Approximation Properties of Generalized Jain Operators

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Abstract. In this paper, we investigate a variant of the Jain operators, which preserve the linear functions. We compute the rate of convergence of these operators with the help of K-functional. We also introduce modifications of the Jain operators based on the models in [4] and [10]. These modified operators yield better error estimates than the Jain operators.

1. Introduction

G.C. Jain [9] constructed a class of linear and positive operators given by the formula for $n \in \mathbb{N}$

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} w_\beta(k,nx) f\left(\frac{k}{n}\right), \ f \in C[0,\infty), \tag{1.1}$$

where

$$w_\beta(k,nx) = nx(nx+k\beta)^{k-1} e^{-(nx+k\beta)}/k!, k \in \mathbb{N}_0 \text{ and } \beta \in [0,1).$$

Recently, Farcaş [5] has proved a Voronovskaja type result for Jain's operators and Agratini has investigated deeper the operators $P_n^{[\beta]}$ in [1].

For $\beta = 0$, the operators $P_n^{[0]}$ reduce to Szasz-Mirakjan operators [17]

$$P_n^{[0]}(f;x) \equiv S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \ x \geq 0.$$

King [10] presented an example of operators of Bernstein type which preserve the test functions $e_0(x) = 1$ and $e_2(x) = x^2$ of Bohman Korovkin theorem. Motivated King's work [10], the Szasz-Mirakjan operators were modified as

$$D_n(f;x) = e^{-nu_n(x)} \sum_{k=0}^{\infty} \frac{(nu_n(x))^k}{k!} f\left(\frac{k}{n}\right),$$

where $\{u_n(x)\}$ is a sequence of real-valued, continuous functions defined on $[0, \infty)$ with $0 \leq u_n(x) < \infty$, one let

$$u_n(x) := \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, n \in \mathbb{N},$$

it has been shown in [4] that the operators preserve the test function e_2 and provide a better error estimation than the operators $S_n(f; x)$ for all $f \in C_B[0, \infty)$ and for all $x \in [0, \infty)$. $C_B[0, \infty)$ indicating the Banach space of all real-valued bounded and continuous functions defined on $[0, \infty)$. The space is endowed with the sub-norm $\|.\|$, where $\|f\| = \sup_{x \geq 0} |f(x)|$, $f \in C_B[0, \infty)$. Moreover, by letting

$$v_n(x) := x - \frac{1}{2n}; \ n \in \mathbb{N},$$

it has been shown in [16] that the operators defined by

$$V_n(f; x) := S_n(f; v_n(x))$$

do not preserve the test functions $e_1(x) = x$ and $e_2(x) = x^2$ but provide the best error estimation among all Szasz-Mirakjan operators for all $f \in C_B[0, \infty)$ and for each $x \in \left[\frac{1}{2}, \infty\right)$.

Favard [7] was the first to introduce the double Szasz-Mirakjan operators

$$S_n(f; x, y) = e^{-n(x+y)} \sum_{k,l=0}^{\infty} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{(nx)^k}{k!} \frac{(ny)^l}{l!}, f \in C[0, \infty) \times [0, \infty)$$

Recently, Dirik and Demirci [2] have investigated different variants of the general double Szasz-Mirakjan operators for $n \in \mathbb{N}$, $x, y \in [0, \infty)$:

$$
\begin{aligned}
D_n(f; x, y) \quad : \quad &= S_n(f; u_n(x), v_n(y)) \\
&= e^{-\left(u_n(x) + v_n(y)\right)} \\
&\quad \times \sum_{k,l=0}^{\infty} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!},
\end{aligned}
$$

where $f \in C[0, \infty) \times [0, \infty)$. In [2], they considered

$$
\begin{aligned}
u_n^1(x) \quad : \quad &= \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \\
v_n^1(y) \quad : \quad &= \frac{-1 + \sqrt{4n^2y^2 + 1}}{2n},
\end{aligned}
$$

which preserve the test function $e_{2,0}(x, y) + e_{0,2}(x, y) = x^2 + y^2$ and provide a better error estimation than the operators $S_n(f; x, y)$ for all $f \in C_B[0, \infty) \times [0, \infty)$ for all $x, y \in [0, \infty)$. On the other hand, in [3], they considered the case

$$
\begin{aligned}
u_n^2(x, \alpha) \quad : \quad &= \frac{-(n\alpha + 1) + \sqrt{4n^2(x^2 + \alpha x) + (n\alpha + 1)^2}}{2n}, \\
v_n^2(y, \beta) \quad : \quad &= \frac{-(n\beta + 1) + \sqrt{4n^2(y^2 + \beta y) + (n\beta + 1)^2}}{2n}, \\
n \quad \in \quad &\mathbb{N}, \alpha, \beta \in \mathbb{R}.
\end{aligned}
$$

The $D_n(f; x, y)$ do not preserve any test function $e_{0,0}(x, y) = 1$, $e_{1,0}(x, y) = x$, $e_{0,1}(x, y) = y$, $e_{2,0}(x, y) + e_{0,2}(x, y) = x^2 + y^2$, but provide a better error estimation than the operators $S_n(f; x, y)$ for all $C_B[0, \infty) \times [0, \infty)$

and $x, y \in [0, 1]$. In [15], the best error estimation among all the general double Szasz-Mirakjan operators obtained from the case:

$$u_n^3(x) := x - \frac{1}{2n}, \ v_n^3(y) := y - \frac{1}{2n}, \ n \in \mathbb{N},$$

for all $f \in C_B[0, \infty) \times [0, \infty)$ and $x, y \in [0, \infty)$. Later on, in [14], a variant of the double Szasz-Mirakjan-Kantorovich operators was introduced and the approximation properties of this modification were investigated.

Some approximation results on Stancu type operators have been investigated by the authors [11–13].

The aim of this to investigate the variant of Jain operators which preserve the linear functions. We compute the rate of convergence of these operators by K-functional. We also introduce modifications of the Jain operators based on the models in [10] and [4]. These type of operators modifications have better error estimation than the Jain operators.

2. Construction of the Operators

The following identities were obtained by Jain [9].

$$P_n^{[\beta]}(e_o; x) = 1,$$
$$P_n^{[\beta]}(e_1; x) = \frac{1}{1 - \beta} x,$$
$$P_n^{[\beta]}(e_2; x) = \frac{1}{(1 - \beta)^2} x^2 + \frac{x}{n(1 - \beta)^3}.$$

We transform the operators defined at (1.1) in order to preserve linear functions. Defining the function

$$u_n(x) := x(1 - \beta), \ x \geq 0,$$

for $n \in \mathbb{N}$, we consider the linear and positive operators

$$D_n^{[\beta]}(f; x) = \sum_{k=0}^{\infty} w_\beta(k, nu_n(x)) f\left(\frac{k}{n}\right), \ f \in C[0, \infty), \tag{2.1}$$

where

$$w_\beta(k, nu_n(x)) = nu_n(x)(nu_n(x) + k\beta)^{k-1} e^{-(nu_n(x) + k\beta)} / k!, k \in \mathbb{N}_0 \text{ and } \beta \in [0, 1).$$

Lemma 2.1. *The operators defined at (2.1) verify for each $x \geq 0$ the following identities*

$$D_n^{[\beta]}(e_o; x) = 1, \tag{2.2}$$
$$D_n^{[\beta]}(e_1; x) = x, \tag{2.3}$$
$$D_n^{[\beta]}(e_2; x) = x^2 + \frac{x}{n(1 - \beta)^2}. \tag{2.4}$$

We omit the proof, since the identities are obtained direct computation.

Now, we recall the concepts of the statistical convergence. The concept was introduced by Fast [6], The density of $S \subseteq \mathbb{N}$ denoted by $\delta(S)$ is defined as follows

$$\delta(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_S(j),$$

where χ_S stands for the characteristic function of the set S. A sequence $(x_n)_{n\geq1}$ of real numbers is said to be statiscally convergent to a real number l, if for every $\varepsilon > 0$,

$$\delta\left(\{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}\right) = 0,$$

the limit being denoted by $st - \lim_n x_n = l$. It is well known that any convergent sequence is statistically convergent, but the converse of this statement is not true. The application of this notation to study of positive linear operators was attempted in 2002 by Gadjiev and Orhan [8]. Their theorem says: if

$$st - \lim_n \left\|L_n e_j - e_j\right\| = 0, \ j = 0, 1, 2,$$

for the test functions, then

$$st - \lim_n \left\|L_n f - f\right\| = 0,$$

where $(L_n)_{n\geq1}$ is a sequence of linear positive operators defined on $C(J)$, and the norm is taken on a compact interval included in J. (see [8], Theorem 1).

In order to transform $\left(D_n^{[\beta]}\right)_{n\geq1}$ into an approximation process, we replace the constant β by a number $\beta_n \in [0, 1)$, $n \in \mathbb{N}$. If

$$st - \lim_n \beta_n = 0, \tag{2.5}$$

then, for any $f \in C[0, \infty)$ and $[a, b] \subset [0, \infty)$, we have

$$st - \lim_n \left\|D_n^{[\beta_n]} f - f\right\|_{[a,b]} = 0, \tag{2.6}$$

where, for $h \in C[0, \infty)$, $\|h\|_{[a,b]}$ represents the sup-norm of the function $h\big|_{[a,b]}$.

Agratini [1] indicated a general technique to construct sequences of operators of discrete type with the same property as in King's [10]. With the help of Agratini's technique, we define the function

$$
\begin{aligned}
\widetilde{u}_n(x) \quad : \quad &= \frac{1}{2}\left(\sqrt{\frac{1}{n^2(1-\beta)^4} + 4x^2} - \frac{1}{n(1-\beta)^2}\right), \\
&= \frac{1}{2n(1-\beta)^2}\left(\sqrt{1 + 4x^2 n^2(1-\beta)^4} - 1\right).
\end{aligned}
$$

After that, we obtain the linear and positive operators

$$\widetilde{D}_n^{[\beta]}(f; x) = \sum_{k=0}^{\infty} w_\beta\left(k, n\widetilde{u}_n(x)\right) f\left(\frac{k}{n}\right), \ f \in C[0, \infty). \tag{2.7}$$

By direct computation, one has

$$\widetilde{D}_n^{[\beta]}(e_o; x) \quad = \quad 1, \tag{2.8}$$
$$\widetilde{D}_n^{[\beta]}(e_1; x) \quad = \quad \widetilde{u}_n(x), \tag{2.9}$$
$$\widetilde{D}_n^{[\beta]}(e_2; x) \quad = \quad x^2. \tag{2.10}$$

For each $n \in \mathbb{N}$, the constant β will be replaced by a number β_n satisfing (2.5). So, for $\left(\widetilde{D}_n^{[\beta_n]}\right)_{n\geq1}$, we can give result as obtained in (2.6), since $st - \lim_n \left\|\widetilde{u}_n - e_1\right\|_{[a,b]} = 0$.

Finally, defining the similar function as used in [16],

$$\widehat{u_n}(x) = (1 - \beta)\left(x - \frac{1}{2n}\right),$$

we have the linear and positive operators, for $n \in \mathbb{N}$,

$$\widehat{D}_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} w_\beta(k, n\widetilde{u}_n(x)) f\left(\frac{k}{n}\right), \quad f \in C[0, \infty). \tag{2.11}$$

For the operators $\widehat{D}_n^{[\beta]}$, $x \geq \frac{1}{2}$, by direct computation, the following identities hold.

$$
\begin{aligned}
\widehat{D}_n^{[\beta]}(e_o;x) &= 1, \\
\widehat{D}_n^{[\beta]}(e_1;x) &= x - \frac{1}{2n}, \\
\widehat{D}_n^{[\beta]}(e_2;x) &= \left(x - \frac{1}{2n}\right)^2 + \frac{x - \frac{1}{2n}}{n(1-\beta)^2}, \\
&= x^2 + x\left(\frac{1}{n(1-\beta)^2} - \frac{1}{n}\right) + \frac{1}{4n^2} - \frac{1}{2n^2(1-\beta)^2}, \\
&= x^2 + x\frac{\beta(2-\beta)}{n(1-\beta)^2} + \frac{1}{4n^2} - \frac{1}{2n^2(1-\beta)^2}.
\end{aligned}
$$

For $\beta = \beta_n$, $n \in \mathbb{N}$, $a \geq \frac{1}{2}$ satisfing (2.5), since

$$st - \lim_n \left\|\widehat{D}_n^{[\beta_n]} e_j - e_j\right\|_{[a,b]} = 0, \quad j = 0, 1, 2,$$

we have

$$st - \lim_n \left\|\widehat{D}_n^{[\beta_n]} f - f\right\|_{[a,b]} = 0, \quad f \in C[0, \infty).$$

3. The Rate of Convergence

In this section, we compute the rate of convergence of the operators $D_n^{[\beta]}$, defined by (2.1). To achive this we use K-functional. Then, we will show that the operators $D_n^{[\beta]}$ have a better error estimation on $\mathbb{R}_+$ than the Jain operators $P_n^{[\beta]}$ given by (1.1). We also compute the rate of convergence the operators $\widetilde{D}_n^{[\beta]}$ defined in (2.7) and $\widehat{D}_n^{[\beta_n]}$ given by (2.11) by means of is the first modulus of smoothness of a function $f \in C_B[0, \infty)$.

The modulus of smoothness of a function $f \in C_B[0, \infty)$ is defined by

$$\omega(f;\delta) = \sup_{x \geq 0; \, 0 \leq h \leq \delta} \left|f(x+h) - f(x)\right|,$$

where $\delta > 0$.

Let $f \in C_B[0, \infty)$ and $g \in C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$ endowed with the sup-norm. The K-functional described by

$$K(f;\delta) = \inf_{g \in C_B^2[0,\infty)} \left(\left\|f - g\right\| + \delta\left\|g''\right\|\right), \quad \delta \geq 0.$$

Theorem 3.1. *Let $D_n^{[\beta]}$, $n \in \mathbb{N}$, be given by (2.1). For every $f \in C_B[0, \infty)$, one has*

$$\left| D_n^{[\beta]}(f; x) - f(x) \right| \le 2K\left(f, \frac{x}{4n(1-\beta)^2} \right), \; x \ge 0.$$

Proof. From the identities (2.2)-(2.4), we obtain

$$
\begin{aligned}
D_n^{[\beta]}\left(\varphi_x^1; x \right) &= 0, \; x \ge 0, \\
D_n^{[\beta]}\left(\varphi_x^2; x \right) &= \frac{(u_n(x) - x(1-\beta))^2}{(1-\beta)^2} + \frac{u_n(x)}{n(1-\beta)^3} \\
&= \frac{x}{n(1-\beta)^2}, \; x \ge 0.
\end{aligned}
$$

For a given $g \in C_B^2[0, \infty)$ and an arbitrarily fixed $x \in [0, \infty)$, by using the Taylor formula, it follows

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du, \; t \ge 0.$$

Since $D_n^{[\beta]}$ is linear and positive operator, we obtain

$$
\begin{aligned}
\left| D_n^{[\beta]}(g; x) - g(x) \right| &= \left| D_n^{[\beta]}(g - g(x)e_0; x) \right| = \left| g'(x)D_n^{[\beta]}\left(\varphi_x^1; x \right) + D_n^{[\beta]}\left(\int_x^t (t-u)g''(u)\,du; x \right) \right| \\
&\le D_n^{[\beta]}\left(\left| \int_x^t (t-u)g''(u)\,du \right|; x \right) \\
&\le \|g''\| D_n^{[\beta]}\left(\left| \int_x^t (t-u)\,du \right|; x \right) \\
&= \frac{\|g''\|}{2} D_n^{[\beta]}\left(\varphi_x^2; x \right) \\
&= \|g''\| \frac{x}{2n(1-\beta)^2}.
\end{aligned}
\tag{3.1}
$$

For $f \in C_B[0, \infty)$, we obtain

$$
\begin{aligned}
\left| D_n^{[\beta]}(f; x) - f(x) \right| &\le \left| D_n^{[\beta]}(f - g; x) \right| + \left| D_n^{[\beta]}(g; x) - g(x) \right| + \left| g(x) - f(x) \right| \\
&\le 2\|f - g\| + \frac{x}{2n(1-\beta)^2} \|g''\|.
\end{aligned}
$$

see (3.1). By using the K-functional described by

$$K(f; \delta) = \inf_{g \in C_B[0, \infty)} \left(\|f - g\| + \delta \|g''\| \right),$$

we can write for $x \ge 0$

$$\left| D_n^{[\beta]}(f; x) - f(x) \right| \le 2K\left(f, \frac{x}{4n(1-\beta)^2} \right).$$

$\square$

Remark 3.1. *The operators* $P_n^{[\beta]}$, $n \in \mathbb{N}$, *have*

$$\left| P_n^{[\beta]}(f;x) - f(x) \right| \le 4K\left(f; \max\left\{x, x^2\right\}\delta_n^2\right) + \omega\left(f; \frac{\beta x}{1-\beta}\right),$$

where

$$\delta_n = \frac{1}{2}\sqrt{\frac{\beta^2}{(1-\beta)^2} + \frac{1}{2n(1-\beta)^3}}. \tag{3.2}$$

(see [1], Theorem 2) . As Theorem 3.1 shows, for $\beta = \beta_n$ *satisfing (2.5), the rate of convergence of the operators* $D_n^{[\beta_n]}$ *by means of the K-functional is better than the error estimation given by (3.2) whenever* $x \ge 0$ *and* $n \in \mathbb{N}$.

Now, we have the following theorems.

Theorem 3.2. *For every* $f \in C_B[0, \infty)$, $\beta = \beta_n$ *satisfing (2.5),* $x \ge 0$ *and* $n \in \mathbb{N}$, *we obtain*

$$\left| \widetilde{D}_n^{[\beta_n]}(f,x) - f(x) \right| \le 2\omega\left(f, \tilde{\delta}(x)\right),$$

where

$$\tilde{\delta}(x) = \sqrt{2x^2 - \frac{x}{n(1-\beta)^2}\left(\sqrt{1 + 4x^2 n^2 (1-\beta)^4} - 1\right)}.$$

Proof. From the identities (2.8)-(2.10), we obtain

$$\begin{aligned} \widetilde{D}_n^{[\beta]}\left(\varphi_x^2; x\right) &= 2x\left(x - \widetilde{u}_n(x)\right). \\ &= 2x^2 - \frac{x}{n(1-\beta)^2}\left(\sqrt{1 + 4x^2 n^2 (1-\beta)^4} - 1\right). \end{aligned} \tag{3.3}$$

Let $f \in C_B[0, \infty)$, $\beta = \beta_n$ satisfing (2.5) and $x \ge 0$. Using linearity and monotonicity of $\widetilde{D}_n^{[\beta_n]}$, we get

$$\left| \widetilde{D}_n^{[\beta_n]}(f,x) - f(x) \right| \le \omega(f, \delta)\left\{1 + \frac{1}{\delta}\sqrt{\widetilde{D}_n^{[\beta_n]}\left(\varphi_x^2; x\right)}\right\}.$$

Applying (3.3) and choosing $\delta = \tilde{\delta}(x)$, the proof is complete. $\square$

Theorem 3.3. *For every* $f \in C_B[0, \infty)$, $\beta = \beta_n$ *satisfing (2.5),* $x \ge \frac{1}{2}$ *and* $n \in \mathbb{N}$, *we obtain*

$$\left| \widehat{D}_n^{[\beta_n]}(f,x) - f(x) \right| \le 2\omega\left(f, \hat{\delta}(x)\right),$$

where

$$\hat{\delta}(x) = \sqrt{\frac{x}{n(1-\beta)^2} + \frac{1}{4n^2} - \frac{1}{2n^2(1-\beta)^2}}.$$

As in the proof of Theorem 3.2, since

$$\widehat{D}_n^{[\beta]}\left(\varphi_x^2; x\right) = \frac{x}{n(1-\beta)^2} + \frac{1}{4n^2} - \frac{1}{2n^2(1-\beta)^2},$$

the modified Jain operators defined by (2.11) satisfy

$$\left| \widehat{D}_n^{[\beta_n]}(f,x) - f(x) \right| \le 2\omega\left(f, \hat{\delta}(x)\right),$$

for every $f \in C_B[0, \infty)$, $\beta = \beta_n$ satisfing (2.5), $x \ge \frac{1}{2}$ and $n \in \mathbb{N}$.

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