(y(h(s))) \nabla s \\ &\leq \lambda \beta \epsilon \|y\| \int_{t_0}^{\delta_+^T(t_0)} b(s) \nabla s \\ &< \|y\|, \end{aligned}$$

which is a contraction. The proof is complete. □

By using Leggett-Williams multiple fixed point theorem [28], we will prove the existence of at least three positive solutions of the problem (1).

Theorem 3.15. *Suppose that there exist constants* $0 < p < q$ *such that*

($H1$) $f(x) \leq \frac{(\theta-1)p}{\theta}$*, for* $x \in [0, p]$*,*

($H2$) $f(x) \geq \frac{q}{\gamma}$*, for* $x \in [q, \frac{q}{\gamma}]$*,*

($H3$) *one of the following conditions holds:*
(a) $\limsup_{x\to\infty} \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \dfrac{f(x)}{a(t)x} > -\xi$
(b) there exists a number $r > \frac{q}{\gamma}$ *such that* $f(x) < \frac{(\theta-1)r}{\theta}$*, for* $x \in [0, r]$*.*

Then Eq. (1) has at least three positive T-periodic solutions for

$$\frac{\theta-1}{\theta B} < \lambda < \frac{1}{\xi \sup\limits_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} b(t)}$$

where $\xi = \delta_+^T(\delta_+^T(t_0)) - \delta_+^T(t_0)$*.*

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Theorem 2.20 hold with respect to A. We first show that if (a) holds, then there exists a number $r > \frac{q}{\gamma}$ such that $A : \overline{K_r} \to \overline{K_r}$. From (H3)-(a), it follows that there exist an $\epsilon \in (-\xi, 0)$ and $\delta > 0$ such that $f(x) \leq a(t)\epsilon x$ for $x > \delta$. Let

$$\Gamma := \max_{0\leq x\leq \delta} f(x).$$

Then $f(x) \leq a(t)\epsilon x + \Gamma$ for $x \geq 0$. Choose

$$r > \max\left\{\frac{q}{\gamma}, \frac{\gamma\theta\xi}{(\theta-1)(\xi+\epsilon)}\right\}.$$

Then for $x \in \overline{K_r}$ we have

$$\begin{aligned}
\|Ax(t)\| &= \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&\leq \lambda \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} b(t) \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \int_t^{\delta_+^T(t)} G(t,s)[a(t)\epsilon x(h(s)) + \Gamma]\nabla s \\
&\leq \lambda \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} b(t) \left\{ \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \int_t^{\delta_+^T(t)} G(t,s)a(t)\epsilon\|x\|\nabla s + \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \int_t^{\delta_+^T(t)} \Gamma\nabla s \right\} \\
&\leq \lambda \sup_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} b(t) \left[\frac{\gamma\theta\xi}{\theta-1} - r\right] \\
&< \frac{1}{\xi}\left[\frac{\gamma\theta\xi}{\theta-1} - r\right] \\
&< r.
\end{aligned}$$

Next, we verify that if there exists a positive number r' such that if $f(x) < \frac{(\theta-1)r'}{\theta}$ for $x \in [0, r']$, then

$A : \overline{K_r} \to \overline{K_r}$. Indeed, if $x \in \overline{K_r}$ then we have

$$\begin{aligned}
\|Ax(t)\| &= \sup_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&\leq \lambda \sup_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} b(t) \sup_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} \int_t^{\delta_+^T(t)} G(t,s)\frac{(\theta-1)r'}{\theta}\nabla s \\
&\leq \lambda r' \sup_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} b(t) \sup_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} \int_t^{\delta_+^T(t)} \nabla s \\
&\leq \lambda r' \sup_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} b(t)\xi \\
&< r'
\end{aligned}$$

thus, $Ax \in K_r$. Hence we have shown in the previous claim if (a) or (b) holds, then there exists a number r with $r > \frac{q}{\gamma}$ such that $A : \overline{K_r} \to K_r$. It is also note from (H1) with $r' = p$ that $A : \overline{K_p} \to K_p$. Therefore the condition (ii) of Leggett-Williams fixed point theorem is satisfied.

Now, we define a nonnegative continuous function ψ on K by $\psi(x) = \min_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} x(t)$. Then $\psi(x) \leq \|x\|$. Let $l = \frac{q}{\gamma}$ and $\vartheta_0(t) = \vartheta_0$ is any given number satisfying $q < \vartheta_0 < l$. Then $\vartheta_0 \in \{x : x \in K(\psi, q, l), \psi(x) > q\}$. Further, for $x \in K(\psi, q, l)$, we have by (H2)

$$\begin{aligned}
\psi(Ax) &= \min_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} \lambda \int_t^{\delta_+^T(t)} G(t,s)b(s)f(x(h(s)))\nabla s \\
&\geq \alpha\lambda \int_{t_0}^{\delta_+^T(t_0)} b(s)f(x(h(s)))\nabla s \\
&\geq \frac{\alpha\lambda q}{\gamma} \int_{t_0}^{\delta_+^T(t_0)} b(s)\nabla s \\
&> q.
\end{aligned}$$

Finally, we assert that if $x \in K(\psi, q, l)$ and $\|Ax\| > \frac{q}{\gamma}$ then $\psi(Ax) > q$.

Suppose that $x \in K(\psi, q, l)$ and $\|Ax\| > \frac{q}{\gamma}$. Then we have

$$l = \frac{q}{\gamma} < \|Ax\| \leq \frac{\theta\lambda}{\theta-1} \int_{t_0}^{\delta_+^T(t_0)} b(s)f(x(h(s)))\nabla s$$

which in turn implies that

$$\begin{aligned}
\psi(Ax) &\geq \frac{a^{②}(t_0)}{a^2(t_0)(\theta-1)}\lambda \int_{t_0}^{\delta_+^T(t_0)} b(s)f(x(h(s)))\nabla s \\
&\geq \frac{a^{②}(t_0)}{\theta a^2(t_0)}\|Ax\| \\
&\geq \gamma\|Ax\| \\
&\geq q.
\end{aligned}$$

Therefore, the condition (iii) of Leggett-Williams fixed point theorem is satisfied.

To sum up, the hypothesises of Leggett-Williams fixed point theorem are satisfied. Hence the Eq. (1) has at least three positive T-periodic solutions x_1, x_2, x_3 in shifts $\delta_\pm$ such that

$$\|x_1\| < p,\ q < \min_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} x_2(t) \text{ and } \|x_3\| > q \text{ with } \min_{t\in[t_0,\delta_+^T(t_0))_{\mathbb{T}}} x_3(t) < q.$$

The proof of Theorem 3.15 is complete. □

Theorem 3.16. *Suppose that there exist constants* $0 < p_1 < q_1 < \frac{q_1}{\gamma} < p_2 < q_2 < \frac{q_2}{\gamma} < p_3 < ... < p_n < q_n,\ \ n \in \mathbb{N}$, *such that the following conditions are satisfied:*

(i) $f(x) \leq \frac{(\theta-1)p_i}{\theta}$, *for* $x \in [0, p_i]$,

(ii) $f(x) \geq \frac{q_i}{\gamma}$, *for* $x \in [q_i, \frac{q_i}{\gamma}]$.

Then Eq. (1) has at least $2n - 1$ *positive periodic solutions in shifts* $\delta_{\pm}$.

Proof. When $n = 1$, it is clear that Theorem 3.2 holds so the operator A has at least one fixed point. When $n = 2$, it is clear that Theorem 3.15 holds. Then we can obtain at least three positive T-periodic solutions in shifts $\delta_{\pm}$. Following this way, we finish the proof by induction. The proof is complete. □

Example 3.17. *Let* $\mathbb{T} = \{3^n\}_{n\in\mathbb{N}_0}$ *be a periodic time scale in shift* $\delta_{\pm}(P, t) = P^{\pm}t$ *with period* $P = 3$. *We consider the dynamic equation (1) with* $a(t) = -1/6t$, $b(t) = 1/8t$ *and* $h(t) = \exp(-1)^{\ln t/\ln 3}$.

The operators $\delta_{-}(s, t) = t/s$ and $\delta_{+}(s, t) = ts$ are backward and forward shift operators for $(s, t) \in D_{\pm}$ and here $\mathbb{T}^{*} = \mathbb{T}$, the initial point $t_0 = 1$.If we consider the functions $a(t)$, $b(t)$ and $h(t)$ in equation (1), we find $T = 3^2$. It is easy to see that $a(t)$ and $b(t)$ functions are ∇-periodic in shifts with period 9 and the nonnegative function $h(t)$ is periodic in shifts with period 9. Also $a(t)$ is negative and ν-regressive function and $b(t)$ is nonnegative and satisfies $\int_1^{\delta_+^T(1)} b(s)\nabla s = \int_1^9 b(s)\nabla s > 0$.
We can calculate

$$B = \int_1^9 \frac{1}{8t}\nabla s = \int_1^3 \frac{1}{8t}\nabla s + \int_3^9 \frac{1}{8t}\nabla s = \frac{1}{6},$$

$$\theta = \hat{e}_{a(t_0)}(t_0, \delta_+^T(t_0)) = \hat{e}_{a(1)}(1, 9) = \exp\left\{-\int_9^1 \frac{3}{2t}\log(1 + \frac{t}{9})\nabla s\right\} = \exp(\log\frac{4}{3} + \log 2) = \frac{8}{3},$$

$$a^{②}(1) = \frac{a^2(1)}{1 - \frac{2a(1)}{3}} = \frac{\frac{1}{36}}{1 + \frac{1}{9}} = \frac{1}{40}.$$

Thus we obtain $\alpha = \frac{27}{50}$, $\beta = \frac{8}{5}$ and $\gamma = \frac{27}{80}$.
(i) Consider the problem (1) with the function $f(u) = u^2$. Since

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to 0^+} \frac{u^2}{u} = 0,\ \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{u^2}{u} = \infty,$$

then using Theorem 3.1, the problem (1) has at least one positive 9-periodic solution in shifts $\delta_{\pm}$.
(ii) Consider the problem (1) with the function $f(u) = \sqrt{u}$. Since

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to 0^+} \frac{\sqrt{u}}{u} = \infty,\ \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{\sqrt{u}}{u} = 0,$$

then using Theorem 3.1, the problem (1) has at least one positive 9-periodic solution in shifts $\delta_{\pm}$.
(iii) Consider the problem (1) with the function

$$f(t, x) = \begin{cases} 72(x-1) + 7, & x \geq 1; \\ \dfrac{14x^2}{1+x}, & 0 \leq x < 1. \end{cases}$$

We can compute the numbers $\Lambda_1 = \frac{15}{4}$ and $\Lambda_2 = \frac{100}{9}$. If we choose $r' = \frac{1}{10}$ the function f satisfies $f(x) < \frac{14}{110} < \frac{15}{40} = r'\Lambda_1$ for $0 < x < \frac{1}{10}$. If we choose $s' = \frac{160}{27}$ the function f satisfies $f(x) > 72 > \frac{160}{27}\frac{100}{9} = s'\Lambda_2$ for $2 < x < \frac{160}{27}$.

All conditions of Theorem 3.2 are satisfied. Thus, the problem (1) has at least one positive 9-periodic solution in shifts $\delta_{\pm}$ such that $\frac{1}{10} < \|x\| < \frac{160}{27}$.

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