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Decomposition of the Laplacian and Pluriharmonic Bloch Functions

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Abstract. We decompose the invariant Laplacian of the deleted unit complex ball by two directional Laplacians, tangential one and radial one. We give a characterization of pluriharmonic Bloch function in terms of the growth of these Laplacians.

1. Preliminaries

Let $B = B_n$ denote the open unit ball of $\mathbb{C}^n$ and S denote the boundary of B : $S = \{z \in \mathbb{C}^n : |z| = 1\}$.

The group of all automorphisms, that is, one to one biholomorphic onto self-maps, of B is denoted by $\mathcal{M}$. It consists of all maps of the form $U\varphi_a$, where U is a unitary operator of $\mathbb{C}^n$ and φ_a is defined by

$$\varphi_a(z) = \begin{cases} \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - <z,a>}, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0. \end{cases}$$

Here $<\ ,\ >$ is the Hermitian inner product of $\mathbb{C}^n$: $< z,\ w >= \sum_{j=1}^n z_j \bar{w}_j,\ z,\ w \in \mathbb{C}^n$, $P_a z$ is the projection of $\mathbb{C}^n$ onto the subspace generated by B :

$$P_a z = \frac{< z, a >}{< a, a >} a, \quad \text{if } a \neq 0 \quad \text{and} \quad P_0 z = 0,$$

and $Q_a(z) = z - P_a z$.

Δ denotes the complex Laplacian of $\mathbb{C}^n$: $\Delta = 4 \sum_{j=1}^n D_j \bar{D}_j$, where $D_j = \frac{\partial}{\partial z_j}$ and $\bar{D}_j = \frac{\partial}{\partial \bar{z}_j}$, $j = 1, 2, \ldots, n$. $\widetilde{\Delta}$ denotes the $\mathcal{M}$-invariant Laplacian defined for $f \in C^2(B)$ and $a \in B$ by

$$\widetilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0).$$

A $C^2(B)$ function f is said to be $\mathcal{M}$-harmonic if $\widetilde{\Delta} f = 0$ in B, pluriharmonic if $D_j \bar{D}_k f = 0$ in B for all $j, k = 1, .., n$.

Compared with the Laplace operator Δ, the Laplace-Beltrami operator is, in a sense, more convenient in descriving properties (of functions) related to the geometric structure of the domain. Accompanying

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the reason, $\mathcal{M}$-harmonic function theory has been progressively developing recent years. It gives rise to interesting and meaningful results not only in the field of function theory but also in the field of potential theory and operator theory etc. See [1 ~ 20] for some of the recent developments of this vein on B.

For $a \in B$, let f_a be defined by $f_a(\lambda) = f(\lambda a),\ \lambda \in B_1$. Then it is straightforward and known that

$$\widetilde{\Delta} f(a) = (1 - r^2)\{\Delta f(a) - \Delta f_a(1)\} \tag{1.1}$$

for $f \in C^2(B)$ (see for example [22, 4.1.3]). Concerning (1.1), W. Rudin (in [22, 19.3.16]) asked why $\widetilde{\Delta}$ is a difference of two ordinary Laplacians. We, in this note, give a viewpoint of (1.1) in connection with the question.

We refer to [21] and [22] for undefined terminologies and notations.

2. Decomposition of $\widetilde{\Delta}$

For $f \in C^2(B)$ and $z = r\zeta \in B$ with $0 < r < 1, \zeta \in S$, the complex radial Laplacian of f, denoted by $\Delta_{rad} f(z)$, is defined to be the Laplacian of the function $\lambda \to f(z + \lambda\zeta)$ at the origin of $\mathbb{C}$ (see [22, 17.3.2]). And the tangential Laplacian of f, denoted by $\Delta_{tan} f(z)$, is defined to be $\Delta_{tan} f(z) = \Delta f(z) - \Delta_{rad} f(z)$. Then Δ can be decomposed into the complex tangential Laplacian and the complex radial Laplacian as

$$\Delta = \Delta_{tan} + \Delta_{rad}.$$

We have a similar decomposition for $\widetilde{\Delta}$ as the following.

Theorem 2.1. *For $f \in C^2(B)$,*

$$\widetilde{\Delta} f(z) = (1 - |z|^2)\Delta_{tan} f(z) + (1 - |z|^2)^2 \Delta_{rad} f(z), \quad z \in B - \{0\}. \tag{2.1}$$

Proof. Let $a = r\zeta, 0 < r < 1, \zeta = (\zeta_1, ..., \zeta_n) \in S$ and Let $F(\lambda) = f(a + \lambda\zeta), \lambda \in B_1$. Then calculating $\Delta F(0)$ via chain rule gives the identity

$$\Delta_{rad} f(a) = 4 \sum_{j,k=1}^{n} \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a) \tag{2.2}$$

and from the fact $\Delta_{tan} = \Delta - \Delta_{rad}$ follows

$$\Delta_{tan} f(a) = 4 \sum_{j,k=1}^{n} (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a), \tag{2.3}$$

where $\delta_{k,j} = 1$ if $k = j$ and $\delta_{k,j} = 0$ if $k \neq j$.

On the other hand, it is straightforward and known [22, 4.1.3]) that

$$\widetilde{\Delta} f(a) = 4(1 - |a|^2) \sum_{i,j=1}^{n} (\delta_{i,j} - \bar{a}_i a_j)(\bar{D}_i D_j f)(a) \tag{2.4}$$

for $f \in C^2(B)$ and $a = (a_1, a_2, ..., a_n) \in B$.

Comparing (2.2), (2.3), with (2.4), it is simple to check that

$$\widetilde{\Delta} f(a) = 4(1 - |a|^2) \sum_{j,k=1}^{n} (\delta_{k,j} - \zeta_k \bar{\zeta}_j + \zeta_k \bar{\zeta}_j - |a|^2 \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a)$$

$$= 4(1 - |a|^2) \sum_{j,k=1}^{n} (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a) + 4(1 - |a|^2)^2 \sum_{j,k=1}^{n} \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a)$$

$$= (1 - |a|^2)\Delta_{tan} f(a) + (1 - |a|^2)^2 \Delta_{rad} f(a).$$

□

It is known that f is pluriharmonic in B if and only if $\Delta f = 0 = \widetilde{\Delta} f$ in B [22, 4.4.9]. Since a $C^2(B)$ function f satisfying $\Delta f = 0 = \widetilde{\Delta} f$ in $B - \{0\}$ also satisfies $\Delta f = 0 = \widetilde{\Delta} f$ at 0, we have

Corollary 2.2. *A $C^2(B)$ function f is pluriharmonic in B if and only if $\Delta_{tan} f = 0 = \Delta_{rad} f$ in $B - \{0\}$.*

Remark 2.3. *(1). Straightforward calculation shows $|a|^2 \Delta_{rad} f(a) = \Delta f_a(1)$, so that*

$$\begin{aligned}
\widetilde{\Delta} f(a) =& (1-|a|^2)\Delta_{tan} f(a) + (1-|a|^2)^2 \Delta_{rad} f(a) \\
=& (1-|a|^2)\left\{\Delta_{tan} f(a) + \Delta_{rad} f(a) - |a|^2 \Delta_{rad} f(a)\right\} \\
=& (1-|a|^2)\left\{\Delta f(a) - \Delta f_a(1)\right\},
\end{aligned}$$

which means the representations (1.1) and (2.1) have same nature. We stress here that the equation (2.1) is a natural and geometrical expression of $\widetilde{\Delta}$ in the sense that $\widetilde{\Delta}$ can be decomposed into two (orthogonal) directional Laplacians with the growth properly controlled (by powers of $1 - r^2$):

$$\widetilde{\Delta} = (1-r^2)\Delta_{tan} + (1-r^2)^2 \Delta_{rad} \quad in \quad B - \{0\}. \tag{2.5}$$

This observation may give an answer to the question of Rudin.

(2). If $n = 1$, then $\Delta_{tan} = 0$ and $\Delta_{rad} = \Delta$ so that Δ can be expressed as

$$\Delta = \frac{\widetilde{\Delta}}{(1-r^2)^2}$$

as is well-known, and is $n = 1$ case of

$$\Delta = \frac{\widetilde{\Delta}}{1-r^2} + r^2 \Delta_{rad} = \frac{\widetilde{\Delta}}{(1-r^2)^2} + \frac{r^2}{1-r^2}\Delta_{tan}$$

which is equivalent to (2.5).

(3) If $f \in C^2(B)$ and $a = r\zeta, 0 \le r < 1, \zeta \in S$, then the Taylor expansion gives

$$\Delta_{rad} f(a) = \lim_{\rho \to 0} \frac{4}{\rho^2} \int_0^{2\pi} \left\{ f(a + \rho e^{i\theta}\zeta) - f(a) \right\} \frac{d\theta}{2\pi}.$$

From this we see that Δ_{rad} commutes with the action of the unitary group. Since Δ commutes with the action of the unitary group, we see that $\Delta, \Delta_{rad}, \Delta_{tan}, \widetilde{\Delta}$ all commutes with the action of the unitary group. On the other hand, if f is radial then

$$\Delta_{rad} f = \frac{\partial f^2}{\partial^2 r} + \frac{1}{r}\frac{\partial f}{\partial r} \text{ and } \Delta_{tan} f = \frac{2(n-1)}{r}\frac{\partial f}{\partial r}.$$

3. Pluriharmonic Bloch Function

The "properly controlled growth" in Remark 2.3-(1) concerned with the growth principle of "twice as well (regular) in the complex tangential direction". We check this for Bloch functions in this section. We let D and ∇ denote respectively the complex gradient of $\mathbb{C}^n$ and the real gradient of $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$:

$$\begin{aligned}
&D = (D_1, D_2, .., D_n), \quad \bar{D} = (\bar{D}_1, \bar{D}_2, .., \bar{D}_n), \\
&\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, .., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right), \quad z_j = x_j + iy_j, \quad j = 1, 2, .., n.
\end{aligned}$$

$\mathcal{M}$-invariant form of ∇ is defined as

$$\widetilde{\nabla} f(a) = \nabla (f \circ \varphi_a)(0), \quad a \in B,$$

for $f \in C^1(B)$. Let R denote the radial derivative : $R = \sum_{j=1}^n z_j D_j$, and let $\bar{R} = \sum_{j=1}^n \bar{z}_j \bar{D}_j$. Let $T_{k,j}$ be usual tangential derivatives $T_{k,j} = \bar{z}_k D_j - \bar{z}_j D_k$ and $\bar{T}_{k,j} = z_k \bar{D}_j - z_j \bar{D}_k$.

Theorem 3.1. *Let f be pluriharmonic in $B = B_n$ with $n \geq 2$ and let $B' = B - \{0\}$. Then the following are mutually equivalent.*

(a) *f is a Bloch function, which means by definition* $\sup_{z \in B} \left|\widetilde{\nabla} f(z)\right| < \infty$

(b) $\sup_{z \in B} \widetilde{\Delta} \left|f(z)\right|^2 < \infty$

(c) $\sup_{z \in B} (1 - |z|^2)^2 \left(|Rf(z)|^2 + |\bar{R}f(z)|^2\right) < \infty$

(d) $\sup_{z \in B'} (1 - |z|^2)^2 \Delta_{rad} |f(z)|^2 < \infty$

(e) $\sup_{z \in B'} (1 - |z|^2) \sum_{k<j} \left(\left|T_{k,j} f\right|^2 + \left|\bar{T}_{k,j} f\right|^2\right)(z) < \infty$

(f) $\sup_{z \in B'} (1 - |z|^2) \Delta_{tan} |f(z)|^2 < \infty$

Proof. That $(a) \iff (b)$ follows from elementary equality

$$4|\widetilde{\nabla} f|^2 = \frac{1}{2} \widetilde{\Delta} |f|^2 - Re\left(\bar{f} \widetilde{\Delta} f\right)$$

with $\mathcal{M}$-harmonicity of f. That $(b) \iff (c)$ follows from [10, Theorem 1]. Simple calculation shows for $r \neq 0$ that

$$\begin{aligned}(\Delta_{rad}|f|^2)(r\zeta) =& 4 \sum_{k,j=1}^{n} \zeta_k \bar{\zeta}_j \left(D_k f \bar{D}_j \bar{f} + D_k \bar{f} \bar{D}_j f + f D_k \bar{D}_j \bar{f} + \bar{f} D_k \bar{D}_j f\right)(r\zeta) \\ =& \frac{4}{r^2} (|Rf|^2 + |\bar{R}f|^2)(r\zeta) + (f \Delta_{rad} \bar{f} + \bar{f} \Delta_{rad} f)(r\zeta)\end{aligned}$$

which equals

$$\frac{4}{r^2} (|Rf|^2 + |\bar{R}f|^2)(r\zeta) \tag{3.1}$$

if f is pluriharmonic. Noting that $|Rf(z)| \leq |z||Df|$ and $|\bar{R}f(z)| \leq |z||\bar{D}f|$, (3.1) is uniformly bounded in a compact neighborhood of 0 in B, so that $(c) \iff (d)$ follows. When f is pluriharmonic, another simple calculation shows for $z \neq 0$ that

$$\begin{aligned}\Delta_{tan}|f|^2(z) =& \Delta|f|^2(z) - \Delta_{rad}|f|^2(z) \\ =& 4\left(|Df|^2 + |\bar{D}f|^2 - \frac{1}{|z|^2}|Rf|^2 - \frac{1}{|z|^2}|\bar{R}f|^2\right)(z) \\ =& \frac{4}{|z|^2} \sum_{k<j} \left(\left|T_{k,j} f\right|^2 + \left|\bar{T}_{k,j} f\right|^2\right)(z).\end{aligned} \tag{3.2}$$

Noting that the last quantity of (3.2) is uniformly bounded in a compact neighborhood of 0 in B, we have

$$\sup_{z \in B'} (1 - |z|^2) \Delta_{tan} |f(z)|^2 < \infty$$

if and only if

$$\sup_{z \in B'} (1 - |z|^2) \sum_{k<j} \left(\left|T_{k,j} f\right|^2 + \left|\bar{T}_{k,j} f\right|^2\right)(z) < \infty,$$

that is, $(e) \iff (f)$.

Since

$$
\begin{aligned}
&|z|^2\widetilde{\Delta}\left|f(z)\right|^2\\
=&4(1-|z|^2)^2\left(|Rf|^2+|\bar{R}f|^2\right)(z)+4(1-|z|^2)\left(\left|T_{k,j}f\right|^2+\left|\bar{T}_{k,j}f\right|^2\right)(z),
\end{aligned}
$$

that $(b) \implies (e)$ is obvious.

We are to prove $(e) \implies (c)$. Suppose f is pluriharmonic satisfying (e) . Fix $z \in B'$ for a moment. There is, say j such that $z_j \neq 0$. Take ζ having j-th coordinate $\frac{\bar{z_k}}{\sqrt{|z_j|^2+|z_k|^2}}$ and k-th coordinate $\frac{-\bar{z_j}}{\sqrt{|z_j|^2+|z_k|^2}}$ and other coordinates 0. Then $\zeta \in \partial B$ with $< z, \zeta >= 0$. Now let

$$
F(\lambda) = \bar{T}_{k,j} f(z + \lambda\zeta), \quad \lambda \in \mathbb{C},\ |\lambda|^2 < 1 - |z|^2 \tag{3.3}
$$

for simplicity. Then by the hypothesis (e)

$$
|F(\lambda)| \leq C\left(1 - |z|^2 - |\lambda|^2\right)^{-1/2}.
$$

Straightforward differentiation with plurisubharmonicity of f shows that $dF/d\bar{\lambda} = 0$, so that F is holomorphic in $|\lambda|^2 < 1 - |z|^2$. Applying the Cauchy formula, we have

$$
F'(0) = \frac{1}{2\pi i}\int_{|\lambda|^2=\frac{1-|z|^2}{2}} \frac{F(\lambda)}{\lambda^2}\, d\lambda,
$$

whence

$$
|F'(0)| \leq C(1 - |z|^2)^{-1}.
$$

But in our case of (3.3)

$$
T_{k,j}\bar{T}_{k,j} f(z) = \sqrt{|z_j|^2 + |z_k|^2}F'(0).
$$

Hence

$$
\left|T_{k,j}\bar{T}_{k,j} f(z)\right| \leq C(1 - |z|^2)^{-1}.
$$

Since f is pluriharmonic, then it follows by a direct computation that

$$
2(n-1)Rf = -\sum_{i\neq j} \bar{T}_{ij}T_{ij} f.
$$

Whence we have

$$
(1 - |z|^2)|Rf(z)| \leq C < \infty.
$$

By a similar argument we have

$$
(1 - |z|^2)|\bar{R}f(z)| = (1 - |z|^2)|R\bar{f}(z)| \leq C < \infty.
$$

We therefore have (c). $\square$

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