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On Some Applications of Noshiro-Warschawski's Theorem

Janusz Sokół[a], Mamoru Nunokawa[b], Nak Eun Cho[c], Huo Tang[d]

[a]*Faculty of Mathematics and Natural Sciences, University of Rzeszów, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland*
[b]*University of Gunma, Hoshikuki-Cho 798-8, Chuou-Ward, Chiba 260-0808, Japan*
[c]*Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea*
[d]*School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China*

Abstract. We apply Noshiro-Warschawski's theorem to prove that if $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and if $\left|\mathfrak{Re}\{zf''(z)\}\right| \le \alpha|z|^\alpha$ in $|z| < 1$, for some $\alpha > 0$, then $f(z)$ is univalent in $|z| < 1$. Also, applying Ozaki's condition, we obtain several sufficient conditions for functions to be p-valent or p-valently starlike function in $|z| < 1$.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of functions being in $\mathcal{H}$ and having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$. Let $\mathcal{A}_p \subset \mathcal{H}$ be the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (z \in \mathbb{D}). \tag{2}$$

So we have $\mathcal{A} = \mathcal{A}_1$. A function $f(z)$ which is analytic in a domain $D \subset \mathbb{C}$ is called p-valent in D if for every complex number w, the equation $f(z) = w$ have at most p roots in D and there will be a complex number w_0 such that the equation $f(z) = w_0$, has exactly p roots in D.

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Email addresses: jsokol@prz.edu.pl (Janusz Sokół), mamoru_nuno@doctor.nifty.jp (Mamoru Nunokawa), necho@pknu.ac.kr (Nak Eun Cho), thth2009@163.com (Huo Tang)

The well known Noshiro-Warschawski univalence condition (see [10] and [17]), indicates that if $f(z)$ is analytic in a convex domain $D \subset \mathbb{C}$ and

$$\Re\mathrm{e}\{e^{i\theta} f'(z)\} > 0 \quad (z \in D), \tag{3}$$

for some real θ, then $f(z)$ is univalent in D. S. Ozaki [11] extended the above result by showing that if $f(z)$ of the form (2) is analytic in a convex domain D and for some real θ we have

$$\Re\mathrm{e}\{e^{i\theta} f^{(p)}(z)\} > 0 \quad (z \in D),$$

then $f(z)$ is at most p-valent in D. Applying Ozaki's theorem, we find that if $f(z) \in \mathcal{A}_p$ and

$$\Re\mathrm{e}\{f^{(p)}(z)\} > 0 \quad (z \in \mathbb{D}), \tag{4}$$

then $f(z)$ is at most p-valent in $\mathbb{D}$. Condition (4) says that $f^{(p)}(z)$ is a Carathéodory function. For several interesting recent developments associated with Carathéodory functions, we refere to the articles [13–16].

In [6] it was proved that if $f(z) \in \mathcal{A}_p$, $p \geq 2$, and

$$|\arg\{f^{(p)}(z)\}| < \frac{3\pi}{4} \quad (z \in \mathbb{D}), \tag{5}$$

then $f(z)$ is at most p-valent in $\mathbb{D}$. Condition (5) says that $f^{(p)}(z)$ is a strongly Carathéodory function of order 3/2, see [13]. If $f \in \mathcal{A}$ satisfies

$$\Re\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{D}),$$

then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{D}$ and it is denoted by $f(z) \in \mathcal{S}^*$. It is known that $\mathcal{S}^* \subset \mathcal{S}$.

2. Main Results

Theorem 2.1. *If $f(z) \in \mathcal{H}$ with $f'(0) = 1$ and if*

$$\left|\Re\mathrm{e}\{zf''(z)\}\right| \leq \alpha |z|^{\alpha} \quad (z \in \mathbb{D}) \tag{6}$$

for some $\alpha > 0$, then $f(z)$ is univalent in $\mathbb{D}$.

Proof. Applying (6) gives

$$\begin{aligned}
\left|\Re\mathrm{e}\{f'(z) - 1\}\right| &= \left|\Re\mathrm{e}\{f'(z) - f'(0)\}\right| = \left|\Re\mathrm{e}\left\{\int_0^z f''(t)\mathrm{d}t\right\}\right| \\
&= \left|\Re\mathrm{e}\left\{\int_0^r f''(\rho e^{i\theta})e^{i\theta}\mathrm{d}\rho\right\}\right| = \left|\Re\mathrm{e}\left\{\int_0^r \rho e^{i\theta} f''(\rho e^{i\theta})\frac{1}{\rho}\mathrm{d}\rho\right\}\right| \\
&= \left|\Re\mathrm{e}\left\{\int_0^r tf''(t)\frac{\mathrm{d}\rho}{\rho}\right\}\right| = \left|\int_0^r \Re\mathrm{e}\{tf''(t)\}\frac{\mathrm{d}\rho}{\rho}\right| \\
&\leq \int_0^r \left|\Re\mathrm{e}\{tf''(t)\}\right|\frac{\mathrm{d}\rho}{\rho} \\
&\leq \int_0^r \frac{\alpha\rho^{\alpha}}{\rho}\mathrm{d}\rho = [\rho^{\alpha}]_0^r = r^{\alpha} < 1,
\end{aligned}$$

where $t = \rho e^{i\theta}$, $z = re^{i\theta}$ and $0 \leq \rho \leq r < 1$. Therefore,

$$\left|\Re\mathrm{e}\{f'(z) - 1\}\right| < 1 \quad (z \in \mathbb{D})$$

and $f'(z)$ satisfies condition (4), which implies the univalence of $f(z)$ in the unit disc $\mathbb{D}$. □

Corollary 2.2. *If $g(z) \in \mathcal{H}$ with $g'(0) \neq 0$ and if*

$$\left|\Re\mathrm{e}\left\{zg''(z)\right\}\right| \leq 2|z|^2 \quad (z \in \mathbb{D}), \tag{7}$$

then $g(z)$ is univalent in $\mathbb{D}$.

If we take $g(z) = z + a_2z^2$, then $zg''(z) = 2a_2z$ and condition (7) becomes

$$|\Re\mathrm{e}\{2za_2\}| \leq |z| \quad (z \in \mathbb{D}),$$

which is satisfied whenever $|2a_2| \leq 1$. Using this way, we can obtain the known and sharp result. If $g(z) = z + xz^{n+1}$, $n \in \mathbb{N}$, then condition (6), with $\alpha = n$, becomes

$$|\Re\mathrm{e}\{n(n+1)xz^n\}| \leq n|z|^n \quad (z \in \mathbb{D}),$$

which is satisfied whenever $|x| \leq 1/(n+1)$. Therefore, if $|x| \leq 1/n$, $n \in \mathbb{N} \setminus \{1\}$, then $h(z) = z + xz^n$ is univalent in $\mathbb{D}$.

Corollary 2.3. *If $g(z) \in \mathcal{H}$ with $g'(0) \neq 0$ and if*

$$\left|\Re\mathrm{e}\left\{\frac{zg''(z)}{g'(0)}\right\}\right| \leq \alpha|z|^\alpha \quad (z \in \mathbb{D}) \tag{8}$$

for some $\alpha > 0$, then $g(z)$ is univalent in $\mathbb{D}$.

Proof. If $g(z) = b_0 + b_1z + b_2z^2 + \cdots$, then

$$f(z) = \frac{g(z)}{g'(0)} = \frac{b_0}{b_1} + z + \frac{b_2}{b_1}z^2 + \cdots$$

with $f'(0) = 1$ and by (8), we have

$$\left|\Re\mathrm{e}\left\{zf''(z)\right\}\right| = \left|\Re\mathrm{e}\left\{\frac{zg''(z)}{g'(0)}\right\}\right| < \alpha|z|^\alpha \quad (z \in \mathbb{D})$$

for some $\alpha \geq 1$. Then Theorem 2.1 implies the univalence of $f(z)$ and $g(z)$ too, in the unit disc $\mathbb{D}$. □

Corollary 2.4. *Assume that $g(z) \in \mathcal{H}$ with $g'(0) \neq 0$. If there exists $0 < \alpha \leq 1$ such that*

$$\left|\Re\mathrm{e}\left\{\frac{zg''(z)}{g'(0)}\right\}\right| \leq \alpha|z| \quad (z \in \mathbb{D}), \tag{9}$$

then $g(z)$ is univalent in $\mathbb{D}$.

Proof. For $0 < \alpha \leq 1$ and $z \in \mathbb{D}$, we have $|z| \leq |z|^\alpha$. Hence

$$\left|\Re\mathrm{e}\left\{\frac{zg''(z)}{g'(0)}\right\}\right| \leq \alpha|z| \leq \alpha|z|^\alpha \quad (z \in \mathbb{D}). \tag{10}$$

Then Corollary 2.3 implies the univalence of $f(z)$ in the unit disc $\mathbb{D}$. □

On the other hand, we have the following known univalence condition.

Lemma 2.5. *[12] Let $f(z) = z + a_2z^2 + \ldots$ be analytic in the unit disc and suppose that*

$$|f''(z)| < 1 \quad (z \in \mathbb{D}). \tag{11}$$

Then $f(z)$ is univalent in $\mathbb{D}$.

Remark 1. If we denote $z = |z|e^{i\gamma}$, $f''(z) = |f''(z)|e^{i\beta}$, then (6) becomes

$$\left|\Re\mathfrak{e}\{|z|e^{i\gamma}|f''(z)|e^{i\beta}\}\right| \leq \alpha|z|^{\alpha} \quad (z \in \mathbb{D}).$$

Hence for $\alpha = 1$, we have that

$$|f''(z)||\cos(\beta + \gamma)| \leq 1 \quad (z \in \mathbb{D}) \tag{12}$$

implies the univalence of $f(z)$ in $\mathbb{D}$. So Theorem 2.1 is a generalization of Lemma 2.5. However condition (12) is not convenient.

Remark 2. Putting

$$h(z) = e^{-i\alpha} f(ze^{i\alpha}) = z + a_2 e^{i\alpha} z^2 + \cdots = z + i|a_2|z^2 + \cdots,$$

where $\alpha = \pi/2 - \arg\{a_2\}$. Therefore without loss of generality, we can consider the coefficient a_2 in Lemma 2.5 which is a pure imaginary number.

Lemma 2.6. *[9, Theorem 2, p. 93] Let $f(z) \in \mathcal{A}_p$, $f^{(k)}(z) \neq 0$ in $0 < |z| < 1$ for $k = 1, 2, \ldots, p$ and suppose that*

$$|\arg\left\{f^{(p)}(z)\right\}| < \frac{\pi}{2}\left(1 + \frac{1}{\pi}\log p\right) \quad (z \in \mathbb{D}). \tag{13}$$

Then $f(z)$ is p-valent in $\mathbb{D}$.

Theorem 2.7. *Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$, $f^{(k)}(z) \neq 0$ in $0 < |z| < 1$ for $k = 1, 2, 3, \ldots, p$ and suppose that*

$$\left|\Im\mathfrak{m}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right| \leq \frac{\pi}{2}\left\{1 + \frac{2}{\pi}\log p\right\}\alpha|z|^{\alpha} \quad (z \in \mathbb{D}), \tag{14}$$

for some $\alpha > 0$. Then $f(z)$ is p-valent in $\mathbb{D}$.

Proof. It follows that

$$\begin{aligned}
|\arg f^{(p)}(z)| &= \left|\Im\mathfrak{m}\left\{\log\{f^{(p)}(z)\} - \log\{f^{(p)}(0)\}\right\}\right| \\
&\leq \int_0^r \left|\Im\mathfrak{m}\left\{\frac{tf^{(p+1)}(t)}{f^{(p)}(t)}\right\}\right| \frac{1}{\rho}\mathrm{d}\rho \leq \frac{\pi}{2}\left\{1 + \frac{2}{\pi}\log p\right\}\int_0^r \frac{\alpha\rho^{\alpha}}{\rho}\mathrm{d}\rho \\
&< \frac{\pi}{2}\left\{1 + \frac{2}{\pi}\log p\right\},
\end{aligned}$$

where $z = re^{i\theta}$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r < 1$. Applying Lemma 2.6 completes the proof. □

A function $f(z) \in \mathcal{A}_p$, $p \in \mathbb{N}$, is said to be *p-valently starlike of order* α, $0 \leq \alpha < p$, if

$$\Re\mathfrak{e}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{D}).$$

The class of all such functions is usually denoted by $\mathcal{S}_p^*(\alpha)$. For $p = 1$, we receive the well known class of normalized starlike univalent functions $\mathcal{S}^*(\alpha)$ of order α, $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$. For further properties of starlike functions and other functions having a geometric property, we refer to [3]. In [7, 8] the second author proved the following theorems.

Lemma 2.8. *[7] Let $f(z) \in \mathcal{A}_p$, with $p \geq 2$ and suppose that*

$$\Re\mathfrak{e}\left\{f^{(p)}(z)\right\} > -\frac{p!\log\{4/e\}}{2\log\{e/2\}} \quad (z \in \mathbb{D}). \tag{15}$$

Then $f(z)$ is p-valently starlike in $\mathbb{D}$.

Lemma 2.9. *[8] Let $f(z) \in \mathcal{A}_p$, with $p \geq 3$ and suppose that*

$$\Re\mathrm{e}\left\{f^{(p)}(z)\right\} > -\frac{p!\,[1-4(\log\{4/e\})\log\{e/2\}]}{4(\log\{4/e\})\log\{e/2\}} \quad (z \in \mathbb{D}). \tag{16}$$

Then $f(z)$ is p-valent in $\mathbb{D}$.

Theorem 2.10. *Let $f(z) \in \mathcal{A}_p$, with $p \geq 2$ and suppose that*

$$\left|\Re\mathrm{e}\left\{zf^{(p+1)}(z)\right\}\right| \leq \frac{p!\alpha|z|^{\alpha}}{2\log\{e/2\}} \quad (z \in \mathbb{D}), \tag{17}$$

for some $\alpha > 0$. Then $f(z)$ is p-valently starlike in $\mathbb{D}$.

Proof. Applying (17), it follows that

$$\begin{aligned}
\left|\Re\mathrm{e}\left\{f^{(p)}(z) - f^{(p)}(0)\right\}\right| &= \left|\Re\mathrm{e}\left\{\int_0^r tf^{(p+1)}(t)\frac{1}{\rho}\mathrm{d}\rho\right\}\right| \\
&\leq \int_0^r \left|\Re\mathrm{e}\left\{tf^{(p+1)}(t)\right\}\right|\frac{1}{\rho}\mathrm{d}\rho \leq \frac{p!}{2\log\{e/2\}}\int_0^r \frac{\alpha\rho^{\alpha}}{\rho}\mathrm{d}\rho \\
&< \frac{p!}{2\log\{e/2\}},
\end{aligned}$$

where $z = re^{i\theta}$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r < 1$. Therefore, applying Lemma 2.8 shows that $f(z)$ is p-valently starlike in $\mathbb{D}$. □

Applying the same method as in the proof of Theorem (2.10) and the result of Lemma 2.9, we obtain the following Theorem 2.11.

Theorem 2.11. *Let $f(z) \in \mathcal{A}_p$, with $p \geq 3$ and suppose that*

$$\left|\Re\mathrm{e}\left\{zf^{(p+1)}(z)\right\}\right| \leq \frac{p!\alpha|z|^{\alpha}}{4(\log\{4/e\})\log\{e/2\}} \quad (z \in \mathbb{D}), \tag{18}$$

for some $\alpha > 0$. Then $f(z)$ is p-valent in $\mathbb{D}$.

Theorem 2.12. *Let $f(z) \in \mathcal{A}_p$, with $p \geq 2$ and suppose that*

$$\left|f^{(p+1)}(z)\right| \leq \frac{p!}{2\log\{e/2\}} \quad (z \in \mathbb{D}). \tag{19}$$

Then $f(z)$ is p-valently starlike in $\mathbb{D}$.

Proof. Applying (19), it follows that

$$\begin{aligned}
\left|\Re\mathrm{e}\left\{f^{(p)}(z) - f^{(p)}(0)\right\}\right| &= \left|\Re\mathrm{e}\left\{\int_0^r f^{(p+1)}(\rho e^{i\theta})e^{i\theta}\mathrm{d}\rho\right\}\right| \\
&\leq \int_0^r \left|f^{(p+1)}(\rho e^{i\theta})e^{i\theta}\right|\mathrm{d}\rho \leq \int_0^r \frac{p!}{2\log\{e/2\}}\mathrm{d}\rho \\
&< \frac{p!}{2\log\{e/2\}},
\end{aligned}$$

where $z = re^{i\theta}$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r < 1$. Therefore, applying Lemma 2.8 shows that $f(z)$ is p-valently starlike in $\mathbb{D}$. □

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