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On Generating Functions for the Special Polynomials

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Abstract. The aim of this paper is to investigate and give a new family of Apell type polynomials, which are related to the Euler, Frobenius-Euler and Apostol-Bernoulli polynomials and numbers and also the generalized Stirling numbers of the second kind *etc*. The results presented in this paper are based upon the theory of the generating functions. By using functional equations of these generating functions, we drive some identities and relations for these numbers and polynomials. Moreover, we give a computation algorithm these numbers.

1. Introduction

The motivation of this paper is to give a new family of the special numbers and polynomials. This family is related very well-known numbers and polynomials, which are the Euler, Frobenius-Euler and Apostol-Bernoulli polynomials and numbers, *etc*. We investigate their properties by using generating functions, related to nonnegative real parameters. It is also well-known that the generating functions have been many applications in almost all branches of mathematics and statistics. These functions are related to the many known and unkonw numbers and polynomials. In the literature there are various studies related to the generating functions for well-known numbers and polynomials: the Bernoulli, Euler and Genocchi numbers and polynomials and also their generalizations, the λ-Stirling numbers of the second kind *etc*. In [8, Definition 4.1, p. 9], we defined the Eulerian type polynomials and numbers, related to nonnegative real parameters by the following definition:

Definition 1.1. *Let* $a, b \in \mathbb{R}^{+}$ *(*$a \neq b$ *and* $a \geq 1$*),* $x \in \mathbb{R}$*,* $\lambda \in \mathbb{C}$ *and* $u \in \mathbb{C} \diagdown \{1\}$*. The generalized Eulerian type polynomials* $\mathcal{H}_n(x; u; a, b, c; \lambda)$ *are defined by means of the following generating function:*

$$F_{\lambda}(t, x; u, a, b, c) = \frac{(a^t - u)\, c^{xt}}{\lambda b^t - u} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!}. \tag{1}$$

By substituting $x = 0$ into (1), the Eulerian type polynomials reduce to the Eulerian type numbers $\mathcal{H}_n(u; a, b; \lambda)$:

$$\mathcal{H}_n(0; u; a, b, c; \lambda) = \mathcal{H}_n(u; a, b; \lambda).$$

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In [8, Eq-(37), p. 21], we also defined a new family of polynomials $Y_n(x, u; a)$ by means of the following generating functions:

Let $a \in \mathbb{R}^+$ with $a \geq 1$ and $u \in \mathbb{C} \setminus \{1\}$. We have

$$\begin{aligned} G_Y(t, a, u) &= \frac{1}{a^t - u} \\ &= \sum_{n=0}^{\infty} Y_n(u; a) \frac{t^n}{n!} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathcal{G}_Y(x, t, a, u) &= G_Y(t, a, u) a^{tx} \\ &= \sum_{n=0}^{\infty} Y_n(x, u; a) \frac{t^n}{n!}. \end{aligned} \tag{3}$$

By using (2), we have

$$Y_0(u; a) = \frac{1}{1 - u}.$$

We define the polynomials $Y_n^{(v)}(x, u; a)$ of higher order as follows:

$$\left(\frac{1}{a^t - u}\right)^v a^{tx} = \sum_{n=0}^{\infty} Y_n^{(v)}(x, u; a) \frac{t^n}{n!},$$

where v is an integer. Furthermore,we have

$$Y_n^{(v)}(u; a) = Y_n^{(v)}(0, u; a).$$

The λ-Stirling type numbers of the second kind $\mathcal{S}(n, v; 1, b; \lambda)$, related to nonnegative real parameters, are defined by means of the following generating function :

$$\frac{(\lambda b^t - 1)^v}{v!} = \sum_{n=0}^{\infty} \mathcal{S}(n, v; 1, b; \lambda) \frac{t^n}{n!}, \tag{4}$$

cf. ([8]). If $b = e$, then $S(n, v; 1, b; \lambda)$ reduces to the λ-Stirling numbers of the second kind:

$$S(n, v; \lambda) = \mathcal{S}(n, v; 1, e; \lambda)$$

and also if $\lambda = 1$, then $S(n, v; \lambda)$ reduces to the Stirling numbers of the second kind:

$$S(n, v) = \mathcal{S}(n, v; 1, e; 1),$$

cf. ([1], [6], [8], [11], [12]).

Other than the introduction, this paper consists of three sections. In Section 2, by using generating functions for the numbers $Y_n(u; a)$, we derive not only recurrence relation and identities, but also a computation algorithm for these numbers. In Section 3, we give some properties and identities of the polynomials $Y_n(x, u; a)$. In Section 4, we give relationships between these numbers and he Euler, Frobenius-Euler and Apostol-Bernoulli numbers, *etc*.

2. Computation Algorithm for the Numbers $Y_n(u;a)$

In this section, by using generating functions for the numbers $Y_n(u;a)$, we derive some new identities, formulas and relations. We give a recurrence relation for this numbers. We also give a computation algorithm of the numbers $Y_n(u;a)$. By using this computation algorithm, we give a few values of these numbers.

By using the *Umbral Calculus convention*, we derive a recurrence relation for the numbers $Y_n(u;a)$. Using (2), we get

$$1=\sum_{n=0}^{\infty}\left((Y(u;a)+\ln a)^n-uY_n(u;a)\right)\frac{t^n}{n!},$$

where $Y^n(u;a)$ is replaced conventionally by $Y_n(u;a)$. By using the above equation, we get the following theorem:

Theorem 2.1. *If $n=0$, we have*

$$Y_0(u;a)=\frac{1}{1-u}.$$

If $n\geq 1$, we have

$$Y_n(u;a)=\frac{1}{u-1}\sum_{j=0}^{n-1}\binom{n}{j}Y_j(u;a)(\ln a)^{n-j}. \tag{5}$$

Theorem 2.2.

$$\begin{aligned}\frac{\partial}{\partial u}Y_n(u;a) &= \sum_{j=0}^{n}\binom{n}{j}Y_j(u;a)Y_{n-j}(u;a)\\ &= Y_n^{(2)}(u;a).\end{aligned}$$

Proof. We differentiate (2) with respect to the variable u to derive the following partial differential equations

$$\frac{\partial}{\partial u}G_Y(t,a,u)=G_Y^2(t,a,u).$$

By using this function with (2), we get

$$\sum_{n=1}^{\infty}\frac{\partial}{\partial u}Y_n(u;a)\frac{t^n}{n!}=\sum_{n=1}^{\infty}Y_n^{(2)}(u;a)\frac{t^n}{n!}.$$

After some elementary calculations, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Now, we are going to differentiate (2) with respect to the variable t to derive a recurrence relation for the numbers $Y_n(u;a)$. Therefore, we obtain the following partial differential equations:

$$\frac{\partial}{\partial t}G_Y(t,a,u)=-a^t(\ln a)G_Y^2(t,a,u).$$

By using this equation, we arrive at the following theorem:

Theorem 2.3.

$$Y_{n+1}(u;a) = -\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k} Y_k(u;a)\, Y_{j-k}(u;a)\, (\ln a)^{n+1-j}.$$

Theorem 2.4. *Let n be a positive integer. Then we have*

$$u^v \sum_{j=0}^{n}\binom{n}{j} Y_{n-j}^{(v)}(u;a)\, S\left(j, v; a; \frac{1}{u}\right) = 0$$

Proof. By using (2), we get

$$\left(a^t - u\right)^v \sum_{n=0}^{\infty} Y_n^{(v)}(u;a)\frac{t^n}{n!} = 1$$

By using (4), we get

$$v!u^v \sum_{n=0}^{\infty} S\left(n, v; a; \frac{1}{u}\right)(\ln a)^n \frac{t^n}{n!}\sum_{n=0}^{\infty} Y_n^{(v)}(u;a)\frac{t^n}{n!} = 1.$$

Therefore

$$\sum_{n=0}^{\infty}\left(u^v v! \sum_{j=0}^{n}\binom{n}{j} Y_{n-j}^{(v)}(u;a)(\ln a)^j S\left(j, v; a; \frac{1}{u}\right)\right)\frac{t^n}{n!} = 1.$$

From the above equation, we get the desired result. □

Algorithms are very important not only in Mathematics, but also in Computer Science and also Communications Systems. There are many applications of the algorithms in the related areas. Here we compute our numbers via an algorithm. Therefore, we are ready to give a computation algorithm for computing the values of the numbers $Y_n(u;a)$. The numbers $Y_n(u;a)$ has the following initial value:

$$Y_0(u;a) = \frac{1}{1-u}$$

as follows:

Algorithm 1 Let a be a real number with $a \geq 1$ and let $u \in \mathbb{C} \setminus \{1\}$. This algorithm will return the value of $Y_n(u,a)$ recursively.

procedure CALCULATE_Y(integer value n, u, a)
  Begin
  Inputs:
  $Y_0 \leftarrow 1/(1-u)$
  $Y_n \leftarrow 0$
  Outputs:
  $Y_n(u,a) \leftarrow Y_n$
  if $n = 0$ **then**
    $Y_n = Y_0$
  end if
  for all j in $\{1, 2, \ldots, n-1\}$ **do**
    $Y_n = Y_n + \text{Binomial\_Coef}(n, j) * Y(j, u, a) * \text{Power}(\ln(a), n-j)$
  end for
  $Y_n = Y_n * 1/(1-u)$
  return Y_n
end procedure

By using this computation algorithm, we compute the following a few values of the numbers $Y_n(u;a)$ as follows:

$$Y_1(u;a) = -\frac{\ln a}{(1-u)^2}$$

$$Y_2(u;a) = \frac{(\ln a)^2(-1-u)}{(1-u)^3},$$

$$Y_3(u;a) = \frac{\left(5+2u-u^2\right)(\ln a)^3}{(1-u)^4}\cdots$$

If we take $a = e$ and $u = -1$ in the above, we have

$$Y_0(-1;e) = \frac{1}{2},$$

$$Y_1(-1;e) = -\frac{1}{4},$$

$$Y_2(-1;e) = 0,$$

$$Y_3(-1;e) = \frac{1}{8}\cdots.$$

3. Some Properties of the Polynomials $Y_n(x,u;a)$

In this section, we investigate some properties of the polynomials $Y_n(x,u;a)$.

By using (3), we easily get the following relation:

$$\sum_{n=0}^{\infty} Y_n(x,u;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_n(u;a)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x\ln a)^n\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Y_n(x,u;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}x^{n-j}(\ln a)^{n-j}Y_j(u;a)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 3.1.

$$Y_n(x,u;a) = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}(\ln a)^{n-j}Y_j(u;a).$$

Now, we are going to differentiate (3) with respect to the variable x to derive a derivative formula for the polynomials $Y_n(x,u;a)$. That is

$$\frac{\partial}{\partial x}\mathcal{G}_Y(x,t,a,u) = t(\ln a)\mathcal{G}_Y(x,t,a,u).$$

By using the above equation with (3), we obtain the following theorem:

Theorem 3.2. *Let n be a positive integer. Then we have*

$$\frac{\partial}{\partial x} Y_n(x,u;a) = n(\ln a)\, Y_{n-1}(x;u;a).$$

The multiplication formula is very important for the normalized polynomials. Multiplication formulas of the Bernoulli and Euler polynomials have been many application in the theory of the Dedekind sums and the Hardy sums, *etc*. Therefore, we are ready to give a *multiplication formula* for the polynomial $Y_n(x,u;a)$ as follows:

Theorem 3.3. *Let m be a positive integer. Then we have*

$$Y_n(mx,u;a) = m^n \sum_{k=0}^{m-1} \frac{1}{u^{k-m+1}} Y_n\left(x + \frac{k}{m}, u^m; a\right).$$

Proof. By using (3), we get

$$\sum_{n=0}^{\infty} Y_n(x,u;a)\frac{t^n}{n!} = \frac{u^{m-1}}{a^{mt}-u^m} \sum_{k=0}^{m-1} \frac{a^{t(x+k)}}{u^k}.$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty} Y_n(x,u;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1} \frac{1}{u^{k-m+1}} Y_n\left(\frac{x+k}{m}, u^m; a\right) m^n\right)\frac{t^n}{n!}.$$

Replacing x by mx into the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 3.4.

$$Y_n(x+1;u;a) - uY_n(x;u;a) = (x\ln a)^n.$$

Proof. By using (3), we have

$$\sum_{n=0}^{\infty}\left(Y_n(x+1;u;a) - uY_n(x;u;a)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(x\ln a)^n\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

4. Further Remarks and Observation

The numbers $Y_n(u;a)$ are related to the many well-known numbers. That is it is easy to give relationships between these numbers and Euler, Frobenius-Euler and Apostol-Bernoulli numbers.

By using (1) and (2), we give the following functional equation:

$$F_1(t,0;u,a,b)G_Y(t,a,u) = G_Y(t,b,u).$$

By combining this equation with (1) and (2), we obtain

$$\sum_{n=0}^{\infty} Y_n(u;b)\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_n(u;a)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_n(u;a,b;1)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Y_n(u;b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} Y_j(u;a)\,\mathcal{H}_{n-j}(u;a,b;1)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4.1. *Let $a, b \in \mathbb{R}^+$ ($a \neq b$ and $a \geq 1$). Then we have*

$$Y_n(u;b) = \sum_{j=0}^{n} \binom{n}{j} Y_j(u;a)\, \mathcal{H}_{n-j}(u;a,b;1). \tag{6}$$

We observe that if we substitute $a = 1$ and $b = e$ into (6), we get recurrence relation for the Frobenius-Euler numbers. That is

$$Y_n(u;e) = \sum_{j=0}^{n} \binom{n}{j} Y_j(u;1)\, H_{n-j}(u),$$

where $H_n(u)$ denotes the Frobenius-Euler numbers. Since $Y_j(u;1) = 0$ if $j > 0$, we get

$$Y_n(u;e) = Y_0(u;1)\, H_n(u).$$

By using (5), we get

$$H_n(u) = -\sum_{j=0}^{n-1} \binom{n}{j} Y_j(u;e)$$

where $n \geq 1$.

If we take $u = -1$ and $a = e$ in (2), we have

$$\sum_{n=0}^{\infty} Y_n(-1;e) \frac{t^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Thus

$$Y_n(-1;e) = \frac{1}{2} E_n,$$

where E_n denotes the Euler numbers *cf.* ([3], [9], [13], [4], [8], [10], [5], [7], [12]).

Setting $a = e$ in (2), we have

$$\sum_{n=0}^{\infty} Y_n(u;e) \frac{t^n}{n!} = \frac{1}{ut} \sum_{n=0}^{\infty} B_n\left(\frac{1}{u}\right) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} unY_{n-1}(u;e) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n\left(\frac{1}{u}\right) \frac{t^n}{n!}.$$

Thus we get

$$B_n\left(\frac{1}{u}\right) = unY_{n-1}(u;e),$$

here $n \geq 1$ and $B_n\left(\frac{1}{u}\right)$ denotes the Apostol-Bernoulli polynomials *cf.* ([2], [3], [13], [4], [7], [8], [12]).

The conclusion of this paper is to combine generating functions for some well-known numbers and polynomials by our generating functions which are given in (2) and (3).

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