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Generalized Fuzzy Hypergraphs and Hypergroupoids

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Abstract. This article first generalizes the ordinary fuzzy hypergraphs to generalized fuzzy hypergraphs and it makes a connection between generalized fuzzy hypergraphs and fuzzy hyperstructures. We construct a partial fuzzy hypergroupoid associated with it, giving some properties of the associated fuzzy hyperstructure. Moreover, we construct higher order fuzzy hypergroupoids and study their properties. Finally, by considering a regular equivalence relation on a (g-f)^p-hypergroupoid, we define a quotient (g-f)^p-hypergroupoid and we investigate some relationships between diagonal product of hypergroupoids and p-product of (g-f)-hypergraphs.

1. Introduction and Preliminaries

The term *hypergraph* was coined by Berge [3, 5], following a remark by Jean-Marie Pal who had used the word *hyperedge* in a seminar. In 1976, Berge enriched the field once more with his lecture notes [6], also see [4]. After that, many researchers in the field of hyperstructure theory tried to make connections between hypergraphs and hyperstructures (see for instance [7, 8, 10–12]). In [14], Sen *et al.* introduced and studied fuzzy semihypergroups by using the concept of fuzzy hyperoperation. Fuzzy hypergraphs as a generalization of fuzzy graphs have been studied by many researchers (see for instance [1, 2, 9]). In [13], there is a very good presentation of fuzzy graph and fuzzy hypergraph theory. In this article, we extend the concept of fuzzy hypergraphs into *generalized fuzzy hypergraphs* and we will present a new connection between generalized fuzzy hypergraphs and fuzzy hyperstructures. First of all, we recall some notions and results of fuzzy set theory and fuzzy hyperstructures which will be used throughout this article.

A *fuzzy subset* of a non-empty set X is a mapping $\mu : X \longrightarrow I$, where I is the unit interval $[0,1] \subseteq \mathbb{R}$. A fuzzy subset μ of X is called empty (denoted by $\mu = \emptyset$) if $\mu(x) = 0$, for all $x \in X$. The set of all fuzzy subsets of X will be denoted by I^X. For each $\mu, \nu \in I^X$ we say $\nu \subseteq \mu$ if $\nu(x) \leq \mu(x)$, for all $x \in X$. Let $\mu_\alpha \in I^X$, in the index set Λ, we define $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$, $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$, $(\mu \backslash \nu)(x) = \begin{cases} \mu(x) & \text{if } \mu(x) > \nu(x), \\ 0 & \text{else} \end{cases}$ and $(\bigcup_{\alpha\in\Lambda} \mu_\alpha)(x) = \bigvee_{\alpha\in\Lambda} \{\mu_\alpha(x)\}$, where $\bigvee_{\alpha\in\Lambda} \{\mu_\alpha(x)\} = \sup_{\alpha\in\Lambda}\{\mu_\alpha(x)\}$. It is easy to verify that if μ, ν and η are fuzzy subsets of a given set X, then from $\nu \subseteq \mu$ it follows that $(\mu\backslash\nu) \cup \nu = \mu$. Moreover, from $\mu \cup \nu = \mu \cup \eta$ it follows that $\nu\backslash\eta \subseteq \mu$. If $\mu \in I^X$, then the *support* of μ, is defined by $\mathrm{supp}(\mu) = \{x \in X \mid \mu(x) > 0\}$. A *fuzzy h-relation* δ on X is a function from $X \times I^{*X}$ to I, where $I^{*X} = I^X - \{\emptyset\}$. Let $p \in I$. The p-domain

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of δ is defined by $\mathrm{Dom}_p(\delta) = \{x \in X \mid \delta(x,\mu) \geq p$ for some $\mu \in I^{*X}\}$ and p-codomain of δ is defined by $\mathrm{Cod}_p(\delta) = \{\mu \in I^{*X} \mid \delta(x,\mu) \geq p$ for some $x \in X\}$. Also, for any $x \in X$, we define $x_\delta^p = \{\mu \in I^{*X} \mid \delta(x,\mu) \geq p\}$.

By a *partial fuzzy hyperoperation* on a non-empty set X we mean a function $\circ$ from $X \times X$ to I^X. In other words, for any $x, y \in X$, $x \circ y$ is a fuzzy subset of X. Every mapping from $X \times X$ to I^{*X} is called a *fuzzy hyperoperation*. If $\mu, \nu \in I^{*X}$, then we define $\mu \circ \nu = \bigcup\{a \circ b \mid a \in \mathrm{supp}(\mu), b \in \mathrm{supp}(\nu)\}$, $x \circ \nu = \chi_{\{x\}} \circ \nu$ and $\mu \circ y = \mu \circ \chi_{\{y\}}$, where χ_X denotes the characteristic function of a given set X. If $\mu = \emptyset$ or $\nu = \emptyset$, then we define $\mu \circ \nu = \emptyset$. A *(partial) fuzzy hypergroupoid* is a pair $(X, \circ)$, where X is a non-empty set and $\circ$ is a (partial) fuzzy hyperoperation. A fuzzy hypergroupoid $(X, \circ)$ is called a *fuzzy semihypergroup* if the associative axiom is valid, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in X$ and it is called *reproductive* if $\mathrm{supp}(x \circ \chi_X) = \mathrm{supp}(\chi_X \circ x) = X$, for all $x \in X$. A *fuzzy hypergroup* is a reproductive fuzzy semihypergroup. The notion of H_v-structures was introduced by Vougiouklis [15]. A fuzzy hypergroupoid $(X, \circ)$ is called *fuzzy H_v-semigroup* if the weak associative axiom is valid, i.e., $x \circ (y \circ z) \bigcap (x \circ y) \circ z \neq \emptyset$, for all $x, y, z \in X$ and it is called *fuzzy H_v-group* if it is reproductive fuzzy H_v-semigroup.

Example 1.1. *Let $X = \{a, b\}$. Consider the following tables:*

$\circ$	a	b
a	$\frac{a}{0.8}, \frac{b}{0}$	$\frac{a}{0}, \frac{b}{0.7}$
b	$\frac{a}{0}, \frac{b}{0.7}$	$\frac{a}{0.8}, \frac{b}{0}$

$*$	a	b
a	$\frac{a}{0.8}, \frac{b}{0}$	$\frac{a}{0}, \frac{b}{0.7}$
b	$\frac{a}{0}, \frac{b}{0.6}$	$\frac{a}{0.9}, \frac{b}{0}$

It is easy to check that $(X, \circ)$ is a fuzzy hypergroup and $(X, *)$ is a fuzzy H_v-group which is not a fuzzy hypergroup.

2. Partial (g-f)^p-Hypergroupoids

In this section we define the concept of generalized fuzzy hypergraphs and then we associate a partial fuzzy hypergroupoid to each generalized fuzzy hypergraph. After defining the notion of separable fuzzy hypergroupoids, we will give a necessary and sufficient condition for a separable fuzzy hypergroupoid to have a separable fuzzy semihypergroup.

Definition 2.1. *A generalized fuzzy hypergraph or, in short, a (g-f)-hypergraph is an ordered pair $\Gamma = (X, \delta)$, where X is a non-empty set and δ is a fuzzy h-relation on X. The elements of X are called vertices and the fuzzy sets in $\mathcal{E} = \{\mu \in I^{*X} \mid \delta(x,\mu) > 0$ for some $x \in X\}$ are called fuzzy hyperedges.*

It is worth mentioning that, in this article, we deal only with (g-f)-hypergraphs $\Gamma = (X, \delta)$ in which X is a finite set. Let $p \in I$. A (g-f)-hypergraph $\Gamma = (X, \delta)$ is called *v_p-linked* if $x_\delta^p \neq \emptyset$, for all $x \in X$ and it is called *p-plenary* if $\bigcup\limits_{\mu \in \mathrm{Cod}_p(\delta)} \mathrm{supp}(\mu) = X$.

Example 2.2. *Assume that $\mu : \frac{1}{0.1}, \frac{2}{0.8}, \frac{3}{0.3}, \frac{4}{0}$ and $\nu : \frac{1}{0.2}, \frac{2}{0.3}, \frac{3}{0}, \frac{4}{0.7}$ are fuzzy subsets of $X = \{1, 2, 3, 4\}$. Let δ be a fuzzy h-relation on X which is denoted in the next figure.*

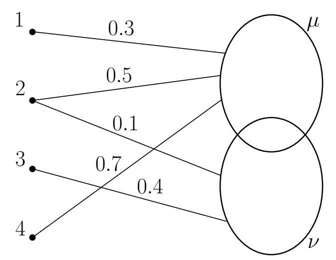

Figure 1: An example of a (g-f)-hypergraph.

For instance, in the above figure we have $\delta(1,\mu) = 0.3$ *and* $\delta(1,\nu) = 0$. *We can see easily that* $\Gamma = (X,\delta)$ *is not* $v_{0.4}$*-linked but it is* 0.4*-plenary.*

Let $\Gamma = (X,\delta)$ be a (g-f)-hypergraph. For $p \in I$, the partial fuzzy hypergroupoid $X_\Gamma^p = (X, \circ^p)$, where $\circ^p$ is defined by

$$x \circ^p y = \mathcal{N}^p(x) \bigcup \mathcal{N}^p(y), \quad \text{for all } x, y \in X,$$

is called the *partial (g-f)^p−hypergroupoid* associated with Γ, where $\mathcal{N}^p(x) = \bigcup\limits_{\delta(x,\mu)\geq p} \mu$. In the case that $\circ^p$ is a fuzzy hyperoperation, X_Γ^p is called the *(g-f)^p−hypergroupoid* associated with Γ.

Lemma 2.3. X_Γ^p *is a (g-f)^p-hypergroupoid if and only if* Γ *is* v_p*-linked.*

Proof. It is obvious. □

Definition 2.4. *A partial fuzzy hypergroupoid* $(X, \circ)$ *is called separable if the following property holds:*

$$x \circ y = x \circ x \bigcup y \circ y, \text{ for all } x, y \in X.$$

Remark 2.5. *Let* $(X, \circ)$ *be a separable fuzzy hypergroupoid and* $p \in (0, 1]$. *For each* $x \in X$ *we define* $\delta(x,\mu) = \begin{cases} p & \text{if } \mu = x \circ x \\ 0 & \text{otherwise.} \end{cases}$. *Then,* $(X, \circ)$ *is the (g-f)^p-hypergroupoid associated with the* v_p*-linked (g-f)-hypergraph* $\Gamma = (X,\delta)$. *Therefore, every separable hypergroupoid can be considered as a (g-f)^p-hypergroupoid, where* $p \in (0, 1]$.

The next lemma can be proved easily by using previously defined notions and thus we omit its proof.

Lemma 2.6. *Let* $(X, \circ^p)$ *be a partial (g-f)^p-hypergroupoid. Then, for all* $x, y \in X$ *and* $\mu \in I^{*X}$ *we have*

(1) $x \circ^p y = y \circ^p x$,

(2) $(x \circ^p x) \circ^p (x \circ^p x) = \bigcup\limits_{t \in supp(x \circ^p x)} t \circ^p t$,

(3) $(\mu \circ^p \mu) \circ^p (\mu \circ^p \mu) = \bigcup\limits_{t \in supp(\mu \circ^p \mu)} t \circ^p t$.

Lemma 2.7. *Every separable fuzzy hypergroupoid is a fuzzy* H_v*-semigroup.*

Proof. Let $(X, \circ)$ be a separable fuzzy hypergroupoid. By Remark 2.5, $(X, \circ)$ can be considered as a (g-f)^p-hypergroupoid, for some $p \in (0, 1]$. Thus, for each $x, y, z \in X$, by using Lemma 2.6, we have

$$(x \circ y) \circ z = (x \circ x \bigcup y \circ y) \circ z = (x \circ x) \circ z \bigcup (y \circ y) \circ z,$$

and

$$x \circ (y \circ z) = (y \circ z) \circ x = (y \circ y \bigcup z \circ z) \circ x = (y \circ y) \circ x \bigcup (z \circ z) \circ x.$$

Moreover,

$$(x \circ x) \circ z = \bigcup_{t \in \text{supp}(x \circ x)} t \circ z = \Big(\bigcup_{t \in \text{supp}(x \circ x)} t \circ t\Big) \bigcup z \circ z = \Big[(x \circ x) \circ (x \circ x)\Big] \bigcup z \circ z.$$

Therefore, we have $(x \circ y) \circ z = \Big[(x \circ x) \circ (x \circ x)\Big] \bigcup z \circ z \bigcup \Big[(y \circ y) \circ (y \circ y)\Big]$ and moreover

$$x \circ (y \circ z) = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup x \circ x \bigcup \Big[(z \circ z) \circ (z \circ z)\Big] .$$

As we see, we have $\emptyset \neq (y \circ y) \circ (y \circ y) \subseteq (x \circ y) \circ z \bigcap x \circ (y \circ z)$ which completes the proof. □

Notice that every partial (g-f)^p-hypergroupoid is separable and so we have the following corollaries.

Corollary 2.8. *Every (g-f)^p-hypergroupoid is a fuzzy H_v-semigroup.*

Corollary 2.9. *A partial (g-f)^p-hypergroupoid X_Γ^p is a fuzzy H_v-semigroup if and only if Γ is v_p-linked.*

Theorem 2.10. *Let $\Gamma = (X, \delta)$ be a v_p-linked (g-f)-hypergraph. Then, the (g-f)^p-hypergroupoid $X_\Gamma^p = (X, \circ^p)$ is a fuzzy H_v-group if and only if Γ is p-plenary.*

Proof. Let X_Γ^p be a fuzzy H_v-group. It suffices to show that $X \subseteq \bigcup\limits_{\mu \in \mathrm{Cod}_p(\delta)} \mathrm{supp}(\mu)$. Let $x \in X$ be an arbitrary element. Since X_Γ^p is a fuzzy H_v-group, we have $\mathrm{supp}(x \circ^p \chi_X) = X$ and therefore there is $y \in X$ such that $x \in \mathrm{supp}(x \circ^p y) = \mathrm{supp}(\mathcal{N}^p(x) \bigcup \mathcal{N}^p(y))$. Thus, there is $\mu \in \mathrm{Cod}_p(\delta)$ such that $x \in \mathrm{supp}(\mu) \subseteq \bigcup\limits_{\mu \in \mathrm{Cod}_p(\delta)} \mathrm{supp}(\mu)$.

Conversely, let Γ be p-plenary. Since Γ is a v_p-linked (g-f)-hypergraph, by Lemma 2.3, X_Γ^p is a (g-f)^p-hypergroupoid and so by Corollary 2.8, it is a fuzzy H_v-semigroup. It is sufficient to show that $\mathrm{supp}(x \circ^p \chi_X) = \mathrm{supp}(\chi_X \circ^p x) = X$, for each $x \in X$. Clearly, we have $\mathrm{supp}(x \circ^p \chi_X) \subseteq X$. We have to show that $X \subseteq \mathrm{supp}(x \circ^p \chi_X)$. Let $z \in X$ be an arbitrary element. Since Γ is p-plenary, there exists $\mu \in \mathrm{Cod}_p(\delta)$ such that $z \in \mathrm{supp}(\mu)$. Since $\mu \in \mathrm{Cod}_p(\delta)$, there is $y \in X$ such that $\delta(y, \mu) \geq p$ and so we have $z \in \mathrm{supp}(x \circ^p y) \subseteq \mathrm{supp}(x \circ^p \chi_X)$. This implies that $X \subseteq \mathrm{supp}(x \circ^p \chi_X)$ and therefore $\mathrm{supp}(x \circ^p \chi_X) = X$. In a similar way we have $\mathrm{supp}(\chi_X \circ^p x) = X$ and therefore X_Γ^p is a fuzzy H_v-group. □

Corollary 2.11. *X_Γ^p is a reproductive (g-f)^p-hypergroupoid if and only if Γ is v_p-linked and p-plenary.*

Theorem 2.12. *Suppose that $(X, \circ)$ is a separable fuzzy hypergroupoid. Then, $\circ$ is associative if and only if the following conditions hold:*

(1) $x \circ x \subseteq (x \circ x) \circ (x \circ x)$, for all $x \in X$,

(2) $((x \circ x) \circ (x \circ x)) \backslash (x \circ x) \subseteq (y \circ y) \circ (y \circ y)$, for all $x, y \in X$.

Proof. Let $\circ$ be associative and x, y be arbitrary elements of X. First, we prove condition (1). Suppose that $\mathrm{supp}(x \circ x) = \{x_1, \ldots, x_n\}$. Since $x \circ x_i = x \circ x \bigcup x_i \circ x_i$, for each $1 \leq i \leq n$, we have

$$
\begin{aligned}
(x \circ x)(x_i) \leq (x \circ x_i)(x_i) \leq (x \circ (x \circ x_i))(x_i) &= ((x \circ x) \circ x_i)(x_i) \\
&= \max\{(x_1 \circ x_i)(x_i), \ldots, (x_n \circ x_i)(x_i)\} \\
&= \max\{(x_1 \circ x_1)(x_i), \ldots, (x_n \circ x_n)(x_i)\} \\
&= ((x \circ x) \circ (x \circ x))(x_i).
\end{aligned}
$$

Thus (1) holds. Now, to prove condition (2) we have

$$
(y \circ y) \circ x = \bigcup_{t \in \mathrm{supp}(y \circ y)} t \circ x = \Big(\bigcup_{t \in \mathrm{supp}(y \circ y)} t \circ t \Big) \bigcup x \circ x = \Big[(y \circ y) \circ (y \circ y) \Big] \bigcup x \circ x,
$$

$$
\begin{aligned}
y \circ (y \circ x) = \bigcup_{t \in \mathrm{supp}(y \circ x)} y \circ t &= \bigcup_{t \in \mathrm{supp}(y \circ x)} (y \circ y \bigcup t \circ t) \\
&= y \circ y \bigcup \Big(\bigcup_{t \in \mathrm{supp}(y \circ y)} t \circ t \Big) \bigcup \Big(\bigcup_{t \in \mathrm{supp}(x \circ x)} t \circ t \Big) \\
&= \Big[(y \circ y) \circ (y \circ y) \Big] \bigcup \Big[(x \circ x) \circ (x \circ x) \Big].
\end{aligned}
$$

Now, associativity of $\circ$ implies that $\Big[(y \circ y) \circ (y \circ y)\Big] \bigcup x \circ x = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup \Big[(x \circ x) \circ (x \circ x)\Big]$. Consequently, (2) holds.

Conversely, suppose that x, y, z are arbitrary elements of X and that conditions (1) and (2) hold. As in the proof of Lemma 2.7, we have

$$
(x \circ y) \circ z = \Big[(x \circ x) \circ (x \circ x) \Big] \bigcup z \circ z \bigcup \Big[(y \circ y) \circ (y \circ y) \Big],
$$

and

$$x \circ (y \circ z) = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup x \circ x \bigcup \Big[(z \circ z) \circ (z \circ z)\Big].$$

By setting $\mu = \Big[(x \circ x) \circ (x \circ x)\Big] \bigcup z \circ z$ and $\nu = \Big[(z \circ z) \circ (z \circ z)\Big] \bigcup x \circ x$ we have $(x \circ y) \circ z = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup \mu$ and $x \circ (y \circ z) = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup \nu$. By using conditions (1) and (2) we have

$$\begin{aligned} \mu = \Big(\Big[(x \circ x) \circ (x \circ x)\Big] \backslash x \circ x\Big) \bigcup x \circ x \bigcup z \circ z \quad &\subseteq \quad \Big[(z \circ z) \circ (z \circ z)\Big] \bigcup z \circ z \bigcup x \circ x \\ &= \quad \Big[(z \circ z) \circ (z \circ z)\Big] \bigcup x \circ x = \nu. \end{aligned}$$

In a similar way the inverse inclusion is proved and then $\circ$ is associative. □

Theorem 2.13. *Suppose that $(X, \circ)$ is a separable fuzzy semihypergroup. Then, $\circ$ is associative if and only if the following conditions hold:*

(1) $\mu \circ \mu \subseteq (\mu \circ \mu) \circ (\mu \circ \mu)$, for all $\mu \in I^{*X}$,

(2) $((\mu \circ \mu) \circ (\mu \circ \mu)) \backslash (\mu \circ \mu) \subseteq (\nu \circ \nu) \circ (\nu \circ \nu)$, for all $\mu, \nu \in I^{*X}$.

Proof. Let $\circ$ be associative and μ, ν be arbitrary non-empty fuzzy subsets of X. Then, by using Theorem 2.12 we have

$$\begin{aligned} \mu \circ \mu = \bigcup_{x \in \mathrm{supp}(\mu)} x \circ x \subseteq \bigcup_{x \in \mathrm{supp}(\mu)} ((x \circ x) \circ (x \circ x)) \quad &= \quad \bigcup_{x \in \mathrm{supp}(\mu)} \Big(\bigcup_{t \in \mathrm{supp}(x \circ x)} t \circ t \Big) \\ &= \quad \bigcup_{t \in \mathrm{supp}(\mu \circ \mu)} t \circ t \\ &= \quad (\mu \circ \mu) \circ (\mu \circ \mu). \end{aligned}$$

Hence (1) is true. To prove condition (2), let $y \in \mathrm{supp}(\nu)$ be an arbitrary element. Then, we have

$$((\mu \circ \mu) \circ (\mu \circ \mu)) \backslash (\mu \circ \mu) \subseteq \bigcup_{x \in \mathrm{supp}(\mu)} \Big(((x \circ x) \circ (x \circ x)) \backslash (x \circ x) \Big) \subseteq (y \circ y) \circ (y \circ y).$$

On the other hand we have $(y \circ y) \circ (y \circ y) \subseteq (\nu \circ \nu) \circ (\nu \circ \nu)$. Hence, condition (2) holds.

Conversely, suppose that conditions (1) and (2) hold. Let x, y be arbitrary elements of X. By setting $\mu = \chi_{\{x\}}$ and $\nu = \chi_{\{y\}}$, conditions (1) and (2) of Theorem 2.12 hold and therefore $\circ$ is associative. □

Corollary 2.14. *If a reproductive (g-f)^p-hypergroupoid $X_\Gamma^p = (X, \circ^p)$ satisfies anyone of the following conditions:*

$$(x \circ^p x) \circ^p (x \circ^p x) = x \circ^p x, \quad \textit{for all } x \in X,$$

$$(x \circ^p x) \circ^p (x \circ^p x) = \bigcup_{t \in X} t \circ^p t, \quad \textit{for all } x \in X,$$

then it is a fuzzy hypergroup.

Example 2.15. *The (g-f)$^{0.3}$-hypergroupoid associated with the (g-f)-hypergraph of Figure 1 has the following table:*

$\circ^{0.3}$	1	2	3	4
1	μ	$\mu \cup \nu$	$\mu \cup \nu$	μ
2	$\mu \cup \nu$	μ	$\mu \cup \nu$	μ
3	$\mu \cup \nu$	$\mu \cup \nu$	ν	$\mu \cup \nu$
4	μ	μ	$\mu \cup \nu$	μ

It can be seen that $3 \circ^{0.3} 3 = \nu$ and $(3 \circ^{0.3} 3) \circ^{0.3} (3 \circ^{0.3} 3) = \nu \circ^{0.3} \nu = \mu$. Therefore, $3 \circ^{0.3} 3 \nsubseteq (3 \circ^{0.3} 3) \circ^{0.3} (3 \circ^{0.3} 3)$. So, by Theorem 2.12, $(X, \circ^{0.3})$ is not a fuzzy hypergroup. In that (g-f)-hypergraph we have $\mathcal{N}^{0.1}(1) = \mathcal{N}^{0.1}(4) = \mu$, $\mathcal{N}^{0.1}(2) = \mu \cup \nu$ and $\mathcal{N}^{0.1}(3) = \nu$. It is easy to verify that $(x \circ^{0.1} x) \circ^{0.1} (x \circ^{0.1} x) = \bigcup_{t \in X} t \circ^{0.1} t$, for all $x \in X$. On the other hand for every $x \in X$ we have $supp(x \circ^{0.1} X) = supp(X \circ^{0.1} x) = X$. So, by Corollary 2.14, $(X, \circ^{0.1})$ is a fuzzy hypergroup.

3. Higher-Order Fuzzy Hypergroupoids

Let $(X, \circ)$ be a separable fuzzy hypergroupoid. We construct a sequence of fuzzy hypergroupoids $X_0 = (X, \circ_0), X_1 = (X, \circ_1), X_2 = (X, \circ_2), \ldots$ recursively as follows: for all $x, y \in X$ we set $x \circ_0 y = x \circ y, x \circ_{k+1} x = (x \circ_k x) \circ_k (x \circ_k x)$ and $x \circ_{k+1} y = x \circ_{k+1} x \bigcup y \circ_{k+1} y$, where $k \geq 0$. Set $\mathcal{N}_k(x) = x \circ_k x$. We define $\mathcal{N}_k(\mu) = \bigcup\limits_{a \in \mathrm{supp}(\mu)} \mathcal{N}_k(a)$, where μ is a fuzzy subset of X.

Lemma 3.1. *Let μ be a fuzzy subset of X. Then, $supp(\mathcal{N}_k(\mu)) = \bigcup\limits_{a \in supp(\mu)} supp(\mathcal{N}_k(a))$.*

Proof. It is straightforward. $\square$

The following properties are immediate for $k \geq 0$:

(1) $\mathcal{N}_k(\mu) = \mu \circ_k \mu$, for all $\mu \in I^{*X}$,

(2) $\mathcal{N}_{k+1}(x) = \mathcal{N}_k(\mathcal{N}_k(x))$, for all $x \in X$,

(3) $\mathcal{N}_k(\mathcal{N}_{k+1}(x)) = \mathcal{N}_{k+1}(\mathcal{N}_k(x))$, for all $x \in X$,

(4) $\mathcal{N}_{k+1}(\mu) = \mathcal{N}_k(\mathcal{N}_k(\mu))$, for all $\mu \in I^{*X}$,

(5) $\mu \subseteq \nu$ implies that $\mathcal{N}_k(\mu) \subseteq \mathcal{N}_k(\nu)$, for all $\mu, \nu \in I^{*X}$,

(6) $\mathcal{N}_k(x) = \mathcal{N}_{k+1}(x)$ implies that $\mathcal{N}_k(x) = \mathcal{N}_r(x)$, for all $r \geq k$.

By Theorem 2.12, X_k is a semihypergroup if and only if the following conditions hold:

(α) $\mathcal{N}_k(x) \subseteq \mathcal{N}_{k+1}(x)$, for all $x \in X$,

(β) $\mathcal{N}_{k+1}(x) \backslash \mathcal{N}_k(x) \subseteq \mathcal{N}_{k+1}(y)$, for all $x, y \in X$.

Lemma 3.2. *The fuzzy hyperoperation $\circ_k$ defined as above has the following properties:*

(1) $\mu \circ_{k+1} \mu = (\mu \circ_k \mu) \circ_k (\mu \circ_k \mu)$, *for all* $\mu \in I^{*X}$,

(2) $x \circ_{k+2} x = \Big((x \circ_{k+1} x) \circ_k (x \circ_{k+1} x)\Big) \circ_k \Big((x \circ_{k+1} x) \circ_k (x \circ_{k+1} x)\Big)$, *for all* $x \in X$.

Proof. (1) Let μ be a non-empty fuzzy subset of X. Then,

$$\mu \circ_{k+1} \mu = \mathcal{N}_{k+1}(\mu) = \mathcal{N}_k(\mathcal{N}_k(\mu)) = \mathcal{N}_k(\mu) \circ_k \mathcal{N}_k(\mu) = (\mu \circ_k \mu) \circ_k (\mu \circ_k \mu).$$

(2) The result follows from part (1) and the definition of $\circ_{k+2}$. $\square$

Theorem 3.3. *Let $(X, \circ)$ be a separable fuzzy hypergroupoid. Then, the following assertions hold:*

(1) If $X_k = (X, \circ_k)$ satisfies condition (α) for some $k \geq 0$, then $\mathcal{N}_r(x) \subseteq \mathcal{N}_{r+1}(x)$, for all $x \in X$ and $r \geq k$.

(2) If $X_k = (X, \circ_k)$ satisfies condition (β) for some $k \geq 0$, then $\mathcal{N}_{r+1}(x) \subseteq \mathcal{N}_r(x)$, for all $x \in X$ and $r > k$.

Proof. (1) Let $x \in X$ be an arbitrary element. We prove the result by induction on r. If $r = k$, then there is nothing to prove. Assume that $\mathcal{N}_{r-1}(x) \subseteq \mathcal{N}_r(x)$ for $r > k$, the induction hypothesis. Then, we have

$$\mathcal{N}_r(x) = \mathcal{N}_{r-1}(\mathcal{N}_{r-1}(x)) \subseteq \mathcal{N}_{r-1}(\mathcal{N}_r(x)) = \mathcal{N}_r(\mathcal{N}_{r-1}(x)) \subseteq \mathcal{N}_r(\mathcal{N}_r(x)) = \mathcal{N}_{r+1}(x).$$

(2) Let $x \in X$ be an arbitrary element. First, we show that $\mathcal{N}_k(\mathcal{N}_{k+1}(x)) \subseteq \mathcal{N}_{k+1}(x)$. Assume to the contrary that $\mathcal{N}_k(\mathcal{N}_{k+1}(x)) \nsubseteq \mathcal{N}_{k+1}(x)$. Then, there exists $t \in X$ such that

$$\mathcal{N}_k(\mathcal{N}_{k+1}(x))(t) > \mathcal{N}_{k+1}(x)(t). \qquad (*)$$

By using Lemma 3.1 we have

$$\begin{aligned}\mathcal{N}_k(\mathcal{N}_{k+1}(x)))(t) &= \bigvee\{\mathcal{N}_k(y)(t) \mid y \in \mathrm{supp}(\mathcal{N}_{k+1}(x))\}\\ &= \bigvee\{\mathcal{N}_k(y)(t) \mid y \in \bigcup_{a\in \mathrm{supp}(\mathcal{N}_k(x))} \mathrm{supp}(\mathcal{N}_k(a))\}\\ &= \bigvee_{a\in \mathrm{supp}(\mathcal{N}_k(x))} \{\mathcal{N}_k(y)(t) \mid y \in \mathrm{supp}(\mathcal{N}_k(a))\}.\end{aligned}$$

So, there exist $a \in \mathrm{supp}(\mathcal{N}_k(x)$ and $y \in \mathrm{supp}(\mathcal{N}_k(a))$ such that $\mathcal{N}_k(\mathcal{N}_{k+1}(x))(t) = \mathcal{N}_k(y)(t)$. Thus, by inequality $(*)$ we have $\mathcal{N}_{k+1}(x)(t) < \mathcal{N}_k(y)(t)$. Obviously, for each $y' \in \mathrm{supp}(\mathcal{N}_k(a))$ we have $\mathcal{N}_(y)(t) \geq \mathcal{N}_k(y')(t)$ and so we have

$$\mathcal{N}_{k+1}(a)(t) = \mathcal{N}_k(\mathcal{N}_k(a))(t) = \bigvee\{\mathcal{N}_k(y')(t) \mid y' \in \mathrm{supp}(\mathcal{N}_k(a))\} = \mathcal{N}_k(y)(t).$$

This implies that $\mathcal{N}_k(a)(t) \leq \mathcal{N}_k(\mathcal{N}_k(x))(t) = \mathcal{N}_{k+1}(x)(t) < \mathcal{N}_k(y)(t) = \mathcal{N}_{k+1}(a)(t)$. Therefore, we have $\big(\mathcal{N}_{k+1}(a)\backslash\mathcal{N}_k(a)\big)(t) = \mathcal{N}_{k+1}(a)(t) = \mathcal{N}_k(y)(t)$. Since X_k satisfies condition (β) we have $\mathcal{N}_{k+1}(a)\backslash\mathcal{N}_k(a) \subseteq \mathcal{N}_{k+1}(x)$ which implies that $\mathcal{N}_k(y)(t) \leq \mathcal{N}_{k+1}(x)(t)$ contradicting to $\mathcal{N}_{k+1}(x)(t) < \mathcal{N}_k(y)(t)$. Now, we prove the result by induction on r. We have $\mathcal{N}_{k+2}(x) = \mathcal{N}_k(\mathcal{N}_k(\mathcal{N}_{k+1}(x))) \subseteq \mathcal{N}_k(\mathcal{N}_{k+1}(x)) \subseteq \mathcal{N}_{k+1}(x)$. So, we are done with the initial step. Assume that $\mathcal{N}_{r+1}(x) \subseteq \mathcal{N}_r(x)$ for $r > k$, the induction hypothesis. We have

$$\mathcal{N}_{r+2}(x) = \mathcal{N}_{r+1}(\mathcal{N}_{r+1}(x)) \subseteq \mathcal{N}_{r+1}(\mathcal{N}_r(x)) = \mathcal{N}_r(\mathcal{N}_{r+1}(x)) \subseteq \mathcal{N}_r(\mathcal{N}_r(x)) = \mathcal{N}_{r+1}(x).$$

□

Corollary 3.4. *If $(X, \circ_k)$ is a separable fuzzy semihypergroup, then $\mathcal{N}_r(x) = \mathcal{N}_{r+1}(x)$, for all $x \in X$ and $r > k$.*

Corollary 3.5. *If $(X, \circ_k)$ is a separable fuzzy semihypergroup, then $\mathcal{N}_r(\mu) = \mathcal{N}_{r+1}(\mu)$, for all $\mu \in I^{*X}$ and $r > k$.*

Next example shows that the converse of Corollary 3.4 is not true.

Example 3.6. *Let $(X = \{1, 2, 3\}, \circ_0)$ be a separable fuzzy hypergroupoid with the following table:*

$\circ_0$	1	2	3
1	$\frac{1}{0}, \frac{2}{0.2}, \frac{3}{0}$	$\frac{1}{0.1}, \frac{2}{0.2}, \frac{3}{0.3}$	$\frac{1}{0}, \frac{2}{0.2}, \frac{3}{0}$
2	$\frac{1}{0.1}, \frac{2}{0.2}, \frac{3}{0.3}$	$\frac{1}{0.1}, \frac{2}{0}, \frac{3}{0.3}$	$\frac{1}{0.1}, \frac{2}{0.2}, \frac{3}{0.3}$
3	$\frac{1}{0}, \frac{2}{0.2}, \frac{3}{0}$	$\frac{1}{0.1}, \frac{2}{0.2}, \frac{3}{0.3}$	$\frac{1}{0}, \frac{2}{0.2}, \frac{3}{0}$

We can see that $\mathcal{N}_0(1)(2) = 0.2$ and $\mathcal{N}_1(1)(2) = \mathcal{N}_0(\mathcal{N}_0(1))(2) = \mathcal{N}_0(2)(2) = 0$. This implies that $\mathcal{N}_0(1) \nsubseteq \mathcal{N}_1(1)$ and so by Theorem 2.12, $(X, \circ_0)$ is not a separable fuzzy semihypergroup. It is not difficult to see that $\mathcal{N}_k(x) = \mathcal{N}_{k+1}(x)$, for all $x \in X$ and $k > 0$. This means that the converse of Corollary 3.4 is not true.

Next proposition is a direct consequence of Theorem 2.12.

Proposition 3.7. *If there exists a natural number k such that $\mathcal{N}_k(x) = \mathcal{N}_{k+1}(x)$, for all $x \in X$, then*

(1) $X_k = (X, \circ_k)$ is a separable fuzzy semihypergroup,

(2) $X_r = X_k$, for all $r \geq k$.

Let $p \in (0, 1]$ and assume that Γ is a v_p-linked (g-f)-hypergraph and X_Γ^p is the (g-f)^p-hypergroupoid associated with Γ. Set $X_0 = X_\Gamma^p$. As in the beginning of this section, we can construct a sequence of separable fuzzy hypergroupoids $X_0 = (X, \circ_0), X_1 = (X, \circ_1), X_2 = (X, \circ_2), \ldots$. For $k > 0$, we define a fuzzy h-relation δ_k on X as follows:

$$\delta_k(x, \mu) = \begin{cases} p & \text{if } \mu = \mathcal{N}_k(x), \\ 0 & \text{otherwise,} \end{cases}$$

and therefore we will have a sequence $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ of (g-f)-hypergraphs where $\Gamma_0 = \Gamma$ and $\Gamma_k = (X, \delta_k)$, for $k > 0$. It is easy to verify that X_k is the (g-f)^p-hypergroupoid associated with Γ_k. Now, by Corollary 3.4 and Proposition 3.7 we conclude that if X_k is an associative (g-f)^p-hypergroupoid, then $\Gamma_r = \Gamma_{k+1}$ and $X_r = X_k$, for all $r > k$. For a given (g-f)-hypergraph Γ we define $n(\Gamma) = \min\Big\{k \mid \mathcal{N}_k(x) = \mathcal{N}_{k+1}(x) \text{ for all } x \in X\Big\}$ and $s(\Gamma) = \min\Big\{k \mid X_k \text{ is a fuzzy semihypergroup}\Big\}$. Obviously, $s(\Gamma) \leq n(\Gamma)$. Consider the (g-f)-hypergraph Γ of Figure 1 . In the Example 2.15 we showed that $X_\Gamma^{0.1}$ is a fuzzy hypergroup and so we have $s(\Gamma) = 0$ whereas $n(\Gamma) = 1$. This means that the inequality $s(\Gamma) \leq n(\Gamma)$ may be hold strictly.

4. Quotient (g-f)^p-Hypergroupoids

In this section, by considering a regular equivalence relation on a (g-f)^p-hypergroupoid, we define a quotient (g-f)^p-hypergroupoid. Next, we investigate some relationships between diagonal product of hypergroupoids and p-product of (g-f)-hypergraphs. In this regards we recall some definitions and results which we need for development of our paper.

Let $(X, *)$ be a fuzzy hypergroupoid and ρ be an equivalence relation on X. If A and B are non-empty subsets of X, then $A\overline{\rho}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\rho b$ and for all $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$. We say that ρ is *regular* if for all $a \in X$, from $x\rho\, y$ it follows that $\mathrm{supp}(a * x)\overline{\rho}\,\mathrm{supp}(a * y)$ and $\mathrm{supp}(x * a)\overline{\rho}\,\mathrm{supp}(y * a)$. For an equivalence relation ρ on X, we may use $\rho(x)$ to denote the equivalence class of $x \in X$. Moreover, generally, if A is a non-empty subset of X, then $\rho(A) = \{\rho(a) \mid a \in A\}$. We let X/ρ denotes the family $\{\rho(x) \mid x \in X\}$ of classes of ρ. It is easy to verify that for a regular relation ρ on a fuzzy hypergroupoid $(X, *)$, the following fuzzy hyperoperation on X/ρ is well defined:

$$\rho(x) \odot \rho(y) = \chi_{\{\rho(z) \mid z \in \mathrm{supp}(x*y)\}}.$$

Let $p \in (0, 1]$ and $\Gamma = (X, \delta)$ be a v_p-linked (g-f)-hypergraph and $(X, \circ^p)$ be the (g-f)^p-hypergroupoid associated with Γ. We define the relation ρ_Γ on X as follows:

$$x\, \rho_\Gamma\, y \text{ if and only if } x_\delta^p = y_\delta^p.$$

Lemma 4.1. *The relation ρ_Γ is a regular equivalence relation.*

Proof. Obviously, ρ_Γ is an equivalence relation. Let $z \in X$ be an arbitrary element and $x\rho_\Gamma\, y$. First, we show that $\mathrm{supp}(x \circ^p z) = \mathrm{supp}(y \circ^p z)$ which will imply that $\mathrm{supp}(x \circ^p z)\overline{\rho}_\Gamma\,\mathrm{supp}(y \circ^p z)$. Let $r \in \mathrm{supp}(x \circ^p z) = \mathrm{supp}(\mathcal{N}^p(x) \bigcup \mathcal{N}^p(z))$ be an arbitrary element. We know that

$$\mathrm{supp}(\mathcal{N}^p(x) \bigcup \mathcal{N}^p(z)) = \mathrm{supp}(\mathcal{N}^p(x)) \bigcup \mathrm{supp}(\mathcal{N}^p(z)) \ .$$

In the case that $r \in \mathrm{supp}(\mathcal{N}^p(z))$, there is nothing to prove. If $r \in \mathrm{supp}(\mathcal{N}^p(x))$, then there is a fuzzy hyperedge $\mu \in x_\delta^p$ such that $r \in \mathrm{supp}(\mu)$. By assumption, we have $x_\delta^p = y_\delta^p$ and therefore we have $\delta(y, \mu) \geq p$. This implies that $r \in \mathrm{supp}(\mathcal{N}^p(y))$. Hence, we have $\mathrm{supp}(x \circ^p z) \subseteq \mathrm{supp}(y \circ^p z)$. The reverse inclusion can be shown similarly. In a similar way we can show that $\mathrm{supp}(z \circ^p x)\overline{\rho}_\Gamma\,\mathrm{supp}(z \circ^p y)$. $\square$

Let μ and ν be non-empty fuzzy subsets of X_1 and X_2, respectively. We define the fuzzy subset $\mu \times \nu$ on $X_1 \times X_2$ as follows:

$$\mu \times \nu(x_1, x_2) = \min\{\mu(x_1), \nu(x_2)\}.$$

It is easy to see that $\operatorname{supp}(\mu \times \nu) = \operatorname{supp}(\mu) \times \operatorname{supp}(\nu)$.

Definition 4.2. *Let $\Gamma_1 = (X_1, \delta_1)$ and $\Gamma_2 = (X_2, \delta_2)$ be two (g-f)-hypergraphs and $p \in (0, 1]$. Set*

$$T(p) = \{\mu \times \nu \mid \delta_1(x_1, \mu) \geq p \text{ and } \delta_2(x_2, \nu) \geq p, \text{ for some } x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$

Then, p-product of Γ_1 and Γ_2 is the (g-f)-hypergraph $\Gamma_1 \times \Gamma_2 = (X_1 \times X_2, \delta_1 \times \delta_2)$ where $\delta_1 \times \delta_2$ is a fuzzy h-relation on $X_1 \times X_2$ which is defined as follows:

$$\delta_1 \times \delta_2\Big((x_1, x_2), \eta\Big) = \begin{cases} \min\{\delta_1(x_1, \mu), \delta_2(x_2, \nu)\} & \text{if } \eta = \mu \times \nu, \text{for some } \mu \times \nu \in T(p), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3. *Let $p \in (0, 1]$. Let $\Gamma_1 = (X_1, \delta_1)$ and $\Gamma_2 = (X_2, \delta_2)$ be two v_p-linked (g-f)-hypergraphs. If $\Gamma_1 \times \Gamma_2$ is the p-product of Γ_1 and Γ_2, then*

(1) $\Gamma_1 \times \Gamma_2$ is a v_p-linked (g-f)-hypergraph,

(2) for $(x, y), (u, v) \in X_1 \times X_2$ we have

$$(x, y)\rho_{\Gamma_1 \times \Gamma_2} (u, v) \Longleftrightarrow x\rho_{\Gamma_1} u \text{ and } y\rho_{\Gamma_2} v.$$

Proof. It is straightforward. □

Definition 4.4. *Let $(X_1, *)$ and $(X_2, \circ)$ be two fuzzy hypergroupoids. We define a fuzzy hyperoperation $\times_d$ on the Cartesian product $X_1 \times X_2$ as follows:*

$$(x_1, y_1) \times_d (x_2, y_2) = \Delta\Big((x_1, y_1)\Big) \bigcup \Delta\Big((x_2, y_2)\Big),$$

where $\Delta\Big((x, y)) = (x * x) \times (y \circ y)$. The fuzzy hypergroupoid $(X_1 \times X_2, \times_d)$ is called diagonal product of $(X_1, *)$ and $(X_2, \circ)$.

Theorem 4.5. *Let $p \in (0, 1]$. Let $(X_1, *)$ and $(X_2, \circ)$ be the (g-f)^p-hypergroupoids associated with the v_p-linked (g-f)-hypergraphs $\Gamma_1 = (X_1, \delta_1)$ and $\Gamma_2 = (X_2, \delta_2)$, respectively. Then, diagonal product of $(X_1, *)$ and $(X_2, \circ)$ is the (g-f)^p-hypergroupoid associated with the p-product of Γ_1 and Γ_2.*

Proof. Let $(X_1 \times X_2, \times_d)$ be the diagonal product of $(X_1, *)$ and $(X_2, \circ)$. It suffices to show that

$$(x, y) \times_d (x, y) = \bigcup \Big\{\mu \times \nu \mid \delta_1 \times \delta_2\Big((x, y), \mu \times \nu\Big) \geq p\Big\},$$

where (x, y) is an arbitrary element of $X_1 \times X_2$. This can be seen by the following argument. Let $(x_1, x_2) \in X_1 \times X_2$ be an arbitrary element. Then,

$$\begin{aligned}
\Big((x, y) \times_d (x, y)\Big)(x_1, x_2) &= \Delta\Big((x, y)\Big)(x_1, x_2) \\
&= \Big((x * x) \times (y \circ y)\Big)(x_1, x_2) \\
&= \min\{(x * x)(x_1), (y \circ y)(x_2)\} \\
&= \min\Big\{\max\{\mu(x_1) \mid \delta_1(x, \mu) \geq p\}, \max\{\nu(x_2) \mid \delta_2(y, \nu) \geq p\}\Big\} \\
&= \max\Big\{\min\{\mu(x_1), \nu(x_2)\} \mid \delta_1(x, \mu) \geq p, \delta_2(y, \nu) \geq p\Big\} \\
&= \max\Big\{\mu \times \nu(x_1, x_2) \mid \delta_1(x, \mu) \geq p, \delta_2(y, \nu) \geq p\Big\} \\
&= \max\Big\{\mu \times \nu(x_1, x_2) \mid \delta_1 \times \delta_2\Big((x, y), \mu \times \nu\Big) \geq p\Big\}.
\end{aligned}$$

□

Definition 4.6. *Let $(X_1, *)$ and $(X_2, \circ)$ be two fuzzy hypergroupoids. A map $\varphi : X_1 \longrightarrow X_2$ is called a homomorphism if for all $x, y \in X_1$ we have $\varphi(supp(x*y)) = supp(\varphi(x)\circ\varphi(y))$. If φ is one to one (onto) we say that φ is a monomorphism (epimorphism). If there exists a one to one epimorphism from X_1 onto X_2 we say that X_1 is isomorphic to X_2 and we write $X_1 \cong X_2$.*

Theorem 4.7. *Let $p \in (0, 1]$. Let $(X_1, *)$ and $(X_2, \circ)$ be the (g-f)^p-hypergroupoids associated with the v_p-linked (g-f)-hypergraphs $\Gamma_1 = (X_1, \delta_1)$ and $\Gamma_2 = (X_2, \delta_2)$, respectively. Then,*

$$X_1/\rho_{\Gamma_1} \times_d X_2/\rho_{\Gamma_2} \cong (X_1 \times_d X_2)/\rho_{\Gamma_1\times\Gamma_2}.$$

Proof. We equip X_1/ρ_{Γ_1}, X_2/ρ_{Γ_2} and $(X_1 \times_d X_2)/\rho_{\Gamma_1\times\Gamma_2}$ with fuzzy hyperoperations $\odot$, $\boxdot$ and $\boxtimes$, respectively. We define $\varphi : X_1/\rho_{\Gamma_1} \times_d X_2/\rho_{\Gamma_2} \longrightarrow (X_1 \times_d X_2)/\rho_{\Gamma_1\times\Gamma_2}$ by

$$\varphi\Big((\rho_{\Gamma_1}(x), \rho_{\Gamma_2}(y))\Big) = \rho_{\Gamma_1\times\Gamma_2}\Big((x, y)\Big), \quad \forall (x, y) \in X_1 \times X_2.$$

We prove firstly that φ is well defined. If $(\rho_{\Gamma_1}(x), \rho_{\Gamma_2}(y)) = (\rho_{\Gamma_1}(x'), \rho_{\Gamma_2}(y'))$, then we have $x^p_{\delta_1} = x'^p_{\delta_2}$ and $y^p_{\delta_2} = y'^p_{\delta_2}$. Since

$$\begin{aligned}
\mu \times \nu \in (x, y)^p_{\delta_1\times\delta_2} &\iff \delta_1 \times \delta_2\Big((x, y), \mu \times \nu\Big) \geq p \\
&\iff \delta_1(x, \mu) \geq p \text{ and } \delta_2(y, \nu) \geq p \\
&\iff \mu \in x^p_{\delta_1} \text{ and } \nu \in y^p_{\delta_2} \\
&\iff \mu \in x'^p_{\delta_1} \text{ and } \nu \in y'^p_{\delta_2} \\
&\iff \delta_1(x', \mu) \geq p \text{ and } \delta_2(y', \nu) \geq p \\
&\iff \delta_1 \times \delta_2\Big((x', y'), \mu \times \nu\Big) \geq p \\
&\iff \mu \times \nu \in (x', y')^p_{\delta_1\times\delta_2},
\end{aligned}$$

we obtain $\rho_{\Gamma_1\times\Gamma_2}\Big((x, y)\Big) = \rho_{\Gamma_1\times\Gamma_2}\Big((x', y')\Big)$, i.e., φ is well defined. Now, we check that φ is one to one. Suppose that $\rho_{\Gamma_1\times\Gamma_2}\Big((x, y)\Big) = \rho_{\Gamma_1\times\Gamma_2}\Big((x', y')\Big)$. We have

$$\begin{aligned}
\mu \in x^p_{\delta_1} \text{ and } \nu \in y^p_{\delta_2} &\iff \delta_1(x, \mu) \geq p \text{ and } \delta_2(y, \nu) \geq p \\
&\iff \delta_1 \times \delta_2\Big((x, y), \mu \times \nu\Big) \geq p \\
&\iff \mu \times \nu \in (x, y)^p_{\delta_1\times\delta_2} \\
&\iff \mu \times \nu \in (x', y')^p_{\delta_1\times\delta_2} \\
&\iff \delta_1 \times \delta_2\Big((x', y'), \mu \times \nu\Big) \geq p \\
&\iff \delta_1(x', \mu) \geq p \text{ and } \delta_2(y', \nu) \geq p \\
&\iff \mu \in x'^p_{\delta_1} \text{ and } \nu \in y'^p_{\delta_2}.
\end{aligned}$$

This implies that $(\rho_{\Gamma_1}(x), \rho_{\Gamma_2}(y)) = (\rho_{\Gamma_1}(x'), \rho_{\Gamma_2}(y'))$. Clearly φ is onto. We need only to show that φ is a homomorphism. Before doing that we show that

$$\varphi\Big(\text{supp}\Big(\Delta\Big((\rho_{\Gamma_1}(x), \rho_{\Gamma_2}(y))\Big)\Big)\Big) = \rho_{\Gamma_1\times\Gamma_2}\Big(\text{supp}(\Delta(x, y))\Big), \quad \forall (x, y) \in X_1 \times X_2.$$

We know that

$$\begin{aligned}
\text{supp}\Big(\Delta\Big((\rho_{\Gamma_1}(x), \rho_{\Gamma_2}(y))\Big)\Big) &= \text{supp}\Big(\rho_{\Gamma_1}(x) \odot \rho_{\Gamma_1}(x) \times \rho_{\Gamma_2}(y) \boxdot \rho_{\Gamma_2}(y)\Big) \\
&= \text{supp}\Big(\rho_{\Gamma_1}(x) \odot \rho_{\Gamma_1}(x)\Big) \times \text{supp}\Big(\rho_{\Gamma_2}(y) \boxdot \rho_{\Gamma_2}(y)\Big) \\
&= \Big\{\rho_{\Gamma_1}(z) \mid z \in \text{supp}(x * x)\Big\} \times \Big\{\rho_{\Gamma_2}(t) \mid t \in \text{supp}(y \circ y)\Big\},
\end{aligned}$$

and so we have

$$\begin{aligned}\varphi\Big(\mathrm{supp}\Big(\Delta\big((\rho_{\Gamma_1}(x),\rho_{\Gamma_2}(y))\big)\Big)\Big) &= \Big\{\varphi\big((\rho_{\Gamma_1}(z),\rho_{\Gamma_2}(t))\big) \mid z\in \mathrm{supp}(x*x) \text{ and } t\in\mathrm{supp}(y\circ y)\Big\}\\ &= \Big\{\rho_{\Gamma_1\times\Gamma_2}\big((z,t)\big)\mid (z,t)\in\mathrm{supp}\big(\Delta(x,y)\big)\Big\}\\ &= \rho_{\Gamma_1\times\Gamma_2}\Big(\mathrm{supp}\big(\Delta(x,y)\big)\Big).\end{aligned}$$

Now, by using the above argument, for every elements $(x,y),(x',y')\in X_1\times X_2$ we have

$$\begin{aligned}&\varphi\Big(\mathrm{supp}\big((\rho_{\Gamma_1}(x),\rho_{\Gamma_2}(y))\times_d(\rho_{\Gamma_1}(x'),\rho_{\Gamma_2}(y'))\big)\Big)\\ &\quad= \varphi\Big(\mathrm{supp}\Big(\Delta\big((\rho_{\Gamma_1}(x),\rho_{\Gamma_2}(y))\big)\bigcup\Delta\big((\rho_{\Gamma_1}(x'),\rho_{\Gamma_2}(y'))\big)\Big)\Big)\\ &\quad= \varphi\Big(\mathrm{supp}\Big(\Delta\big((\rho_{\Gamma_1}(x),\rho_{\Gamma_2}(y))\big)\Big)\Big)\bigcup\varphi\Big(\mathrm{supp}\Big(\Delta\big((\rho_{\Gamma_1}(x'),\rho_{\Gamma_2}(y'))\big)\Big)\Big)\\ &\quad= \rho_{\Gamma_1\times\Gamma_2}\Big(\mathrm{supp}\big(\Delta(x,y)\big)\Big)\bigcup\rho_{\Gamma_1\times\Gamma_2}\Big(\mathrm{supp}\big(\Delta(x',y')\big)\Big)\\ &\quad= \rho_{\Gamma_1\times\Gamma_2}\Big(\mathrm{supp}\Big(\Delta\big((x,y)\big)\Big)\bigcup\mathrm{supp}\Big(\Delta\big((x',y')\big)\Big)\Big)\\ &\quad= \rho_{\Gamma_1\times\Gamma_2}\Big(\mathrm{supp}\Big(\Delta\big((x,y)\big)\bigcup\Delta\big((x',y')\big)\Big)\Big)\\ &\quad= \rho_{\Gamma_1\times\Gamma_2}\Big(\mathrm{supp}\big((x,y)\times_d(x',y')\big)\Big)\\ &\quad= \mathrm{supp}\Big(\rho_{\Gamma_1\times\Gamma_2}\big((x,y)\big)\boxtimes\rho_{\Gamma_1\times\Gamma_2}\big((x',y')\big)\Big)\\ &\quad= \mathrm{supp}\Big(\varphi\big((\rho_{\Gamma_1}(x),\rho_{\Gamma_2}(y))\big)\boxtimes\varphi\big((\rho_{\Gamma_1}(x'),\rho_{\Gamma_2}(y'))\big)\Big).\end{aligned}$$

Hence φ is an isomorphism. □

Theorem 4.8. *Let $p\in(0,1]$. Let Γ be a v_p-linked (g-f)-hypergraph and $(X_2,\circ)$ be the (g-f)^p-hypergroupoid associated with Γ. If $(X_1,*)$ is a separable fuzzy hypergroupoid and $\varphi: X_1\longrightarrow X_2$ is an epimorphism, then there exists a regular equivalence relation ρ on X_1 such that*

$$X_1/\rho\cong X_2/\rho_\Gamma.$$

Proof. We define a relation ρ on X_1 as follows:

$$x\rho\, y\Longleftrightarrow \varphi(x)\rho_\Gamma\varphi(y),\quad \forall x,y\in X_1.$$

By Lemma 4.1, ρ_Γ is an equivalence relation on X_2 which implies that ρ is an equivalence relation on X_1. Let $x,y,z\in X_1$ be arbitrary elements such that $x\rho\, y$. We show that $\mathrm{supp}(x*z)\overline{\rho}\,\mathrm{supp}(y*z)$. From $x\rho\, y$ it follows that $\varphi(x)\circ\varphi(x)=\varphi(y)\circ\varphi(y)$ which implies that $\varphi(\mathrm{supp}(x*x))=\varphi(\mathrm{supp}(y*y))$. Let $r\in\mathrm{supp}(x*z)$ be an arbitrary element. Then, we have $\varphi(r)\in\varphi(\mathrm{supp}(x*z))=\varphi(\mathrm{supp}(x*x))\bigcup\varphi(\mathrm{supp}(z*z))=\varphi(\mathrm{supp}(y*y))\bigcup\varphi(\mathrm{supp}(z*z))=\varphi(\mathrm{supp}(y*z))$. Therefore, there is $t\in\mathrm{supp}(y*z)$ such that $\varphi(r)=\varphi(t)$. This means that $r\rho\, t$ and so $\mathrm{supp}(x*z)\overline{\rho}\,\mathrm{supp}(y*z)$. In a similar way we can show that $\mathrm{supp}(z*x)\overline{\rho}\,\mathrm{supp}(z*y)$. Thus ρ is regular. Now, let $\psi: X_1/\rho\longrightarrow X_2/\rho_\Gamma$ is defined by $\psi\big(\rho(x)\big)=\rho_\Gamma\big(\varphi(x)\big)$. Let $x,y\in X_1$. Then, we have

$$\rho(x)=\rho(y)\Longleftrightarrow\varphi(x)\rho_\Gamma\varphi(y)\Longleftrightarrow\rho_\Gamma(\varphi(x))=\rho_\Gamma(\varphi(y))\Longleftrightarrow\psi\big(\rho(x)\big)=\psi\big(\rho(y)\big).$$

Thus, ψ is well-defined and one to one. Since φ is onto, ψ is onto. We equip X_1/ρ and X_2/ρ_Γ with the fuzzy hyperoperations $\odot$ and $\boxdot$, respectively. Let $x,y\in X_1$. The following argument shows that ψ is a homomorphism.

$$\begin{aligned}
\operatorname{supp}\Big(\psi\big(\rho(x)\big) \boxdot \psi\big(\rho(y)\big)\Big) &= \operatorname{supp}\Big(\rho_{_\Gamma}(\varphi(x)) \boxdot \rho_{_\Gamma}(\varphi(y))\Big) \\
&= \rho_{_\Gamma}\Big(\operatorname{supp}(\varphi(x) \circ \varphi(y))\Big) \\
&= \rho_{_\Gamma}\Big(\varphi\big(\operatorname{supp}(x * y)\big)\Big) \\
&= \Big\{\rho_{_\Gamma}(\varphi(z)) \mid z \in \operatorname{supp}(x * y)\Big\} \\
&= \psi\Big(\big\{\rho(z) \mid z \in \operatorname{supp}(x * y)\big\}\Big) \\
&= \psi\Big(\operatorname{supp}(\rho(x) \odot \rho(y))\Big).
\end{aligned}$$

□

Theorem 4.9. *Let $p \in (0,1]$. Let Γ be a v_p-linked (g-f)-hypergraph and $(X_1, *)$ be the (g-f)^p-hypergroupoid associated with Γ. If $(X_2, \circ)$ is a separable fuzzy hypergroupoid and $\varphi : X_1 \longrightarrow X_2$ is a monomorphism, then there exists a regular equivalence relation ρ' on $\varphi(X_1)$ such that*

$$X_1/\rho_{_\Gamma} \cong \varphi(X_1)/\rho'.$$

Proof. We define a relation ρ' on $\varphi(X_1)$ as follows:

$$\varphi(x)\rho'\varphi(y) \Longleftrightarrow x\rho_{_\Gamma} y, \quad \forall x, y \in X_1.$$

It is obvious that ρ' is a regular equivalence relation. We define $\psi : X_1/\rho_{_\Gamma} \longrightarrow \varphi(X_1)/\rho'$ by $\psi\big(\rho_{_\Gamma}(x)\big) = \rho'(\varphi(x))$. One easily checks that ψ is an isomorphism. □

Lemma 4.10. *Let ρ be a regular equivalence relation on a fuzzy hypergroupoid $(X, \circ)$. Then, $\pi : X \longrightarrow X/\rho$ which is defined by $\pi(x) = \rho(x)$, for all $x \in X$, is an epimorphism which is called canonical epimorphism.*

Proof. It is straightforward. □

Theorem 4.11. *Let $p \in (0,1]$. Let $(X_1, *)$ and $(X_2, \circ)$ be the (g-f)^p-hypergroupoids associated with the v_p-linked (g-f)-hypergraphs $\Gamma_1 = (X_1, \delta_1)$ and $\Gamma_2 = (X_2, \delta_2)$, respectively. Let $\varphi : X_1 \longrightarrow X_2$ be an epimorphism such that $\varphi(x)\rho_{_{\Gamma_2}}\varphi(y)$ implies $x\rho_{_{\Gamma_1}} y$. If $\rho = \{(x, y) \in X_1^2 \mid \varphi(x)\rho_{_{\Gamma_2}}\varphi(y)\big\}$ and $\rho' = \{(\varphi(x), \varphi(y)) \in X_2^2 \mid x\rho_{_{\Gamma_1}} y\big\}$, then there exists a unique homomorphism $\varphi^* : X_1/\rho \longrightarrow X_2/\rho'$ such that the following diagram is commutative:*

$$\begin{tikzcd}
X_1 \arrow[r, "\varphi"] \arrow[d, "\pi"'] & X_2 \arrow[d, "\pi'"] \\
X_1/\rho \arrow[r, "\varphi^*"] & X_2/\rho'
\end{tikzcd}$$

i.e., $\pi' \circ \varphi = \varphi^* \circ \pi$, where π and π' denote the canonical epimorphisms.

Proof. The proof of the fact that ρ and ρ' are regular equivalence relations is analogous to the corresponding part of the proof of Theorem 4.8 and we omit the details. We equip X_1/ρ and X_2/ρ' with the fuzzy hyperoperations $\odot$ and $\boxdot$, respectively. Let $\varphi^* : X_1/\rho \longrightarrow X_2/\rho'$ is defined by $\varphi^*\big(\rho(x)\big) = \rho'(\varphi(x))$, for all $x \in X_1$. First, we show that φ^* is well-defined. Let $x, y \in X_1$ and $\rho(x) = \rho(y)$. Then, $\varphi(x)\rho_{_{\Gamma_2}}\varphi(y)$ and so by assumption $x\rho_{_{\Gamma_1}} y$. Therefore, φ^* is well-defined. Moreover, it is easy to prove that $\varphi^*\big(\operatorname{supp}(\rho(x) \odot \rho(y)\big) = \operatorname{supp}\Big(\varphi^*\big(\rho(x)\big) \boxdot \varphi^*\big(\rho(y)\big)\Big)$ and $\pi' \circ \varphi = \varphi^* \circ \pi$. Now, we show that φ^* is unique. Let $g : X_1/\rho \longrightarrow X_2/\rho'$ be a homomorphism such that $\pi' \circ \varphi = g \circ \pi$. Then, for all $x \in X_1$, we have $g(\rho(x)) = g(\pi(x)) = \pi' \circ \varphi(x) = \varphi^* \circ \pi(x) = \varphi^*\big(\rho(x)\big)$. □

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