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A Family of the Incomplete Hypergeometric Functions and Associated Integral Transform and Fractional Derivative Formulas

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Abstract. Recently, Srivastava *et al.* [*Integral Transforms Spec. Funct.* **23** (2012), 659–683] introduced the incomplete Pochhammer symbols that led to a natural generalization and decomposition of a class of hypergeometric and other related functions as well as to certain potentially useful closed-form representations of definite and improper integrals of various special functions of applied mathematics and mathematical physics. In the present paper, our aim is to establish several formulas involving integral transforms and fractional derivatives of this family of incomplete hypergeometric functions. As corollaries and consequences, many interesting results are shown to follow from our main results.

1. Introduction, Definitions and Preliminaries

Throughout this presentation, we shall denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. We also set

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}; \quad \mathbb{Z}^- := \{-1,-2,-3,\cdots\}; \quad \mathbb{N} := \{1,2,3,\cdots\} = \mathbb{N}_0 \setminus \{0\},$$

In terms of the familiar (Euler's) gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\,\mathrm{d}t & \big(\Re(z) > 0\big) \\[2ex] \dfrac{\Gamma(z+n)}{\displaystyle\prod_{j=0}^{n-1}(z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}), \end{cases} \tag{1}$$

a generalized binomial coefficient $\dbinom{\lambda}{\mu}$ is defined (for real or complex parameters λ and μ) by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \quad (\lambda, \mu \in \mathbb{C}), \tag{2}$$

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so that, in the special case when

$$\mu = n \qquad (n \in \mathbb{N}_0; \;\; \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

we have

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}_0). \tag{3}$$

Here, and in what follows, $(\lambda)_n$ denotes the Pochhammer symbol which is defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases} \tag{4}$$

The familiar incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ have proved to be important for physicists and engineers as well as mathematicians. These widely-investigated incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, defined by

$$\gamma(s,x) := \int_0^x t^{s-1}\, e^{-t}\, \mathrm{d}t \qquad \big(\Re(s) > 0;\; x \geqq 0\big) \tag{5}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}\, e^{-t}\, \mathrm{d}t \qquad \big(x \geqq 0;\; \Re(s) > 0 \quad \text{when} \;\; x = 0\big), \tag{6}$$

respectively, satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s) \qquad \big(\Re(s) > 0;\; x \geqq 0\big). \tag{7}$$

The function $\Gamma(z)$, and its incomplete versions $\gamma(s,x)$ and $\Gamma(s,x)$, are known to play important rôles in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, for example, [1], [5], [18] and [29]; see also the recent papers [10], [23], [24], [26], [27], [16], [17], [34], [36], [37] and [38]).

The theory of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, as a part of the theory of confluent hypergeometric functions, received its presumably first systematic exposition by Tricomi [41] in the early 1950s. On the other hand, in his investigation of asymptotic expansions of a family of branch-cut integrals occurring in diffraction theory by means of the Wiener-Hopf technique, Kobayashi (see [13] and [14]) encountered an integral representing a function class $\Gamma_m(u,v)$ $(m \in \mathbb{N}_0)$, which was referred to as a class of generalized gamma functions occurring in diffraction theory. Subsequently, by using many different functions of hypergeometric type in the kernel, various further generalizations and extensions of Kobayashi's integral were studied by several authors including (for example) Al-Musallam and Kalla (see [3] and [4]), Srivastava *et al.* [33], Prieto *et al.* [19], and others. In fact, Srivastava *et al.* [33] made the unnoticed observation that the so-called generalized gamma function $\Gamma_m(u,v)$ is closely related as follows [33, p. 932, Eq. (1.6)]:

$$\Gamma_m(u,v) = v^{u-m}\, \Gamma(u)\Psi(u, u-m+1; v) = \Gamma(u)\Psi(m, 1-u+m; v) \tag{8}$$

with the Tricomi function $\Psi(a,c;z)$, which provides a solution of the confluent hypergeometric equation:

$$z\, \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (c-z)\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0. \tag{9}$$

More importantly, Srivastava *et al.* [33] made use of the Fox-Wright hypergeometric function ${}_p\Psi_q$ in the kernel of the aforementioned Kobayashi's integral for $\Gamma_m(u,v)$ in order to present a systematic and unified study of these gamma-type functions, together with their applications in the theory of probability and statistics. Recently, Srivastava *et al.* [27] introduced and systematically studied several fundamental properties and characteristics of a family of two potentially useful and generalized incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, which are defined as follows [27, p. 675, Eqs. (4.1) and (4.2)]:

$$ {}_p\gamma_q[z] = {}_p\gamma_q\left[\begin{array}{c} (a_1, x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z\right] := \sum_{n=0}^{\infty} \frac{(a_1; x)_n \, (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{10} $$

and

$$ {}_p\Gamma_q[z] = {}_p\Gamma_q\left[\begin{array}{c} (a_1, x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z\right] := \sum_{n=0}^{\infty} \frac{[a_1; x]_n \, (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{11} $$

where $(a_1; x)_n$ and $[a_1; x]_n$ are interesting generalizations of the Pochhammer symbol $(\lambda)_n$, which are defined by

$$ (\lambda; x)_\nu := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \qquad (x \geqq 0;\ \lambda, \nu \in \mathbb{C}) \tag{12} $$

and

$$ [\lambda; x]_\nu := \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \qquad (x \geqq 0;\ \lambda, \nu \in \mathbb{C}) \tag{13} $$

in terms of the incomplete gamma type functions $\gamma(\lambda, x)$ and $\Gamma(\lambda, x)$. These incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ satisfy the following decomposition relation:

$$ (\lambda; x)_\nu + [\lambda; x]_\nu = (\lambda)_\nu \qquad (x \geqq 0;\ \lambda, \nu \in \mathbb{C}). $$

In the definitions (5), (6), (10), (11), (12) and (13), and throughout our present investigation, the argument $x \geqq 0$ is independent of the argument $z \in \mathbb{C}$ which occurs in the definitions (1), (10) and (11) and also elsewhere in this paper (see [35]). Moreover, as already pointed out by Srivastava *et al.* [27, p. 675, Remark 7], since

$$ |(\lambda; x)_\nu| \leqq |(\lambda)_\nu| \quad \text{and} \quad |[\lambda; x]_\nu| \leqq |(\lambda)_\nu| \qquad (x \geqq 0;\ \lambda, \nu \in \mathbb{C}), \tag{14} $$

the precise (sufficient) conditions under which the infinite series in the definitions (10) and (11) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ (see, for details, [20, pp. 72–73] and [28, p. 20]). Indeed, in their special case when $x = 0$, ${}_p\Gamma_q$ $(p, q \in \mathbb{N}_0)$ would reduce immediately to the extensively-investigated generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$. Furthermore, as an immediate consequence of the definitions (10) and (11), we have the following decomposition formula (see, for details, [27]):

$$ \begin{aligned} &{}_p\gamma_q\left[\begin{array}{c} (a_1, x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z\right] + {}_p\Gamma_q\left[\begin{array}{c} (a_1, x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z\right] \\ &\qquad = {}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z\right] \end{aligned} \tag{15} $$

in terms of the familiar generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$.

Definition 1 below makes use of the classical orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ defined by (see, for details, [39, Chapter 4] and [30, Chapters 1 and 2])

$$\begin{aligned} P_n^{(\alpha,\beta)}(z) &= (-1)^n \, P_n^{(\beta,\alpha)}(-z) \\ &= \binom{\alpha+n}{n} {}_2F_1\left[\begin{array}{c} -n, \alpha+\beta+n+1; \\ \alpha+1; \end{array} \frac{1-z}{2}\right], \end{aligned} \tag{16}$$

where ${}_2F_1$ denotes the familiar Gauss hypergeometric function.

Definition 1. (see, for example, [8, p. 501]) The Jacobi transform of a function $f(z)$ is defined as follows:

$$\mathbb{J}^{(\alpha,\beta)}[f(z);n] = \int_{-1}^{1} (1-z)^{\alpha}(1+z)^{\beta} \, P_n^{(\alpha,\beta)}(z) \, f(z)dz \tag{17}$$

$$\Big(\min\{\Re(\alpha), \Re(\beta)\} > -1; \; n \in \mathbb{N}_0\Big),$$

provided that the function $f(z)$ is so constrained that the integral in (17) exists.

The Jacobi transform of the power function $z^{\rho-1}$ (see, for example, [25, Eq. (20)]) is given by

$$\begin{aligned} &\int_{-1}^{1} (1-z)^{\xi-1}(1+z)^{\eta-1} \, P_n^{(\alpha,\beta)}(z) \, z^{\rho-1} \, \mathrm{d}z \\ &= 2^{\xi+\eta-1}\binom{\alpha+n}{n} B(\xi,\eta) \\ &\quad \cdot F_{1:1;0}^{1:2;1}\left[\begin{array}{rr} \xi: & -n, \alpha+\beta+n+1; 1-\rho; \\ \xi+\eta: & \alpha+1; \text{——}; \end{array} 1,2\right] \end{aligned} \tag{18}$$

$$\Big(\min\{\Re(\xi), \Re(\eta)\} > 0; \; \rho \in \mathbb{C}; \; n \in \mathbb{N}_0\Big),$$

where $F_{q:m;v}^{p:\,\ell;u}$ denotes the familiar Kampé de Fériet function (see, for details, [28, p. 27 *et seq.*]). Indeed, in its *further* special case when $\rho = m+1 \;\; (m \in \mathbb{N}_0)$, (18) yields the following well-known result for the Jacobi transform of $z^m \;\; (m \in \mathbb{N}_0)$, which is given by (see, for example, [20, p. 261, Eqs. (14) and (15)])

$$\begin{aligned} &\mathbb{J}^{(\alpha,\beta)}[z^m;n] \\ &:= \int_{-1}^{1} (1-z)^{\alpha}(1+z)^{\beta} \, P_n^{(\alpha,\beta)}(z) \, z^m \, \mathrm{d}z \\ &= \begin{cases} 0 & (m = 0,1,2,\cdots,n-1) \\ 2^{\alpha+\beta+n+1} \, B(\alpha+n+1, \beta+n+1) & (m = n) \\ 2^{\alpha+\beta+n+1}\dbinom{m}{n} \, B(\alpha+n+1,\beta+n+1) & \\ \qquad \cdot {}_2F_1\left[\begin{array}{c} n-m, \alpha+n+1; \\ \alpha+\beta+2n+2; \end{array} 2\right] & (m = n+1, n+2, n+3, \cdots) \end{cases} \end{aligned} \tag{19}$$

$$\Big(\min\{\Re(\alpha), \Re(\beta)\} > -1;\ m, n \in \mathbb{N}_0\Big).$$

Many classical orthogonal polynomials (such as the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(z)$, the Legendre (or spherical) polynomials $P_n(z)$, and the Tchebycheff polynomials $T_n(z)$ and $U_n(z)$ of the first and second kind) follow as special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ for various choices of the parameters α and β. For example, we have

$$C_n^\nu(z) = \binom{\nu + n - \frac{1}{2}}{n}^{-1} \binom{2\nu + n - 1}{n} P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z) \tag{20}$$

and

$$P_n(z) = C_n^{\frac{1}{2}}(z) = P_n^{(0,0)}(z), \tag{21}$$

which, in conjunction with Definition 1, yields the corresponding Gegenbauer transform $\mathbb{G}^{(\nu)}[f(z); n]$ given by

$$\begin{aligned} &\mathbb{G}^{(\nu)}[f(z); n] \\ &\quad = \binom{\nu + n - \frac{1}{2}}{n}^{-1} \binom{2\nu + n - 1}{n} \mathbb{J}^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}[f(z); n] \\ &\quad := \int_{-1}^{1} \left(1 - z^2\right)^{\nu - \frac{1}{2}} C_n^\nu(z)\ f(z)\mathrm{d}z \qquad \left(\Re(\nu) > -\frac{1}{2};\ n \in \mathbb{N}_0\right) \end{aligned} \tag{22}$$

and the corresponding Legendre transform $\mathbb{L}[f(z); n]$ defined by

$$\mathbb{L}[f(z); n] = \mathbb{G}^{(\frac{1}{2})}[f(z); n] := \int_{-1}^{1} P_n(z)\ f(z)\mathrm{d}z \qquad (n \in \mathbb{N}_0)\,. \tag{23}$$

Definition 2. (see [15]) The $\mathcal{P}_\delta$-transform $\mathcal{P}_\delta[f(t); s]$ of a function $f(t)$ $(t \in \mathbb{R})$ is a function $F_{\mathcal{P}}(s)$ of a complex variable s, which is defined by

$$\mathcal{P}_\delta[f(t); s] = F_{\mathcal{P}}(s) := \int_0^\infty [1 + (\delta - 1)s]^{-\frac{t}{\delta-1}}\ f(t)\mathrm{d}t \qquad (\delta > 1), \tag{24}$$

provided that the sufficient existence conditions given by Lemma 1 below are satisfied.

Lemma 1. (see [15]) *Let the function $f(t)$ be integrable over any finite interval (a, b) $(0 < a < t < b)$. Suppose also that there exists a real number c such that each of the following assertions holds true*:

(i) *For any arbitrary $b > 0$, $\int_b^\varrho e^{-ct}\ f(t)\mathrm{d}t$ tends to a finite limit as $\varrho \to \infty$;*

(ii) *For any arbitrary $a > 0$, $\int_\varsigma^a |f(t)|\mathrm{d}t$ tends to a finite limit as $\varsigma \to 0+$.*

Then the $\mathcal{P}_\delta$-transform $\mathcal{P}_\delta[f(t); s]$ exists whenever

$$\Re\left(\frac{\ln[1 + (\delta - 1)s]}{\delta - 1}\right) > c \qquad (s \in \mathbb{C}).$$

The $\mathcal{P}_\delta$-transform of the power function $t^{\rho-1}$ is given by

$$\mathcal{P}_\delta\big[t^{\rho-1}; s\big] = \left(\frac{\delta - 1}{\ln[1 + (\delta - 1)s]}\right)^\rho \Gamma(\rho) \qquad \big(\Re(\rho) > 0;\ \delta > 1\big). \tag{25}$$

Furthermore, upon letting $\delta \to 1+$ in the definition (24), the $\mathcal{P}_\delta$-transform is immediately reduced to the classical Laplace transform (see, for example, [21]):

$$\mathfrak{L}[f(t);s] := \int_0^\infty e^{-st} f(t)\mathrm{d}t, \tag{26}$$

provided that the integral exists.

Remark 1. By closely comparing the definitions in (24) and (26), it is easily observed that the so-called $\mathcal{P}_\delta$-transform is essentially the same as the classical Laplace transform *with* the following rather trivial parameter change in (26):

$$s \longmapsto \frac{\ln[1+(\delta-1)s]}{\delta-1} \qquad (\delta > 1). \tag{27}$$

Nevertheless, the current literature on integral transforms, special functions and fractional calculus is flooded by investigations claiming at least implicitly that the $\mathcal{P}_\delta$-transform $\mathcal{P}_\delta[f(t);s]$ defined by (24) is a generalization of the classical Laplace transform defined by (26).

Remark 2. Agarwal *et al.* [2] made use of the $\mathcal{P}_\delta$-transform given by Definition 2 in order to solve a fractional Volterra type integral equation and a non-homogeneous time-fractional heat equation involving a so-called pathway-type integral transform which is, in fact, the same as the extensively- and widely-investigated Riemann-Liouville fractional integral with, of course, some obvious parameter and variable changes. On the other hand, Srivastava *et al.* [25] found many results involving a family of generalized hypergeometric functions by using the $\mathcal{P}_\delta$-transform given by Definition 2.

Various families of integral transforms and fractional calculus operators play important rôles from the application viewpoint in several areas of mathematical, physical and engineering sciences. A lot of work has been done on the theory and applications of integral transforms (see, for example, [7], [8], [9], [15] and [25]). In recent years, integral transforms involving fractional integral and fractional derivative formulas and various classes of special functions were investigated by many authors (see, for example, [2], [12], [22] and [23]). In the present sequel to the aforementioned recent work [25], by using essentially the same techniques as those that are detailed by Srivastava *et al.* [25], we establish several (presumably new) integral transform and fractional derivative formulas involving the generalized incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ given by the equations (10) and (11), respectively.

2. Jacobi and Related Integral Transforms of the Incomplete Hypergeometric Functions

In this section, we prove three results which exhibit the connections between the Jacobi, Gegenbauer and Legendre transforms with the following incomplete hypergeometric functions:

$${}_p\gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right] \quad \text{and} \quad {}_p\Gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right],$$

which are given by the equations (10) and (11), respectively.

Theorem 1. *Under the conditions stated already with* (10) *and* (11), *the following Jacobi transform formulas hold true*:

$$\begin{aligned}
&\mathbb{J}^{(\alpha,\beta)}\left[z^{\rho-1}\,{}_p\gamma_q[yz];n\right]\\
&\quad= 2^{\alpha+\beta+1}\binom{\alpha+n}{n}B(\alpha+1,\beta+1)\sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\\
&\quad\cdot F^{1:2;1}_{1:1;0}\left[\begin{array}{rr}\alpha+1: & -n,\alpha+\beta+n+1;1-\rho-k;\\ \\ \alpha+\beta+2: & \alpha+1;\ \text{———};\end{array} 1,2\right]\frac{y^k}{k!}
\end{aligned} \tag{28}$$

$$\left(x \geqq 0;\ \ n \in \mathbb{N}_0;\ \ \min\{\Re(\alpha), \Re(\beta)\} > -1;\ \ \rho \in \mathbb{C};\ p, q \in \mathbb{N}_0\right)$$

and

$$\begin{aligned}
&\mathbb{J}^{(\alpha,\beta)}\left[z^{\rho-1}\ {}_p\Gamma_q[yz]; n\right] \\
&\quad = 2^{\alpha+\beta+1}\binom{\alpha+n}{n} B(\alpha+1,\beta+1) \sum_{k=0}^{\infty} \frac{[a_1;x]_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \\
&\quad \cdot F^{1:2;1}_{1:1;0}\left[\begin{array}{rl} \alpha+1: & -n, \alpha+\beta+n+1; 1-\rho-k; \\ & \qquad\qquad\qquad\qquad\qquad 1,2 \\ \alpha+\beta+2: & \qquad \alpha+1;\ \text{———};\end{array}\right] \frac{y^k}{k!}
\end{aligned} \tag{29}$$

$$\left(x \geqq 0;\ n \in \mathbb{N}_0;\ \min\{\Re(\alpha), \Re(\beta)\} > -1;\ \rho \in \mathbb{C};\ p, q \in \mathbb{N}_0\right),$$

where the coefficients of the incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ are given by (10) *and* (11) *and the Jacobi transforms in* (28) *and* (29) *are assumed to exist.*

Proof. In order to prove the assertion (28) of Theorem 1, we first apply the definition (17) in conjunction with (10). Then, upon changing the order of integration and summation (which can be justified easily by absolute convergence), we make use of the Jacobi transform formula (18) with the parameter ρ replaced by $\rho + k$ $(\rho \in \mathbb{C};\ k \in \mathbb{N}_0)$.

The assertion (29) of Theorem 1 can be proven similarly by using the definition (11) in place of (10). The details involved are being left as an exercise for the interested reader. □

By applying the Jacobi transform formula (19), we can simplify the assertions (28) and (29) of Theorem 1 in their special case when $\rho = m + 1$ $(m \in \mathbb{N}_0)$. Moreover, in view of the relationship (20), Theorem 1 yields the following corollary by setting $\alpha = \beta = \nu - \frac{1}{2}$.

Corollary 1. *Under the conditions stated already with* (10) *and* (11), *the following Gegenbauer transform formulas hold true*:

$$\begin{aligned}
&\mathbb{G}^{(\nu)}\left[z^{\rho-1}\ {}_p\gamma_q[yz]; n\right] \\
&\quad = 2^{2\nu}\binom{2\nu+n-1}{n} B\left(\nu+\frac{1}{2}, \nu+\frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(a_1;x)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \\
&\quad \cdot F^{1:2;1}_{1:1;0}\left[\begin{array}{rl} \nu+\frac{1}{2}: & -n, 2\nu+n; 1-\rho-k; \\ & \qquad\qquad\qquad\qquad 1,2 \\ 2\nu+1: & \qquad \nu+\frac{1}{2};\ \text{———};\end{array}\right] \frac{y^k}{k!}
\end{aligned} \tag{30}$$

$$\left(x \geqq 0;\ n \in \mathbb{N}_0;\ \rho \in \mathbb{C};\ p, q \in \mathbb{N}_0\right)$$

and

$$\begin{aligned}
&\mathbb{G}\left[z^{\rho-1}\ {}_p\Gamma_q[yz]; n\right] \\
&\quad = 2^{2\nu}\binom{2\nu+n-1}{n} B\left(\nu+\frac{1}{2}, \nu+\frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{[a_1;x]_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \\
&\quad \cdot F^{1:2;1}_{1:1;0}\left[\begin{array}{rl} \nu+\frac{1}{2}: & -n, 2\nu+n; 1-\rho-k; \\ & \qquad\qquad\qquad\qquad 1,2 \\ 2\nu+1: & \qquad \nu+\frac{1}{2};\ \text{———};\end{array}\right] \frac{y^k}{k!}
\end{aligned} \tag{31}$$

$$\left(x \geqq 0;\ n \in \mathbb{N}_0;\ \min\{\Re(\nu)\} > -\frac{1}{2};\ \rho \in \mathbb{C};\ p, q \in \mathbb{N}_0\right),$$

where it is assumed that the Gegenbauer transforms in (30) *and* (31) *exist.*

For the Legendre transform defined by (23), a special case of Theorem 1 when $\alpha = \beta = 0$ (or, alternatively, a *further* special case of Corollary 1 when $\nu = \frac{1}{2}$) yields the following result.

Corollary 2. *Under the conditions stated already with* (10) *and* (11), *the following Legendre transform formulas hold true*:

$$\mathbb{L}\left[z^{\rho-1}\ {}_p\gamma_q[yz];n\right] = 2\sum_{k=0}^{\infty} \frac{(a_1;x)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k} \cdot F^{1:2;1}_{1:1;0}\left[\begin{array}{ll} 1: & -n,n+1;1-\rho-k; \\ & \qquad\qquad\qquad\qquad 1,2 \\ 2: & 1;\ \text{———};\end{array}\right]\frac{y^k}{k!} \tag{32}$$

$$\left(x \geqq 0;\ n \in \mathbb{N}_0;\ \rho \in \mathbb{C};\ p,q \in \mathbb{N}_0\right)$$

and

$$\mathbb{L}\left[z^{\rho-1}{}_p\Gamma_q[yz];n\right] = 2\sum_{k=0}^{\infty} \frac{[a_1;x]_k(a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k} \cdot F^{1:2;1}_{1:1;0}\left[\begin{array}{ll} 1: & -n,n+1;1-\rho-k; \\ & \qquad\qquad\qquad\qquad 1,2 \\ 2: & 1;\ \text{———};\end{array}\right]\frac{y^k}{k!} \tag{33}$$

$$\left(x \geqq 0;\ n \in \mathbb{N}_0;\ \rho \in \mathbb{C};\ p,q \in \mathbb{N}_0\right),$$

where it is assumed that the Legendre transforms in (32) *and* (33) *exist.*

3. $\mathcal{P}_\delta$-Transforms of the Incomplete Hypergeometric Functions

As we have already mentioned in Remark 1, the following relationship holds true between the so-called $\mathcal{P}_\delta$-transform defined by (24) and the classical Laplace transform given by (26):

$$\mathcal{P}_\delta[f(t):s] = \mathfrak{L}\left[f(t):\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)\right] \qquad (\delta > 1) \tag{34}$$

or, equivalently,

$$\mathfrak{L}[f(t):s] = \mathcal{P}_\delta\left[f(t):\frac{e^{(\delta-1)s}-1}{\delta-1}\right] \qquad (\delta > 1), \tag{35}$$

which can indeed be applied reasonably simply to convert the table of Laplace transforms into the corresponding table of the $\mathcal{P}_\delta$-transform and *vice versa*.

Theorem 2. *Under the conditions stated already with* (10) *and* (11), *the following* $\mathcal{P}_\delta$*-transform formulas hold true*:

$$\mathcal{P}_\delta\left[t^{\rho-1}\ {}_p\gamma_q[zt];s\right] = \frac{\Gamma(\rho)}{[\Lambda(\delta;s)]^\rho}\ {}_{p+1}\gamma_q\left[\begin{array}{c}\rho,(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{array} z\right] \tag{36}$$

$$\left(|z| < 1;\ \min\{\Re(s),\Re(\rho)\} > 0;\ \delta > 1;\ p,q \in \mathbb{N}_0\right)$$

and

$$\mathcal{P}_\delta\left[t^{\rho-1}\ {}_p\Gamma_q[zt];s\right] = \frac{\Gamma(\rho)}{[\Lambda(\delta;s)]^\rho}\ {}_{p+1}\Gamma_q\left[\begin{array}{c}\rho,(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{array} z\right] \tag{37}$$

$$\Big(|z| < 1;\ \min\{\Re(s), \Re(\rho)\} > 0;\ \delta > 1;\ p, q \in \mathbb{N}_0\Big),$$

where

$$\Lambda(\delta; s) := \frac{\ln[1 + (\delta - 1)s]}{\delta - 1} \tag{38}$$

and it is assumed that the $\mathcal{P}_\delta$-transforms in (36) *and* (37) *exist.*

Proof. Applying the definitions (24) and (10) on the left-hand side of (36), we find that

$$\begin{aligned}
&\mathcal{P}_\delta\left[t^{\rho-1}\ {}_p\gamma_q\left[\begin{array}{c}(a_1, x), a_2, \cdots, a_p;\\ \\ b_1, \cdots, b_q;\end{array} zt\right]; s\right]\\
&\quad = \int_0^\infty t^{\rho-1}[1 + (\delta - 1)s]^{-\frac{t}{\delta-1}}\ {}_p\gamma_q\left[\begin{array}{c}(a_1, x), a_2, \cdots, a_p;\\ \\ b_1, \cdots, b_q;\end{array} zt\right] \mathrm{d}t\\
&\quad = \int_0^\infty t^{\rho-1}[1 + (\delta - 1)s]^{-\frac{t}{\delta-1}} \left(\sum_{k=0}^\infty \frac{(a_1; x)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{(zt)^k}{k!}\right) \mathrm{d}t.
\end{aligned} \tag{39}$$

If, upon changing the order of integration and summation in Eq. (39), we make use of Eq. (25) with the parameter ρ replaced by $\rho + k$ $(k \in \mathbb{N}_0)$, we obtain

$$\begin{aligned}
&\mathcal{P}_\delta\left[t^{\rho-1}\ {}_p\gamma_q\left[\begin{array}{c}(a_1, x), a_2, \cdots, a_p;\\ \\ b_1, \cdots, b_q;\end{array} zt\right]; s\right]\\
&\quad = \sum_{k=0}^\infty \frac{(a_1; x)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \int_0^\infty t^{\rho+k-1}[1 + (\delta - 1)s]^{-\frac{t}{\delta-1}}\ \mathrm{d}t\\
&\quad = \sum_{k=0}^\infty \frac{(a_1; x)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \frac{\Gamma(\rho + k)}{[\Lambda(\delta; s)]^{\rho+k}},
\end{aligned} \tag{40}$$

where $[\Lambda(\delta; s)]$ is given by (38). The assertion (36) of Theorem 2 now follows when we interpret the last member of (40) by means of the definition (10).

A similar argument as in the proof of (36) will establish the result (37). This completes the proof of Theorem 2. □

The following corollary is a limit case of Theorem 2 when $\delta \to 1+$.

Corollary 3. *Under the conditions stated already with* (10) *and* (11), *the following Laplace transform formulas hold true*:

$$\mathfrak{L}\left[t^{\rho-1}\ {}_p\gamma_q[zt]; s\right] = \frac{\Gamma(\rho)}{s^\rho}\ {}_{p+1}\gamma_q\left[\begin{array}{c}\rho, (a_1, x), a_2, \cdots, a_p;\\ \\ b_1, \cdots, b_q;\end{array} \frac{z}{s}\right] \tag{41}$$

and

$$\mathfrak{L}\left[t^{\rho-1}\ {}_p\Gamma_q[zt]; s\right] = \frac{\Gamma(\rho)}{s^\rho}\ {}_{p+1}\Gamma_q\left[\begin{array}{c}\rho, (a_1, x), a_2, \cdots, a_p;\\ \\ b_1, \cdots, b_q;\end{array} \frac{z}{s}\right], \tag{42}$$

$$\Big(|z| < 1;\ \min\{\Re(s), \Re(\rho)\} > 0;\ p, q \in \mathbb{N}_0\Big),$$

where it is assumed that the Laplace transforms in (41) *and* (42) *exist.*

Remark 3. By appealing to the relationship (34), it is rather straightforward to deduce the assertions (36) and (37) of Theorem 2 from the Laplace transform formulas (41) and (42) in Corollary 3 by trivially setting $s \mapsto \Lambda(\delta; s)$ for $\Lambda(\delta; s)$ given by (38).

4. Fractional Derivative Formulas for the Incomplete Hypergeometric Functions

Here, in this section, we establish several fractional derivative formulas for each of the following incomplete hypergeometric functions:

$$ {}_p\gamma_q\left[\begin{array}{c} (a_1,x),a_2,\cdots,a_p; \\ \\ b_1,\cdots,b_q; \end{array} z\right] \qquad \text{and} \qquad {}_p\Gamma_q\left[\begin{array}{c} (a_1,x),a_2,\cdots,a_p; \\ \\ b_1,\cdots,b_q; \end{array} z\right] $$

which are given by (10) and (11). With this purpose in view, we recall the pairs of hypergeometric fractional derivative operators $D_{0+}^{\omega,\nu,\eta}$ and $D_{\infty-}^{\omega,\nu,\eta}$, which are defined below in terms of the corresponding pairs of hypergeometric fractional integral operators $I_{0+}^{\omega,\nu,\eta}$ and $I_{\infty-}^{\omega,-\nu,\eta}$, respectively.

Definition 3. (see, for details, [32] and [31]) In terms of the Gauss hypergeometric function ${}_2F_1$, the left-sided hypergeometric fractional integral operator $I_{0+}^{\omega,\nu,\eta}$ and the corresponding left-sided hypergeometric fractional derivative operator $D_{0+}^{\omega,\nu,\eta}$ are defined, for $x > 0$ and $\omega,\nu,\eta \in \mathbb{C}$, by

$$ \begin{aligned} \left(I_{0+}^{\omega,\nu,\eta} f\right)(x) := \frac{x^{-\omega-\nu}}{\Gamma(\omega)} & \int_0^x (x-t)^{\omega-1} \\ & \cdot\ {}_2F_1\left(\omega+\nu,-\eta;\omega;1-\frac{t}{x}\right) f(t)\mathrm{d}t \qquad \left(\Re(\omega) > 0\right) \end{aligned} \tag{43} $$

and

$$ \begin{aligned} \left(D_{0+}^{\omega,\nu,\eta} f\right)(x) &= \left(I_{0+}^{-\omega,-\nu,\omega+\eta} f\right)(x) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\left(I_{0+}^{-\omega+\eta,-\nu-\eta,\omega+\eta-n} f\right)(x)\right\} \end{aligned} \tag{44} $$

$$ \left(\Re(\omega) \geqq 0;\ n = [\Re(\omega)] + 1\right), $$

where, and in what follows, $[\kappa]$ denotes the largest integer in the real number κ.

The left-sided hypergeometric fractional derivative operator $D_{0+}^{\omega,\nu,\eta}$ unifies both the Riemann-Liouville fractional derivative operator ${}_{\mathrm{RL}}\mathcal{D}_{0+}^{\omega}$ and the left-sided Erdélyi-Kober fractional derivative operator ${}_{\mathrm{EK}}\mathfrak{D}_{0+}^{\omega,\eta}$. In fact, we have the following relationships:

$$ {}_{\mathrm{RL}}\mathcal{D}_{0+}^{\omega} = D_{0+}^{\omega,-\omega,\eta} \qquad \text{and} \qquad {}_{\mathrm{EK}}\mathfrak{D}_{0+}^{\omega,\eta} = D_{0+}^{\omega,0,\eta}, \tag{45} $$

where (see, for details, [9, Chapter 13])

$$ \left({}_{\mathrm{RL}}\mathcal{D}_{0+}^{\omega} f\right)(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\omega)} \int_0^x \frac{f(t)}{(x-t)^{\omega-n+1}}\, \mathrm{d}t\right\} \tag{46} $$

$$ \left(x > 0;\ n = [\Re(\omega)] + 1;\ \Re(\omega) \geqq 0\right) $$

and

$$ \left({}_{\mathrm{EK}}\mathfrak{D}_{0+}^{\omega,\eta} f\right)(x) := x^{\eta} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\omega)} \int_0^x \frac{t^{\omega+\eta}\, f(t)}{(x-t)^{\omega-n+1}}\, \mathrm{d}t\right\} \tag{47} $$

$$ \left(x > 0;\ n = [\Re(\omega)] + 1;\ \Re(\omega) \geqq 0\right). $$

Definition 4. (see, for details, [32] and [31]; see also [11]) In terms of the Gauss hypergeometric function ${}_2F_1$, the right-sided hypergeometric fractional integral operator $I_{\infty-}^{\omega,\nu,\eta}$ and the corresponding right-sided hypergeometric fractional derivative operator $D_{\infty-}^{\omega,\nu,\eta}$ are defined, for $x > 0$ and $\omega, \nu, \eta \in \mathbb{C}$, by

$$\begin{aligned}\left(I_{\infty-}^{\omega,\nu,\eta} f\right)(x) := \frac{1}{\Gamma(\omega)} \int_x^\infty (t-x)^{\omega-1}\, t^{-\omega-\nu} \\ \cdot\; {}_2F_1\left(\omega+\nu, -\eta; \omega; 1-\frac{x}{t}\right) f(t)\mathrm{d}t \qquad \Big(\Re(\omega) > 0\Big)\end{aligned} \tag{48}$$

and

$$\begin{aligned}\left(D_{\infty-}^{\omega,\nu,\eta} f\right)(x) &= \left(I_{\infty}^{-\omega,-\nu,\omega+\eta} f\right)(x) \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\left(I_{\infty-}^{-\omega+\eta,-\nu-\eta,\omega+\eta-n} f\right)(x)\right\}\end{aligned} \tag{49}$$

$$\Big(\Re(\omega) \geqq 0;\ n = [\Re(\omega)] + 1\Big).$$

The right-sided hypergeometric fractional derivative operator $D_{\infty-}^{\omega,\nu,\eta}$ unifies both the Weyl fractional derivative operator ${}_{\mathrm{W}}\mathcal{D}_{\infty-}^{\omega}$ and the right-sided Erdélyi-Kober fractional derivative operator ${}_{\mathrm{EK}}\mathfrak{D}_{\infty-}^{\omega,\eta}$. In fact, we have the following relationships:

$${}_{\mathrm{W}}\mathcal{D}_{\infty-}^{\omega} = D_{\infty-}^{\omega,-\omega,\eta} \qquad \text{and} \qquad {}_{\mathrm{EK}}\mathfrak{D}_{\infty-}^{\omega,\eta} = D_{\infty-}^{\omega,0,\eta}, \tag{50}$$

where (see, for details, [9, Chapter 13])

$$\left({}_{\mathrm{W}}\mathcal{D}_{\infty-}^{\omega} f\right)(x) := \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\omega)} \int_0^x \frac{f(t)}{(t-x)^{\omega-n+1}}\, \mathrm{d}t\right\} \tag{51}$$

$$\Big(x > 0;\ \Re(\omega) \geqq 0;\ n = [\Re(\omega)] + 1\Big)$$

and

$$\left({}_{\mathrm{EK}}\mathfrak{D}_{\infty-}^{\omega} f\right)(x) := x^{\omega+\eta} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\omega)} \int_x^\infty \frac{t^{-\eta}\, f(t)}{(t-x)^{\omega-n+1}}\, \mathrm{d}t\right\} \tag{52}$$

$$\Big(x > 0;\ \Re(\omega) \geqq 0;\ n = [\Re(\omega)] + 1\Big).$$

Lemma 2. (see, for example, [11, pp. 327–328]) *Each of the following hypergeometric fractional derivative formulas holds true*:

$$\left(D_{0+}^{\omega,\nu,\eta}\, t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\omega+\nu+\eta)}{\Gamma(\rho+\nu)\Gamma(\rho+\eta)}\, x^{\rho+\nu-1} \tag{53}$$

$$\Big(x > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) > -\min\{0, \Re(\omega+\nu+\eta)\}\Big)$$

and

$$\left(D_{\infty-}^{\omega,\nu,\eta}\, t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\nu)(1-\rho+\omega+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\nu)}\, x^{\rho+\nu-1} \tag{54}$$

$$\Big(x > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) < 1 + \min\{-\Re(\nu+\eta), \Re(\omega+\eta)\}\Big).$$

Now, if we appeal appropriately to the assertions (53) and (54) of Lemma 2, we can easily derive each of the following results.

Theorem 3. *Under the conditions stated with* (10) *and* (11), *the following left-sided hypergeometric fractional derivative formulas hold true*:

$$\left(D_{0+}^{\omega,\nu,\eta}\; t^{\rho-1}\; {}_p\gamma_q\left[\begin{array}{c} (a_1,x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} zt\right]\right)(\xi)$$
$$= \xi^{\rho+\nu-1}\, \frac{\Gamma(\rho)\Gamma(\rho+\omega+\nu+\eta)}{\Gamma(\rho+\nu)\Gamma(\rho+\eta)}$$
$$\cdot\, {}_{p+2}\gamma_{q+2}\left[\begin{array}{c} \rho, \rho+\omega+\nu+\eta, (a_1,x), a_2, \cdots, a_p; \\ \\ \rho+\nu, \rho+\eta, b_1, \cdots, b_q; \end{array} z\xi\right] \tag{55}$$
$$\Big(\xi > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) > -\min\{0, \Re(\omega+\nu+\eta)\}\Big)$$

and

$$\left(D_{0+}^{\omega,\nu,\eta}\; t^{\rho-1}\; {}_p\Gamma_q\left[\begin{array}{c} (a_1,x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} zt\right]\right)(\xi)$$
$$= \xi^{\rho+\nu-1}\, \frac{\Gamma(\rho)\Gamma(\rho+\omega+\nu+\eta)}{\Gamma(\rho+\nu)\Gamma(\rho+\eta)}$$
$$\cdot\, {}_{p+2}\Gamma_{q+2}\left[\begin{array}{c} \rho, \rho+\omega+\nu+\eta, (a_1,x), a_2, \cdots, a_p; \\ \\ \rho+\nu, \rho+\eta, b_1, \cdots, b_q; \end{array} z\xi\right] \tag{56}$$
$$\Big(\xi > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) > -\min\{0, \Re(\omega+\nu+\eta)\}\Big),$$

where it is assumed that the left-sided hypergeometric fractional derivatives in (55) *and* (56) *exist.*

Proof. Our demonstration of the hypergeometric fractional derivative formulas (55) and (56) is based upon the known result (53). The details are fairly straightforward and are, therefore, omitted. □

Analogously to the proof of Theorem 3 above, Theorem 4 below can be proven by applying the hypergeometric fractional derivative formula (54).

Theorem 4. *Under the conditions stated already with* (10) *and* (11), *the following right-sided hypergeometric fractional derivative formulas hold true*:

$$\left(D_{\infty-}^{\omega,\nu,\eta}\; t^{\rho-1}\; {}_p\gamma_q\left[\begin{array}{c} (a_1,x), a_2, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} \frac{z}{t}\right]\right)(\xi)$$
$$= \xi^{\rho+\nu-1}\, \frac{\Gamma(1-\rho-\nu)1-\rho+\omega+\eta}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\nu)}$$
$$\cdot\, {}_{p+2}\gamma_{q+2}\left[\begin{array}{c} 1-\rho-\nu, 1-\rho+\omega+\eta, (a_1,x), a_2, \cdots, a_p; \\ \\ 1-\rho, 1-\rho+\eta-\nu, b_1, \cdots, b_q; \end{array} \frac{z}{\xi}\right] \tag{57}$$
$$\Big(\xi > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) < 1 + \min\{-\Re(\nu+\eta), \Re(\omega+\eta)\}\Big)$$

and

$$\left(D_{\infty-}^{\omega,\nu,\eta}\ t^{\rho-1}\ {}_p\Gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array}\ \frac{z}{t}\right]\right)(\xi)$$
$$=\xi^{\rho+\nu-1}\ \frac{\Gamma(1-\rho-\nu)(1-\rho+\omega+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\nu)}$$
$$\cdot\ {}_{p+2}\Gamma_{q+2}\left[\begin{array}{c}1-\rho-\nu,1-\rho+\omega+\eta,(a_1,x),a_2,\cdots,a_p;\\ \\ 1-\rho,1-\rho+\eta-\nu,b_1,\cdots,b_q;\end{array}\ \frac{z}{\xi}\right] \tag{58}$$
$$\Big(\xi>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\min\{-\Re(\nu+\eta),\Re(\omega+\eta)\}\Big),$$

where it is assumed that the right-sided hypergeometric fractional derivatives in (57) *and* (58) *exist.*

Upon setting $\nu=-\omega$ and $\nu=0$ in Theorem 3, if we use the relationships in (45), we can deduce Corollary 4 and Corollary 5, respectively.

Corollary 4. *Under the conditions stated with* (10) *and* (11), *the following Riemann-Liouville fractional derivative formulas hold true*:

$$\left({}_{\mathrm{RL}}D_{0+}^{\omega}\ t^{\rho-1}\ {}_p\gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array}\ zt\right]\right)(\xi)$$
$$=\xi^{\rho+\omega-1}\ \frac{\Gamma(\rho)}{\Gamma(\rho-\omega)}\cdot{}_{p+1}\gamma_{q+1}\left[\begin{array}{c}\rho,(a_1,x),a_2,\cdots,a_p;\\ \\ \rho-\omega,b_1,\cdots,b_q;\end{array}\ z\xi\right] \tag{59}$$
$$\Big(\xi>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)>0\Big)$$

and

$$\left({}_{\mathrm{RL}}D_{0+}^{\omega}\ t^{\rho-1}\ {}_p\Gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array}\ zt\right]\right)(\xi)$$
$$=\xi^{\rho+\omega-1}\ \frac{\Gamma(\rho)}{\Gamma(\rho-\omega)}\cdot{}_{p+1}\Gamma_{q+1}\left[\begin{array}{c}\rho,[a_1,x],a_2,\cdots,a_p;\\ \\ \rho-\omega,b_1,\cdots,b_q;\end{array}\ z\xi\right] \tag{60}$$
$$\Big(\xi>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)>0\Big),$$

where it is assumed that the Riemann-Liouville fractional derivatives in (59) *and* (60) *exist.*

Corollary 5. *Under the conditions stated already with* (10) *and* (11), *the following left-sided Erdélyi-Kober fractional derivative formulas hold true*:

$$\left({}_{\mathrm{EK}}\mathcal{D}_{0+}^{\omega,\eta}\ t^{\rho-1}\ {}_p\gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array}\ zt\right]\right)(\xi)$$
$$=\xi^{\rho-1}\ \frac{\Gamma(\rho+\omega+\eta)}{\Gamma(\rho+\eta)}\cdot{}_{p+1}\gamma_{q+1}\left[\begin{array}{c}\rho+\omega+\eta,(a_1,x),a_2,\cdots,a_p;\\ \\ \rho+\eta,b_1,\cdots,b_q;\end{array}\ z\xi\right] \tag{61}$$

$$\Big(\xi > 0; \Re(\omega) \geqq 0;\ \Re(\rho) > -\min\{0, \Re(\eta)\}\Big)$$

and

$$\begin{aligned}&\left({}_{\mathrm{EK}}\mathcal{D}^{\omega,\eta}_{0+}\ t^{\rho-1}\ {}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{matrix}\ zt\right]\right)(\xi)\\ &\quad= \xi^{\rho-1}\ \frac{\Gamma(\rho+\omega+\eta)}{\Gamma(\rho+\eta)}\cdot {}_{p+1}\Gamma_{q+1}\left[\begin{matrix}\rho+\omega+\eta,[a_1,x],a_2,\cdots,a_p;\\ \\ \rho+\eta,b_1,\cdots,b_q;\end{matrix}\ z\xi\right]\end{aligned}\tag{62}$$

$$\Big(\xi > 0; \Re(\omega) \geqq 0;\ \Re(\rho) > -\min\{0, \Re(\eta)\}\Big),$$

where it is assumed that the left-sided Erdélyi-Kober fractional derivatives in (61) *and* (62) *exist.*

Corollary 6 and Corollary 7 below would follow from the Theorem 4 by first setting $\nu = -\omega$ and $\nu = 0$, respectively, and then making use of the relationships given by (50).

Corollary 6. *Under the conditions stated already with* (10) *and* (11), *the following Weyl fractional derivative formulas hold true*:

$$\begin{aligned}&\left({}_{\mathrm{W}}D^{\omega}_{\infty-}\ t^{\rho-1}\ {}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{matrix}\ \frac{z}{t}\right]\right)(\xi)\\ &\quad= \xi^{\rho-\omega-1}\ \frac{\Gamma(1-\rho+\omega)}{\Gamma(1-\rho)}\cdot {}_{p+1}\gamma_{q+1}\left[\begin{matrix}1-\rho+\omega,(a_1,x),a_2,\cdots,a_p;\\ \\ 1-\rho,b_1,\cdots,b_q;\end{matrix}\ \frac{z}{\xi}\right]\end{aligned}\tag{63}$$

$$\Big(\xi > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) < 1 + \Re(\omega)\Big)$$

and

$$\begin{aligned}&\left({}_{\mathrm{W}}D^{\omega}_{\infty-}\ t^{\rho-1}\ {}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{matrix}\ \frac{z}{t}\right]\right)(\xi)\\ &\quad= \xi^{\rho-\omega-1}\ \frac{\Gamma(1-\rho+\omega)}{\Gamma(1-\rho)}\cdot {}_{p+1}\Gamma_{q+1}\left[\begin{matrix}1-\rho+\omega,(a_1,x),a_2,\cdots,a_p;\\ \\ 1-\rho,b_1,\cdots,b_q;\end{matrix}\ \frac{z}{\xi}\right]\end{aligned}\tag{64}$$

$$\Big(\xi > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) < 1 + \Re(\omega)\Big),$$

where it is assumed that the Weyl fractional derivatives in (63) and (64) exist.

Corollary 7. *Under the conditions stated with* (10) *and* (11), *the following right-sided Erdélyi-Kober fractional derivative formulas hold true*:

$$\begin{aligned}&\left({}_{\mathrm{EK}}D^{\omega,\eta}_{\infty-}\ t^{\rho-1}\ {}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{matrix}\ \frac{z}{t}\right]\right)(\xi) = \xi^{\rho-1}\ \frac{\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho+\eta)}\\ &\qquad\cdot {}_{p+1}\gamma_{q+1}\left[\begin{matrix}1-\rho+\omega+\eta,(a_1,x),a_2,\cdots,a_p;\\ \\ 1-\rho+\eta,b_1,\cdots,b_q;\end{matrix}\ \frac{z}{\xi}\right]\end{aligned}\tag{65}$$

$$\Big(\xi > 0;\ \Re(\omega) \geqq 0;\ \Re(\rho) < 1 + \min\{-\Re(\eta), \Re(\omega+\eta)\}\Big)$$

and

$$\left({}_{\mathrm{EK}}D_{\infty-}^{\omega,\eta}\ t^{\rho-1}\ {}_p\Gamma_q\left[\begin{array}{c}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array}\ \frac{z}{t}\right]\right)(\xi)=\xi^{\rho-1}\ \frac{\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho+\eta)}$$
$$\cdot\ {}_{p+1}\Gamma_{q+1}\left[\begin{array}{c}1-\rho+\omega+\eta,(a_1,x),\ a_2,\cdots,a_p;\\ 1-\rho+\eta,b_1,\cdots,b_q;\end{array}\ \frac{z}{\xi}\right] \tag{66}$$
$$\Big(\xi>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\min\{-\Re(\eta),\Re(\omega+\eta)\}\Big),$$

where it is assumed that the right-sided Erdélyi-Kober fractional derivatives in (65) *and* (66) *exist.*

5. Concluding Remarks and Observations

The family of the incomplete hypergeometric functions, which were introduced and investigated systematically by Srivastava *et al.* [27], possess the advantage that most of the known and widely-studied special functions are expressible in terms of these incomplete hypergeometric functions. In conclusion, therefore, we remark that the results derived in this paper are sufficiently significant and sufficiently general in nature *and* are capable of yielding numerous other integral transform and fractional derivative formulas involving various special functions by some appropriate choices of the essentially arbitrary parameters which are involved in these results. Moreover, various applications of this pair of incomplete hypergeometric functions in Communication Theory, Probability Theory and Groundwater Pumping Modeling are already shown by Srivastava *et al.* [27]. Thus, naturally, the results presented in this paper are expected to lead to some potential applications in several diverse fields of mathematical, physical, statistical and engineering sciences. Such results may also find applications in the solutions of integral and integro-differential equations occurring in applied mathematics (see also the recent works [6], [40] and [42]).

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