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# On Connectedness of the Set of Efficient Solutions for Generalized Systems

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**Abstract.** In this paper, we first give a density theorem. We will see that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of the efficient solutions. Finally, we discuss about the connectedness for the set of the efficient solutions of a generalized system.

## 1. Introduction and Preliminaries

Throughout this paper, let $X$ be a real Hausdorff topological vector space and let $Y$ be a real Hausdorff topological vector space. Let $Y^*$ be the topological dual space of $Y$. Let $C$ be a closed convex pointed cone in $Y$. The cone $C$ induces a partial ordering in $Y$ defined by

$$x \leq y, \text{ if and only if } y - x \in C.$$

Let

$$C^* = \{f \in Y^* : f(y) \geq 0, \text{ for all } y \in C\}$$

be the dual cone of $C$. Denote the quasi-interior of $C^*$ by $C^\sharp$, i.e.

$$C^\sharp := \{f \in Y^* : f(y) > 0 \text{ for all } y \in C \setminus \{0\}\}.$$

Let $D$ be a nonempty subset of $Y$. The cone hull of $D$ is defined as

$$\text{cone}(D) = \{td : t \geq 0, d \in D\}.$$

Denote the closure of $D$ by $\text{cl}(D)$. A nonempty convex subset $M$ of the convex cone $C$ is called a base of $C$ if $C = \text{cone}(M)$. It is easy to see that $C^\sharp \neq \emptyset$ if and only if $C$ has a base.

Let $A$ be a nonempty subset of $X$ and $F : A \times A \to 2^Y \setminus \{\emptyset\}$ be a set-valued mapping. A vector $x \in A$ is called an efficient solution if

$$F(x, y) \notin -C \setminus \{0\}, \text{ for all } y \in A.$$

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The set of efficient solutions is denoted by $V(A, F)$.

If int $C \neq \emptyset$, a vector $x \in A$ is called a weakly efficient solution if

$$F(x, y) \notin -\text{int } C, \text{ for all } y \in A.$$

The set of weakly efficient solutions is denoted by $V_W(A, F)$ ( see, for instance, [7]).

Let $f \in C^* \setminus \{0\}$. A vector $x \in A$ is called an $f-$efficient solution if

$$f(F(x, y)) \geq 0, \text{ for all } y \in A.$$

The set of $f-$efficient solutions is denoted by $V_f(A, F)$.

**Definition 1.1.** *A vector $x \in A$ is called a positive proper efficient solution if there exists $f \in C^\sharp$ such that*

$$f(F(x, y)) \geq 0, \textit{ for all } y \in A.$$

By Definition 1.1, we can get easily the following Proposition.

**Proposition 1.2.** *If int $C \neq \emptyset$, then*

$$V(A, F) \subset V_W(A, F)$$

*and*

$$\bigcup_{f \in C^* \setminus \{0\}} V_f(A, F) \subset V_W(A, F).$$

**Lemma 1.3.** *Suppose that int $C \neq \emptyset$ and for each $x \in A$, $F(x, A) = \bigcup_{y \in A} F(x, y)$ is $C-$convex, that is $F(x, A) + C$ is a convex set. Then*

$$V_W(A, F) = \bigcup_{f \in C^* \setminus \{0\}} V_f(A, F).$$

*Proof.* In view of Proposition 1.2, it suffices to prove that

$$V_W(A, F) \subset \bigcup_{f \in C^* \setminus \{0\}} V_f(A, F).$$

Let $x \in V_W(A, F)$. By definition,

$$F(x, y) \notin -\text{int } C, \text{ for all } y \in A.$$

Thus,

$$F(x, A) \cap (-\text{int } C) = \emptyset,$$

and hence,

$$(F(x, A) + C) \cap (-\text{int } C) = \emptyset.$$

Because $F(x, A) + C$ is a convex set, by the separation theorem of convex sets (see Theorem 3.21 in [13]), we can find a function $f \in Y^* \setminus \{0\}$ such that

$$\inf\{f(F(x, y) + c) : y \in A, c \in C\} > \sup\{f(-c) : c \in \text{int } C\}.$$

We obtain that $f \in C^* \setminus \{0\}$ and

$$f(F(x, y)) \geq 0, \text{ for all } y \in A.$$

Therefore, $x \in V_f(A, F)$. □

**Definition 1.4.** *([3]) Let $E$ be a topological vector space and $K$ be a subset of it. A mapping $F : K \to 2^E$ is said to be a KKM-mapping, if for any $\{x_1, x_2, \ldots, x_n\} \subset K$, co$\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$, where $2^E \setminus \{\emptyset\}$ denotes the family of all nonempty subsets of $E$.*

**Lemma 1.5.** *([3]) Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ a KKM mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B} F(x)$ is compact. Then*

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

## 2. Density and Connectedness

In this section, we first give a density theorem. We will see that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of the efficient solutions. Finally, we discuss about the connectedness for the set of the efficient solutions.

Let $\varphi : A \times A \to Y$ be a mapping. The mapping $\varphi$ is called $C-$monotone on $A \times A$ if

$$\varphi(x, y) + \varphi(y, x) \in -C, \text{ for all } x, y \in A.$$

**Proposition 2.1.** *Let $\varphi : A \times A \to Y$ be a mapping. If $\varphi$ is $C-$monotone, then, for any $f \in C^* \setminus \{0\}$, the mapping $f \circ \varphi$ is $\mathbb{R}^+-$ monotone.*

*Proof.* Assume that $\varphi$ is $C-$monotone. Then

$$\varphi(x, y) + \varphi(y, x) \in -C, \text{ for all } x, y \in A.$$

This implies that

$$f(\varphi(x, y) + \varphi(y, x)) \leq 0.$$

So

$$\begin{aligned}(f \circ \varphi)(x, y) + (f \circ \varphi)(y, x) &= f(\varphi(x, y)) + f(\varphi(y, x)) \\ &= f(\varphi(x, y) + \varphi(y, x)) \\ &\leq 0.\end{aligned}$$

This completes the proof. □

The mapping $\varphi$ is called $C-$strongly monotone on $A \times A$ if $\varphi$ is $C-$monotone and if $x, y \in A, x \neq y$, then

$$\varphi(x, y) + \varphi(y, x) \in -\text{int } C.$$

**Remark 2.2.** *It is obvious from the definition that the $C-$strong monotonicity of $\varphi$ implies the $C-$monotonicity of $\varphi$. While if we take $X = \mathbb{R}$, $C = [0, \infty)$ and define $\varphi : X \times X \to \mathbb{R}$ by $\varphi(x, y) = \{0\}$ then one can check that $\varphi$ is $C-$monotonicity and is not $C-$strongly monotone.*

Let $\psi : A \to Y$ be a mapping. The mapping $\psi$ is called $C-$lower ($C-$upper) semicontinuous at $x_0 \in A$ if, for any neighborhood $U$ of 0( the zero vector) in $Y$, there is a neighborhood $U(x_0)$ of $x_0$ such that

$$\psi(x) \in \psi(x_0) + U + C, \text{ for all } x \in U(x_0) \cap A,$$

$$(\psi(x) \in \psi(x_0) + U - C, \text{ for all } x \in U(x_0) \cap A).$$

The following lemma establishes a link between lower semicontinuity and $C-$ lower semicontinuity.

**Lemma 2.3.** *Let $\psi : A \to Y$ be a mapping and $G : A \to 2^Y$ be defined by*

$$G(x) = \psi(x) - C, \quad \forall x \in A.$$

*Then G is lower semicontinuous if and only if $\psi$ is $C-$lower semicontinuous.*

*Proof.* (⇒) Let $G$ is lower semicontinuous at $x_0 \in A$ and $U_0$ is a neighborhood of 0. Then

$$(\psi(x) - C) \cap (\psi(x_0) + U_0 + C) \neq \emptyset.$$

Since $G$ is lower semicontinuous at $x_0$, there exists $U(x_0)$ such that for all $x \in U_{x_0}$,

$$(\psi(x) - C) \cap (\psi(x_0) + U_0 + C) \neq \emptyset.$$

Hence there is an $\alpha \in C$ such that

$$\psi(x) - \alpha \in \psi(x_0) + U_0 + C,$$

and so

$$\psi(x) \in \psi(x_0) + U_0 + \alpha + C \subseteq \psi(x_0) + U_0 + C + C \subseteq \psi(x_0) + U_0 + C.$$

Therefore $\psi$ is $C-$lower semicontinuous.

(⇐) Let $G(x_0) \cap W = (\psi(x_0) - C) \cap W \neq \emptyset$, where $W$ is an open set. Hence, there is an $\alpha \in C$ such that

$$\psi(x_0) - \alpha \in W.$$

Then there exists a balanced neighborhood $U_0$ of 0 such that

$$\psi(x_0) - \alpha + U_0 \subseteq W.$$

Since $\psi$ is $C-$lower semicontinuous, then there is a neighborhood $U_{x_0}$ of $x_0$ such that for all $x \in U_{x_0}$, we have

$$\psi(x) \subseteq \psi(x_0) + U_0 + C.$$

Thus

$$\psi(x) - C \subseteq \psi(x_0) + U_0.$$

This implies that

$$\psi(x) - C - \alpha \subseteq \psi(x_0) - \alpha_0 + U_0 \subseteq W.$$

So $(\psi(x) - C - \alpha) \cap W$ and

$$\psi(x) - C - \alpha_0 \subseteq \psi(x) - C - C = \psi(x) - C.$$

Therefore

$$W \cap (\psi(x) - C) \neq \emptyset.$$

This completes the proof. □

**Proposition 2.4.** *For any $\psi : A \to Y$, if $f \in C^* \setminus \{0_{Y^*}\}$ and $\psi$ is $C-$lower semicontinuous, then $f \circ \psi : A \to \mathbb{R}$ is lower semicontinuous.*

*Proof.* Assume that $f \in C^* \setminus \{0_{Y^*}\}$. For any $\lambda \in \mathbb{R}$, we prove that

$$\{x \in A | (f \circ \psi)(x) \leq \lambda\} \text{ is a closed set.}$$

To see this, it suffices that we show that the set

$$M = \{x \in A | (f \circ \psi)(x) > \lambda\} = \{x \in A | (f \circ \psi)(x) \leq \lambda\}^C \text{ is open.}$$

Let $\bar{x} \in M$. Then $f(\psi(\bar{x})) > \lambda$. This implies that

$$\psi(\bar{x}) \in f^{-1}(\lambda, \infty).$$

Since $f^{-1}(\lambda, \infty) - \psi(\bar{x})$ is an open set which contains 0, there exists a neighborhood $U$ of 0 such that

$$U + \psi(\bar{x}) \subseteq f^{-1}(\lambda, \infty).$$

Since $\psi$ is $C-$lower semicontinuous, there is a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$\psi(x) \in U + \psi(\bar{x}) + C \subseteq f^{-1}(\lambda, +\infty) + C, \quad \text{for all } x \in U(\bar{x}).$$

Hence, for each $x \in U(\bar{x})$, there is a $c \in C$ such that

$$\begin{aligned} \psi(x) - c \in f^{-1}(\lambda, +\infty) \quad &\Rightarrow \quad f(\psi(x) - c) \in (\lambda, +\infty) \\ &\Rightarrow \quad f(\psi(x)) - f(c) \in (\lambda, +\infty) \\ &\Rightarrow \quad f(\psi(x)) \geq f(\psi(x)) - f(c) > \lambda. \end{aligned}$$

Therefore

$$U(\bar{x}) \subseteq M.$$

Then $\bar{x}$ is an interior point of $M$ and so $M$ is open. This completes the proof. □

**Proposition 2.5.** *For any $\psi : A \to Y$, if $f \in C^* \setminus \{0_{Y^*}\}$ and $\psi$ is $C-$lower semicontinuous, then $f \circ \psi : A \to \mathbb{R}$ is upper semicontinuous.*

*Proof.* Assume that $f \in C^* \setminus \{0_{Y^*}\}$. For any $\lambda \in \mathbb{R}$, we have to prove that the set

$$\{x \in A | - (f \circ \psi)(x) \leq \lambda\} \text{ is a closed}$$

or equivalently the set

$$Q = \{x \in A | - (f \circ \psi)(x) > \lambda\} = \{x \in A | - (f \circ \psi)(x) \leq \lambda\}^C \text{ is open.}$$

To see this, let $\bar{x} \in Q$. Then $-f(\psi(\bar{x})) > \lambda$ and so

$$f(\psi(\bar{x})) < -\lambda.$$

This implies that

$$\psi(\bar{x}) \in f^{-1}(-\infty, -\lambda).$$

Since $f^{-1}(-\infty, -\lambda) - \psi(\bar{x})$ is an open set which contains 0, there exists a neighborhood $U$ of 0 such that

$$U + \psi(\bar{x}) \subseteq f^{-1}(-\infty, -\lambda).$$

It follows from the $-C-$lower semicontinuity of $\psi$ that there exists a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$\psi(x) \in U + \psi(\bar{x}) - C \subseteq f^{-1}(-\infty, -\lambda) - C, \quad \text{for all } x \in U(\bar{x}).$$

Consequently, For each $x \in U(\bar{x})$, there is a $c \in C$ such that

$$\begin{aligned} \psi(x) + c \in f^{-1}(-\infty, -\lambda) \quad &\Rightarrow \quad f(\psi(x) + c) \in (-\infty, -\lambda) \\ &\Rightarrow \quad f(\psi(x)) + f(c) \in (-\infty, -\lambda) \\ &\Rightarrow \quad f(\psi(x)) \leq f(\psi(x)) + f(c) < -\lambda, \end{aligned}$$

and so

$$-(f \circ \psi)(x) > \lambda.$$

Hence

$$U(\bar{x}) \subseteq Q.$$

Therefore $\bar{x}$ is an interior point of $Q$. This implies that each point of $Q$ is an interior open ( note $\bar{x}$ was an arbitrary element of $Q$) and so $Q$ is an open set. This completes the proof. □

**Remark 2.6.** *For any $\psi : A \to Y$, if $f \in C^*$ and $\psi$ is $C-$convex, then $f \circ \psi : A \to \mathbb{R}$ is $\mathbb{R}^+-$convex.*

**Proposition 2.7.** *Let $Y$ be a locally convex space and $\psi : A \to Y$ a mapping. If, for any $f \in C^* \setminus \{0_{Y^*}\}$, $f \circ \psi$ is $\mathbb{R}^+-$convex, then $\psi$ is $C-$convex.*

*Proof.* If the result is false then there exist $x_1, x_2 \in A$ and $t \in [0,1]$ such that

$$z = t\psi(x_1) + (1-t)\psi(x_2) - \psi(tx_1 + (1-t)x_2) \notin C.$$

By the separation theorem ( see, for instance, [[8], Theorem 3.4]) there exist $f \in Y^*$ and two real numbers $\alpha, \beta$ such that

$$f(z) < \alpha < \beta < f(x), \quad \forall x \in C. \tag{1}$$

We deduce from the properties of $C$ and (1) that $f \in C^* \backslash \{0_{Y^*}\}$ and by taking $x = 0$ in (1) we get $f(z) < 0$ and this means that

$$f(t\psi(x_1) + (1-t)\psi(x_2)) = tf(\psi(x_1)) + (1-t)f(\psi(x_2)) < f(\psi(tx_1 + (1-t)x_2))$$

which is contradicted by the $\mathbb{R}^+-$convexity of $f \circ \psi$. This completes the proof. □

**Lemma 2.8.** *(See Theorem 3.1 of [5]) Let $A \subset X$ be a nonempty compact convex set. Let $\psi : A \to Y$ and $\varphi : A \times A \to Y$ be two mappings. Assume that the following conditions are satisfied:*

1. *$\psi$ is $C-$lower semicontinuous;*
2. *$\varphi(x,x) \geq 0$ for all $x \in A$ and $\varphi$ is $C-$monotone;*
3. *for each $x \in A, \varphi(x,y)$ is $C-$lower semicontinuous in $y$ and for each $y \in A, \varphi(x,y)$ is $C-$upper semicontinuous in $x$;*
4. *for each $x \in A, \psi(y) + \varphi(x,y)$ is $C-$convex mapping in $y$.*

*Then, for each $f \in C^* \setminus \{0_{Y^*}\}$, $V_f(A,F)$ is a nonempty compact convex set, where*

$$F(x,y) = \psi(y) + \varphi(x,y) - \psi(x), \quad \text{for} \quad x, y \in A.$$

The following result establishes an existence and uniqueness theorem for an efficient solution for bifunctions which one can consider it as an extension of Lemma 2.8 and Theorem 3.1 in [5] by relaxing the $C-$lower semicontinuity of the mapping $\varphi$ in the second variable and compactness of the set as well extending the result for the mapping $\psi$ is a bifunction, that is from one variable to two variables in the setting of topological vector spaces ( more exact, we replace the locally convex topological vector space $Y$ by topological vector space). Further, the coercivity ( that is condition (5) in the next result is more general than the coercivity condition used in Theorem 3.1 of [12].

**Lemma 2.9.** *Let $A \subset X$ be a nonempty convex set. Let $\psi : A \times A \to Y$ and $\varphi : A \times A \to Y$ be two mappings. Assume that the following conditions are satisfied:*

1. *for each $y \in A, \psi(x,y) + \varphi(x,y)$ is $C-$ upper semicontinuous ( or $(-C)-$ lower semicontinuous) in $x$;*
2. $$\psi(x,x) + \varphi(x,x) = 0, \text{ for all } x \in A;$$
3. *for each $x \in A, \psi(x,y) + \varphi(x,y)$ is $C-$convex mapping in $y$.*
4. *$\varphi, \psi$ are $C-$strongly monotone on $A \times A$.*
5. *There exist a nonempty compact convex subset $B$ and a compact subset $D$ of $A$ such that*

$$\forall y \in A \backslash D, \exists x \in B : \quad \psi(x,y) + \varphi(x,y) \in -intC.$$

*Then, for each $f \in C^* \setminus \{0_{Y^*}\}$, the set of $f-$ efficient solutions, that is $V_f(A,F)$ is singleton and so convex and compact, where*

$$F(x,y) = \psi(x,y) + \varphi(x,y), \text{ for all } x, y \in A.$$

*Proof.* Define $\Gamma : A \to 2^A$ as follows

$$\Gamma(y) = \{x \in A : f \circ F(x, y) \geq 0\}, \quad \forall y \in A.$$

By Proposition 2.5, $\Gamma(y)$ is closed for each $y \in A$. We claim that $\Gamma$ is a KKM mapping. Indeed, let $B = \{y_1, ..., y_n\} \subset A$ and $z = \sum_{i=1}^{n} \lambda_i y_i \in coB$. It follows from (2) and (3) that

$$\sum_{i=1}^{n} \lambda_i F(z, y_i) = \sum_{i=1}^{n} \lambda_i F(z, y_i) - F(z, z) \in C.$$

So

$$\sum_{i=1}^{n} f(F(z, y_i)) \geq 0.$$

Therefore, there is an $i \in \{1, 2, ..., n\}$ such that $f(F(z, y_i)) \geq 0$ and so $z \in \Gamma(y_i)$ and this completes the proof of the assertion. Moreover, it follows from (5) that $\bigcap_{y \in B} \Gamma(y) \subset D$ and so $\Gamma$ satisfies all the assumptions of Lemma 1.5. Hence it follows from Lemma 1.5 that $\bigcap_{x \in A} \Gamma(x) \neq \emptyset$. Then there exists $\overline{x} \in \bigcap_{x \in A} \Gamma(x)$. This means

$$f(F(\overline{x}, y)) \geq 0, \quad \forall y \in A.$$

Hence $\overline{x} \in V_f(F, A)$. It is obvious from the definitions of $\Gamma$ and $V_f(F, A)$ that $V_f(F, A) = \bigcap_{x \in A} \Gamma(x) \subset \bigcap_{y \in B} \Gamma(y) \subset D$ and since $V_f(F, A)$ is a closed subset of the compact set $D$ ( note that the values of $\Gamma$ are closed) we get that $V_f(A, F)$ is a compact subset of $D$. Now we are going to show that $V_f(A, F)$ is singleton. To verify this, let $x_1, x_2 \in V_f(A, F)$. If we assume that $x_1 \neq x_2$ then it follows from (4) that

$$F(x_1, x_2) + F(x_2, x_1) = \psi(x_1, x_2) + \varphi(x_1, x_2) + \psi(x_2, x_1) + \varphi(x_2, x_1) =$$

$$\psi(x_1, x_2) + \psi(x_2, x_1) + \varphi(x_1, x_2) + \varphi(x_2, x_1)$$
$$\in (-intC + (-intC)) \subseteq -intC.$$

Thus, $f(F((x_1, x_2) + F(x_2 + x_1) < 0$ which is contradicted by $x_1, x_2 \in V_f(A, F)$. This completes the proof. □

Note that if the mappings $\psi$ and $\varphi$ are $C-$ upper semicontinuous ( $(-C)-$ lower semicontinuous and $C-$ convex) then $\psi + \varphi$ is $C-$ upper semicontinuous ( $(-C)-$ lower semicontinuous and $C$-convex). Also we can omit condition (5) in Lemma 2.9 when $A$ is compact.

The following result is the main goal of the paper that provides a density theorem between the solution set of efficient solutions and properly $f-$ efficient solutions which extends the corresponding result in [5, 6, 8–10, 10, 11].

**Theorem 2.10.** *Let $A \subset X$ be a nonempty compact convex set. Let $\psi : A \times A \to Y$ and $\varphi : A \times A \to Y$ be two mappings. Assume that the following conditions are satisfied:*

1. *for each $y \in A$, $\psi(x, y) + \varphi(x, y)$ is $C-$ upper semicontinuous ( or $(-C)-$ lower semicontinuous) in $x$;*
2. $$\psi(x, x) + \varphi(x, x) = 0, \text{ for all } x \in A;$$
3. *for each $x \in A$, $\psi(x, y) + \varphi(x, y)$ is $C-$convex mapping in $y$.*
4. *$\varphi, \psi$ are $C-$strongly monotone on $A \times A$.*
5. *$\Psi(A \times A)$ and $D = \{\varphi(x, y) : x, y \in A\}$ are bounded subsets of $Y$.*
6. *$C^\sharp \neq \emptyset$ and $intC \neq \emptyset$*

*Then,*

$$\bigcup\nolimits_{f \in C^\sharp} V_f(A, F) \subset V(A, F) \subset cl(\bigcup\nolimits_{f \in C^\sharp} V_f(A, F))$$

*where*

$$F(x, y) = \psi(x, y) + \varphi(x, y), \text{ for all } x, y \in A.$$

*Proof.* It follows from our hypothesis and Lemma 2.9 that

$$V_f(A,F) \neq \emptyset, \qquad \forall f \in C^* \setminus \{0\}.$$

It is obvious from the definitions of an efficient solution, $f-$ proper solution and weakly solution that,

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset V_W(A,F).$$

We claim that

$$V(A,F) \subset \bigcup_{f\in C^*\setminus\{0\}} V_f(A,F).$$

Indeed, let $x \in V(A,F)$. It is easy to check from the condition (3) that the set $F(x,A)+C$ is convex and since $x \in V(A,F)$ and $-C - intC \subset -intC$ we get that $(F(x,A)+C) \cap -intC = \emptyset$. Then by applying the separation theorem ( note that we can separate the convex sets $F(x,A)+C$ and $-C$) that there exist $f \in X^*$ and real number $\alpha$ such that

$$f(a) \leq \alpha \leq f(F(x,y)+b), \quad \forall (a,b,y) \in (-C)\times C \times A.$$

Now by taking $(a,b) = (0,0)$ one can deduce that

$$0 \leq \alpha \leq f(F(x,y)), \quad \forall y \in A.$$

This means that $x \in V_f(A,F)$ which completes the proof of the assertion.

Now we are going to prove that

$$\bigcup_{f\in C^*\setminus\{0\}} V_f(A,F) \subset cl(\bigcup_{f\in C^\sharp} V_f(A,F)).$$

To verify the inclusion let us define the mapping $H : C^* \setminus \{0\} \to 2^A$ by

$$H(f) = V_f(A,F), \quad \text{f} \in C^* \setminus \{0\}.$$

By using the proof as given in Lemma 2.9 one can see that the values of $H$ are nonempty and compact. We show that the graph of $H$ is closed and so it is upper semicontinuous ( note that the values of $H$ are compact). Let $\{(f_\alpha, x_\alpha) : \alpha \in I\}$ be a net in the graph of $H$, i.e.,

$$\{(f_\alpha, x_\alpha) : \alpha \in I\} \subset Graph(H) = \{(f,x) \in C^* \setminus \{0\} \times A : x \in H(f)\},$$

with

$$(f_\alpha, x_\alpha) \to (f,x) \in C^* \setminus \{0\} \times A,$$

where $f_\alpha \to f$ means that $\{f_\alpha\}$ converges to $f$ with respect to the topology induced by the bounded subsets of $Y$ on $Y^*$ which is denoted by $\beta(Y^*,Y)$ and a neighborhood base of the zero is as

$$\{\bigcap_{i=1}^{m} \{f \in Y^* : \sup_{y\in B_i} \mid f(y) \mid \ < \varepsilon\}\}_{B_i \in \mathbf{B}(Y)},$$

where $\mathbf{B}(Y)$ denotes all the nonempty bounded subsets of $Y$. From

$$x_\alpha \in H(f_\alpha) = V_{f_\alpha}(A,F), \alpha \in I,$$

we have

$$f_\alpha(\psi(x_\alpha,y)) + f_\alpha(\varphi(x_\alpha,y)) \geq 0$$

Next, we show that

$$f(\psi(x,y)) + f(\varphi(x,y)) \geq 0$$

By (5) the sets $\psi(A \times A)$ and $D = \{\varphi(x, y) : x, y \in A\}$ are bounded subsets of $Y$. Define

$$P_{\psi(A\times A)+D}(y^*) = \sup{}_{u\in\psi(A\times A)+D} \mid y^*(u) \mid \quad \text{for} \quad y^* \in Y^*.$$

It is easy to check that $P_{\psi(A\times A)+D}$ is a seminorm of $Y^*$ ( note that $y^*(B)$ is bounded when $B$ is a bounded subset of $Y$ because if we take $\{x_n\} \subset B$ then $\frac{1}{n}y^*(x_n) = y^*(\frac{1}{n}x_n)$ and $(\frac{1}{n}x_n) \to 0$. Hence it follows from the continuity of $y^*$ that $y^*(\frac{1}{n}x_n)$ converges to zero and this completes the proof of the claim). Let $\varepsilon$ be an arbitrary positive real number. Put

$$U = \{y^* \in Y^* : P_{\psi(A\times A)+D}(y^*) < \varepsilon\}$$

which is a neighborhood of 0 with respect to $\beta(Y^*, Y)$. Since $f_\alpha - f \to 0$ ( in $\beta(Y^*, Y)$) there exists $\alpha_0 \in I$ such that $f_\alpha - f \in U$ for all $\alpha \geq \alpha_0$. It implies that

$$P_{\psi(A\times A)+D} = \sup{}_{u\in\psi(A\times A)+D} \mid (f_\alpha - f)(u) \mid \ < \varepsilon, \quad \text{whenever } \alpha \geq \alpha_0.$$

Therefore, for any $y \in A$ and $\alpha \geq \alpha_0$ we have, for all $(y, \beta) \in A \times I$,

$$\mid (f_\alpha - f)(\psi(x_\beta, y) + \varphi(x_\beta, y)) \mid \ = \ \mid f_\alpha(\psi(x_\beta, y) + \varphi(x_\beta, y)) - f(\psi(x_\beta, y) + \varphi(x_\beta, y)) \mid < \varepsilon.$$

Hence

$$\lim_\alpha f_\alpha(\psi(x_\beta, y) + \varphi(x_\beta, y)) = f(\psi(x_\beta, y) + \varphi(x_\beta, y)) \quad \forall(y, \beta) \in A \times I.$$

So

$$\limsup_\alpha f_\alpha(\psi(x_\beta, y) + \varphi(x_\beta, y)) = f(\psi(x_\beta, y) + \varphi(x_\beta, y)) \quad \forall(y, \beta) \in A \times I.$$

Then

$$0 \leq \limsup_\alpha f_\alpha(\psi(x_\beta, y) + \varphi(x_\beta, y)) = f(\psi(x_\beta, y) + \varphi(x_\beta, y)) \quad \forall(y, \beta) \in A \times I.$$

Then we deduce , since $\psi$ and $\varphi$ are $C-$ lower semicontinuous in the second variable, that

$$0 \leq \liminf_\beta f(\psi(x_\beta, y) + \varphi(x_\beta, y)) \leq f(\psi(x, y) + \varphi(x, y)),$$

That is,

$$x \in V_f(A, F) = H(F)$$

Thus, H is closed ( i.e., $H$ has closed graph).

Now we assume that $x_0 \in \bigcup_{f\in C^*\setminus\{0\}} V_f(A, F)$. Then, there exists $f_0 \in C^* \setminus \{0\}$ such that

$$\{x_0\} = V_{f_0}(A, F) = H(f_0).$$

Since $C^\sharp \neq \emptyset$, let $g \in C^\sharp$ and set

$$f_n = f_0 + (1 \setminus n)g.$$

Then, it is obvious that $f_n \in C^\sharp$. Because $Y^*$ together with $\beta(Y^*, Y)$ is a topological vector space and so it follows from the continuity of scalar multiplication and sum we get that $\{f_n\}$ converges to $f_0$ with respect to the topology $\beta(Y^*, Y)$. In the other words, for any neighborhood $U$ of 0 with respect to $\beta(Y^*, Y)$, there exist bounded subsets $B_i \subset Y(i = 1, 2, ..., m)$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{m}\{f \in Y^* : \sup_{y\in B_i} \mid f(y) \mid \ < \varepsilon\} \subset U.$$

Since $B_i$ is bounded and $g \in Y^*$, $\mid g(B_i) \mid$ is bounded for $i = 1, 2, ..., m$. Thus, there exists $N$ such that

$$\sup_{y\in B_i} \mid (1 \setminus n)g(y) \mid \ < \varepsilon, \quad i = 1, 2, ..., m, n \geq N.$$

Hence $(1 \setminus n)g(y) \in U$, that is, $f_n - f_0 \in U$. This means that $\{f_n\}$ converges to $f_0$ with respect to $\beta(Y^*, Y)$.

Since, for each natural number $n$, the set $V_{f_n}(A, F)$ is nonempty we can choose $x_n \in V_{f_n}(A, F) \subset A$. Hence $\{x_n\}$ is a sequence in the compact set $A$ and so there exists a converges subsequence $x_{n_k}$ of $\{x_n\}$. Let $x_{n_k} \to x$. Because $H$ is closed, $(f_{n_k}, x_{n_k}) \in Graph(H)$ and $(f_{n_k}, x_{n_k}) \to (f_0, x)$ then

$$x \in H(f_0) = V_{f_0}(F, A).$$

Hence, since $x_0 \in V_{f_0}(A, F)$ and $V_{f_0}(A, F)$ is singleton we obtain

$$x_0 = x \in cl(\bigcup_{f \in C^\sharp} V_f(A, F))$$

and this completes the proof. □

**Theorem 2.11.** *Let $X, Y, A, C, \psi, \varphi, F$ be as in Theorem 2.10. Then, the set $V(A, F)$ is connected.*

*Proof.* Define the set-valued mapping $H : C^\sharp \to 2^A$ by

$$H(f) = V_f(A, F), \quad \mathrm{f} \in C^\sharp.$$

By Lemma 2.9, for each$f \in C^* \setminus \{0\}, V_f(A, F) \neq \emptyset$. It is easy to see that $C^\sharp$ is a convex set. Hence, it is a path connected set. It is easy from our hypothesis that the mapping $H$ is upper semicontinuous on $C^\sharp$. We show that, $\bigcup_{f \in C^\sharp} V_f(A, F)$ is connected.
Suppose that $\bigcup_{f \in C^\sharp} V_f(A, F)$ is separated; that is, there exist open sets $A$ and $B$ such that

$$\bigcup_{f \in C^\sharp} V_f(A, F) \subset A \cup B, \quad A_1 = A \cap (\bigcup_{f \in C^\sharp} V_f(A, F)),$$

$$B_1 = B \cap (\bigcup_{f \in C^\sharp} V_f(A, F)), \quad A_1 \cap B_1 = \emptyset.$$

Let $f \in C^\sharp$. Since $H(f)$ is connected by Lemma 2.9 and

$$\emptyset \neq H(f) \subset A_1 \cup B_1,$$

it follows that either $H(f) \subset A_1$ or $H(f) \subset B_1$, but not both. Let

$$V = \{f \in C^\sharp | H(f) \subset A_1\},$$

$$W = \{f \in C^\sharp | H(f) \subset B_1\}.$$

Clearly,

$$C^\sharp = V \cup W.$$

Let $\overline{f}$ belong to the closure of $V$. If $\overline{f} \in W$, we have

$$H(\overline{f}) \subset B_1 \subset B.$$

Thus, by the upper semicontinuity of $H$ there exists a neighborhood $U$ of $\overline{f}$ such that

$$H(f) \subset B_1, \quad \forall f \in U \cap V.$$

But $U$ must contain $f$ of $W$, since $\overline{f}$ in the closure of $V_f(A, F)$. While $H(f) \subset A_1$ for such points, contradicting $H(f) \subset B_1$. Hence,

$$V \cap \overline{W} = \emptyset.$$

That is, the intersection of the set $V$ with the closure of $W$ is empty. Similarly, no point of the closure of $V$ belongs to $W$. We then have that

$$C^\sharp = V \cup W,$$

is separated which is contradicted by the connectedness of $C^\sharp$ ( note that $C^\sharp$ is convex). Consequently, the set

$$\bigcup_{f\in C^\sharp} V_f(A,F)$$

is connected. Hence it follows from Theorem 2.10 that

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset cl(\bigcup_{f\in C^\sharp} V_f(A,F)).$$

and so $V(A,F)$ is a connected set. □

## References


[10] Y.H. Cheng, On the connectedness of the solution set for the weak vector variational inequality, Journal of Mathematical Analysis and Applications 260 (2001) 1–5.
[2] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.
[3] K. Fan, Some properties of convex sets related to fixed point theorems, Mathematische Annalen 266(1984)519–537.
[4] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[5] X.H. Gong, Connectedness of the solution sets and scalarization for vector equilibrium problems, Journal of Optimization Theory and Applications 133 (2007) 151–161.
[6] X.H. Gong, Efficiency and Henig efficiency for vector equilibrium problems, Journal of Optimization Theory and Applications 108 (2001)139–154.
[7] X.H. Gong, Strong vector equilibrium problems, Journal of Global Optimization Theory 36(2006) 339–349.
[8] X.H. Gong, W.T. Fu, W. Liu, Superefficiency for a vector equilibrium in locally convex topological vector spaces. In: Giannessi, F. (ed.) Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, pp. 233–252. Kluwer, Dordrecht (2000).
[9] X.H. Gong, J.C. Yao, Connectedness of the set of efficient solutions for generalized systems, Journal of optimization 138 (2008) 189–196.
[10] G.M. Lee, D.S. Kim, B.S. Lee, N.D. Yun, Vector variational inequality as a tool for studying vector optimization problems. Nonlinear Analaysis: Theory Methods Applications 34 (1998) 745–765
[11] Q.Y. Liu, X.J. Long, N.J. Huang, Connectedness of the sets of weak efficient solutions for generalized vector equilibrium problems, Mathematica Slovaca 62 (2012) 123–136.
[12] X.J. Long, J.W. Peng, Connectedness and compactness of weak efficient solutions for vector equilibrium problems, Bulltien of Korean Mathematical Soceity 48 (2011) 1225–1233.
[13] W. Rudin, Functional Analysis, McGraw-Hill company, United States of America, 1972.
[14] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.