



\mathcal{I} –Lacunary Summability Function of Order α

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Abstract. In this paper, we further generalize recently introduced summability methods in [23](where ideals of \mathbb{N} were used to extend certain important summability methods) and introduce new notions, namely, \mathcal{I} –statistical convergence of order α , where $0 < \alpha < 1$ by taking nonnegative real-valued Lebesgue measurable function in the interval $(1, \infty)$. We mainly investigate their relationship and also make some observations about these classes. The study leaves a lot of interesting open problems.

1. Introduction

The idea of statistical convergence was formerly given under the name almost convergence by Zygmund [25] in the first edition of his celebrated monograph published in Warsaw in 1935. The concept was formally introduced by Steinhaus [24] and Fast [8] and later was introduced by Schoenberg [26]. If K is subset of \mathbb{N} , \mathbb{N} , the set of natural numbers, then the asymptotic density of K , denoted by $\delta(K)$, is given by

$$\delta(K) = \lim_n \frac{1}{n} | \{k \leq n : k \in K\} |$$

when ever the limits exists, where $|A|$ denotes the cardinality of the set A . A sequence $x = (x_k)$ of numbers is statistically convergent to L if

$$\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [9] and Šalát [15]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory.

The idea of statistical convergence was further extended to \mathcal{I} -convergence in [11] using the notion of ideals of \mathbb{N} with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [7, 16–21].

A lacunary sequence $\theta = (k_r)_{k=0}^{\infty}$ is an increasing sequence of integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$, as $r \rightarrow \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

Lacunary statistical convergence was introduced in [10] as follows.

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A sequence (x_k) of reals numbers is said to be lacunary statistical convergent to L (or, S_θ -convergent to L) if for any $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0.$$

In [10] the relation between lacunary statistical convergence and statistical convergence was established among other things.

Recently in [6], we used ideals to introduce the concepts of \mathcal{I} -statistical convergence and \mathcal{I} -lacunary statistical convergence which naturally extend the notions of the above mentioned convergence.

A generalized statistically convergent function via ideals was given by Savas in [22]. On the other hand in [1, 4, 5] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α , $0 < \alpha < 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence.

In this short paper we shall introduce the concept of \mathcal{I} -statistical convergence and \mathcal{I} -lacunary statistical convergence of order α , where $0 < \alpha < 1$ by taking nonnegative real-valued Lebesgue measurable function in the interval $(1, \infty)$. It should be note that Braha et al. [2] introduced a new Λ^2 -weighted statistical convergence and also Braha et al. [3] proved a Krovkin type theorem for the test function 1 , $\cos x$ and $\sin x$ in the space $C_{2\pi}(\mathbb{R})$ of all continuous 2π -periodic functions on the real line \mathbb{R} respectively. Further Mursaleen et al. [13] defined certain new spaces of statistical convergence and strongly summable sequences of fuzzy numbers.

In this paper we mainly investigate \mathcal{I} -statistical convergence of order α , where $0 < \alpha < 1$ by taking non-negative real-valued Lebesgue measurable function in the interval $(1, \infty)$ and also make some observations about these classes and most importantly the study leaves a lot of interesting open problems.

2. Main Results

The following definitions and notions will be needed.

Definition 2.1. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of \mathbb{N} if the following conditions hold:

- (a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
- (b) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

Definition 2.2. A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of \mathbb{N} if the following conditions hold:

- (a) $\emptyset \notin F$,
- (b) $A, B \in F$ implies $A \cap B \in F$,
- (c) $A \in F$, $A \subset B$ implies $B \in F$.

If \mathcal{I} is a proper ideal of \mathbb{N} (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of \mathbb{N} . It is called the filter associated with the ideal.

Definition 2.3. A proper ideal \mathcal{I} is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout \mathcal{I} will stand for a proper admissible ideal of \mathbb{N} .

Definition 2.4. (See [11]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in \mathbb{N} . The sequence $x = (x_n)$ of elements of \mathbb{R} is said to be \mathcal{I} -convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in \mathcal{I}$.

We now ready to give the main definitions.

Definition 2.5. A function $x(t)$ is said to be \mathcal{I} -statistically convergent of order α to L or $S(\mathcal{I})^\alpha$ -convergent to L , where $0 < \alpha \leq 1$, if for each $\epsilon > 0$ and $\delta > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{t \leq n : |x(t) - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $x(t) \rightarrow L(S(\mathcal{I})^\alpha)$. The class of all \mathcal{I} -statically convergent function of order α will be denoted by simply $S(\mathcal{I})^\alpha$.

Remark 2.6. If we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$. Then $S(\mathcal{I})^\alpha$ -convergence coincides with statistical convergence function of order α . For an arbitrary ideal \mathcal{I} and for $\alpha = 1$ it coincides with \mathcal{I} -statistical convergence of function. When $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes only statistical convergence of function.

Definition 2.7. Let θ be a lacunary sequence. A function $x(t)$ is said to be \mathcal{I} -lacunary statistically convergent of order α to L or $S_\theta(\mathcal{I})^\alpha$ -convergent to L if for any $\epsilon > 0$ and $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \leq I_r : |x(t) - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $x(t) \rightarrow L(S_\theta(\mathcal{I})^\alpha)$. The class of all \mathcal{I} -lacunary statistically convergent function of order α will be denoted by $S_\theta(\mathcal{I})^\alpha$.

Remark 2.8. For $\alpha = 1$ the definition coincides with \mathcal{I} -lacunary statistically convergent function (see, [23]). If we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$ and $\alpha = 1$, lacunary statistical convergence is a special case of \mathcal{I} -lacunary statistical convergence of order α , (see, [14]). Further it must be noted in this context that lacunary statistical convergence function of order α has not been studied till now. Moreover, if we take $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$, obviously lacunary statistical convergence of order α is a special case of \mathcal{I} -lacunary statistical convergence of order α . So properties of lacunary statistical convergence of order α can be easily obtained from our results with obvious modifications.

Theorem 2.9. Let $0 < \alpha \leq \beta \leq 1$. Then $S(\mathcal{I})^\alpha \subset S(\mathcal{I})^\beta$ and the inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $\mathcal{I} = \mathcal{I}_{fin}$.

Proof. Let $0 < \alpha \leq \beta \leq 1$. Then

$$\frac{|\{t \leq n : |x(t) - L| \geq \epsilon\}|}{n^\beta} \leq \frac{|\{t \leq n : |x(t) - L| \geq \epsilon\}|}{n^\alpha}$$

and so for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{|\{t \leq n : |x(t) - L| \geq \epsilon\}|}{n^\beta} \geq \delta \right\} \subset \left\{ n \in \mathbb{N} : \frac{|\{t \leq n : |x(t) - L| \geq \epsilon\}|}{n^\alpha} \geq \delta \right\}.$$

Hence if the set on the right hand side belongs to the ideal \mathcal{I} then obviously the set on the left hand side also belongs to \mathcal{I} . This shows that $S(\mathcal{I})^\alpha \subset S(\mathcal{I})^\beta$. To prove that the inclusion is strict for the above mentioned α, β consider the sequence $x = x(t)$ defined by

$$x(t) = 1, \text{ if } t = j^k$$

$$x(t) = 0, \text{ if } t \neq j^k, j \in \mathbb{N}.$$

Then $S(\mathcal{I})^\beta - \lim x(t) = 0$ i.e. $x \in S(\mathcal{I})^\beta$ but $x \notin S(\mathcal{I})^\alpha$ where $\mathcal{I} = \mathcal{I}_{fin}$. \square

Corollary 2.10. If a sequence is \mathcal{I} -statistically convergent of order α to L for some $0 < \alpha \leq 1$ then it is \mathcal{I} -statistically convergent to L i.e. $S(\mathcal{I})^\alpha \subset S(\mathcal{I})$

Similarly we can show that

Theorem 2.11. Let $0 < \alpha \leq \beta \leq 1$. Then

(i) $S_\theta(\mathcal{I})^\alpha \subset S_\theta(\mathcal{I})^\beta$.

(ii) In particular $S_\theta(\mathcal{I})^\alpha \subset S_\theta(\mathcal{I})$,

and the inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $\mathcal{I} = \mathcal{I}_{fin}$.

Definition 2.12. Let θ be a lacunary sequence. Then $x = x(t)$ is said to be $N_\theta(\mathcal{I})^\alpha$ -convergent to L if for any $\epsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \int_{t \in I_r} |x(t) - L| \geq \epsilon \right\} \in \mathcal{I}.$$

It is denoted by $x(t) \rightarrow L (N_\theta(\mathcal{I})^\alpha)$ and the class of such sequences will be denoted by simply $N_\theta(\mathcal{I})^\alpha$.

In the following we prove a result in line of Theorem 1 in [23] regarding \mathcal{I} -lacunary statistical convergence of order α .

Theorem 2.13. Let $\theta = (k_r)$ be a lacunary sequence, then

- (i)(a) $x(t) \rightarrow L (N_\theta(\mathcal{I})^\alpha) \Rightarrow x(t) \rightarrow L (S_\theta(\mathcal{I})^\alpha)$, and
 (b) $N_\theta(\mathcal{I})^\alpha$ is a proper subset of $S_\theta(\mathcal{I})^\alpha$.

Proof. (i)(a) If $\epsilon > 0$ and $x(t) \rightarrow L (N_\theta(\mathcal{I})^\alpha)$, we can write

$$\int_{t \in I_r} |x(t) - L| dt \geq \int_{t \in I_r, |x(t) - L| \geq \epsilon} |x(t) - L| dt \geq \epsilon |\{t \in I_r : |x(t) - L| \geq \epsilon\}|$$

and so

$$\frac{1}{\epsilon \cdot h_r^\alpha} \int_{t \in I_r} |x(t) - L| dt \geq \frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - L| \geq \epsilon\}|.$$

Then for any $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - L| \geq \epsilon\}| \geq \delta \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \int_{t \in I_r} |x(t) - L| dt \geq \epsilon \cdot \delta \right\} \in \mathcal{I}.$$

This proves the result.

(b) In order to establish that the inclusion $N_\theta(\mathcal{I})^\alpha \subset S_\theta(\mathcal{I})^\alpha$ is proper, let θ be given and $x(t)$ to be $1, 2, \dots, \lfloor \sqrt{h_r^\alpha} \rfloor$ at first $\lfloor \sqrt{h_r^\alpha} \rfloor$ integers in I_r and $x(t) = 0$ otherwise for all $r = 1, 2, 3, \dots$. Then for any $\epsilon > 0$,

$$\frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - 0| \geq \epsilon\}| \leq \frac{\lfloor \sqrt{h_r^\alpha} \rfloor}{h_r^\alpha}$$

and any $\delta > 0$ we get

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - 0| \geq \epsilon\}| \geq \delta \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{\lfloor \sqrt{h_r^\alpha} \rfloor}{h_r^\alpha} \geq \delta \right\}.$$

Since the set on the right hand side is a finite set and so belongs to \mathcal{I} , it follows that $x(t) \rightarrow 0 (S_\theta(\mathcal{I})^\alpha)$. On the other hand

$$\frac{1}{h_r^\alpha} \int_{t \in I_r} |x(t) - 0| dt = \frac{1}{h_r^\alpha} \cdot \frac{\lfloor \sqrt{h_r^\alpha} \rfloor (\lfloor \sqrt{h_r^\alpha} \rfloor + 1)}{2}.$$

Then

$$\begin{aligned} \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \int_{t \in I_r} |x(t) - 0| dt \geq \frac{1}{4} \right\} &= \left\{ r \in \mathbb{N} : \frac{\lfloor \sqrt{h_r^\alpha} \rfloor (\lfloor \sqrt{h_r^\alpha} \rfloor + 1)}{h_r^\alpha} \geq \frac{1}{2} \right\} \\ &= \{m, m + 1, m + 2, \dots\} \end{aligned}$$

for some $m \in \mathbb{N}$ which belongs to $F(\mathcal{I})$ since \mathcal{I} is admissible. So $x(t) \not\rightarrow 0 (N_\theta(\mathcal{I})^\alpha)$. \square

Remark 2.14. In Theorem 1 [23] it was further proved that

- (ii) $x(t) \in B(X, Y)$ and $x(t) \rightarrow L(S_\theta(\mathcal{I})) \Rightarrow x(t) \rightarrow L(N_\theta(\mathcal{I}))$,
- (iii) $S_\theta(\mathcal{I}) \cap B(X, Y) = N_\theta(\mathcal{I}) \cap B(X, Y)$, $B(X, Y)$, is set of bounded functions.

However whether these results remain true for $0 < \alpha < 1$ is not clear and we leave them as open problems.

In the following we investigate relationship between \mathcal{I} -statistical and \mathcal{I} -lacunary statistical convergence of order α for function.

Theorem 2.15. Let \mathcal{I} be an ideal and $\theta = (k_r)$ be a lacunary sequence with $\liminf q_r^\alpha > 1$, then \mathcal{I} -statistical convergence of order α implies \mathcal{I} -lacunary statistical convergence of order α .

Proof. Suppose first that $\liminf q_r^\alpha > 1$. Then there exists $\sigma > 0$ such that $q_r^\alpha \geq 1 + \sigma$ for sufficiently large r which implies that

$$\frac{h_r^\alpha}{k_r^\alpha} \geq \frac{\sigma}{1 + \sigma}.$$

Since $x(t) \rightarrow L(S(\mathcal{I})^\alpha)$, then for every $\epsilon > 0$ and for sufficiently large r , we have

$$\begin{aligned} \frac{1}{k_r^\alpha} |\{t \leq k_r : |x(t) - L| \geq \epsilon\}| &\geq \frac{1}{k_r^\alpha} |\{k \in I_r : |x(t) - L| \geq \epsilon\}| \\ &\geq \frac{\sigma}{1 + \sigma} \cdot \frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - L| \geq \epsilon\}|. \end{aligned}$$

Then for any $\delta > 0$, we get

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - L| \geq \epsilon\}| \geq \delta \right\} \\ &\subseteq \left\{ r \in \mathbb{N} : \frac{1}{k_r^\alpha} |\{t \leq k_r : |x(t) - L| \geq \epsilon\}| \geq \frac{\delta\sigma}{(1 + \sigma)} \right\} \in \mathcal{I}. \end{aligned}$$

This completes the proof. \square

Remark 2.16. The converse of this result is true for $\alpha = 1$ (see Theorem 2 [23]). However for $\alpha < 1$ it is not clear and we leave it as an open problem.

When \mathcal{I} -lacunary statistical convergence of order α implies \mathcal{I} -statistical convergence of order α ? The following theorem gives the answer: For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(\mathcal{I})$, $\cup \{n : k_{r-1} < n < k_r, r \in C\} \in F(\mathcal{I})$.

Theorem 2.17. For a lacunary sequence θ satisfying the above condition, \mathcal{I} -lacunary statistical convergence of order α implies \mathcal{I} -statistical convergence of order α , $0 < \alpha < 1$, if $\sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^\alpha}{(k_{r-1})^\alpha} = B(\text{say}) < \infty$.

Proof. Suppose that $x(t) \rightarrow L(S_\theta(\mathcal{I})^\alpha)$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{t \leq I_r : |x(t) - L| \geq \epsilon\}| < \delta \right\}$$

and

$$T = \left\{ r \in \mathbb{N} : \frac{1}{n^\alpha} |\{t \leq n : |x(t) - L| \geq \epsilon\}| < \delta_1 \right\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal \mathcal{I} . Further observe that

$$A_j = \frac{1}{h_j^\alpha} |\{t \in I_j : |x(t) - L| \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned} \frac{1}{n^\alpha} |\{t \leq n : |x(t) - L| \geq \epsilon\}| &\leq \frac{1}{k_{r-1}^\alpha} |\{t \leq k_r : |x(t) - L| \geq \epsilon\}| \\ &= \frac{1}{k_{r-1}^\alpha} |\{t \in I_1 : |x(t) - L| \geq \epsilon\}| + \dots + \frac{1}{k_{r-1}^\alpha} |\{t \in I_r : |x(t) - L| \geq \epsilon\}| \\ &= \frac{k_1^\alpha}{k_{r-1}^\alpha} \frac{1}{h_1^\alpha} |\{t \in I_1 : |x(t) - L| \geq \epsilon\}| + \frac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} \frac{1}{h_2^\alpha} |\{t \in I_2 : |x(t) - L| \geq \epsilon\}| + \dots + \\ &\quad + \frac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \frac{1}{h_r^\alpha} |\{t \in I_r : |x(t) - L| \geq \epsilon\}| \\ &= \frac{k_1^\alpha}{k_{r-1}^\alpha} A_1 + \frac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} A_2 + \dots + \frac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} A_r \\ &\leq \sup_{j \in C} A_j \cdot \sup_r \sum_{i=0}^{r-1} \frac{(k_{i+1} - k_i)^\alpha}{k_{r-1}^\alpha} < B\delta. \end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\cup \{n : k_{r-1} < t < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. \square

Theorem 2.18. Let $\theta = (\theta_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$,

(i) If

$$\liminf_{r \rightarrow \infty} \frac{h_r^\alpha}{\ell_r^\beta} > 0 \tag{1}$$

then $S_{\theta'}^\beta(I) \subseteq S_\theta^\alpha(\mathcal{I})$,

(ii) If

$$\lim_{r \rightarrow \infty} \frac{\ell_r}{h_r^\beta} = 1 \tag{2}$$

then $S_\theta^\alpha(\mathcal{I}) \subseteq S_{\theta'}^\beta(\mathcal{I})$.

Proof. (i) Easy, and omitted.

(ii) Let $x = x(t) \in S_{\theta}^{\alpha}(\mathcal{I})$ and (2) be satisfied. Since $I_r \subset J_s$, for $\varepsilon > 0$ we may write

$$\begin{aligned} \frac{1}{\ell_r^{\beta}} |\{k \in J_r : |x(t) - L| \geq \varepsilon\}| &= \frac{1}{\ell_r^{\beta}} |\{s_{r-1} < k \leq k_{r-1} : |x(t) - L| \geq \varepsilon\}| \\ &+ \frac{1}{\ell_r^{\beta}} |\{k_r < k \leq s_r : |x(t) - L| \geq \varepsilon\}| \\ &+ \frac{1}{\ell_r^{\beta}} |\{k_{r-1} < k \leq k_r : |x(t) - L| \geq \varepsilon\}| \\ &\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^{\beta}} + \frac{s_r - k_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}} |\{k \in I_r : |x(t) - L| \geq \varepsilon\}| \\ &= \frac{\ell_r - h_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}} |\{k \in I_r : |x(t) - L| \geq \varepsilon\}| \\ &\leq \frac{\ell_r - h_r}{\ell_r^{\beta}} + \frac{1}{h_r^{\beta}} |\{k \in I_r : |x(t) - L| \geq \varepsilon\}| \\ &\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1 \right) + \frac{1}{h_r^{\alpha}} |\{k \in I_r : |x(t) - L| \geq \varepsilon\}| \end{aligned}$$

for all $r \in \mathbb{N}$. Hence we have

$$\left\{ r \in \mathbb{N} : \frac{1}{\ell_r^{\beta}} |\{k \in J_r : |x(t) - L| \geq \varepsilon\}| \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{h_r^{\alpha}} |\{k \in I_r : |x(t) - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

This implies that $S_{\theta}^{\alpha}(\mathcal{I}) \subseteq S_{\theta'}^{\beta}(\mathcal{I})$. \square

From Theorem 2.8 we have the following.

Corollary 2.19. Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If (1) holds, then

- (i) $S_{\theta'}^{\alpha}(I) \subseteq S_{\theta}^{\alpha}(\mathcal{I})$ for each $\alpha \in (0, 1]$,
- (ii) $S_{\theta'}(\mathcal{I}) \subseteq S_{\theta}^{\alpha}(I)$ for each $\alpha \in (0, 1]$,
- (iii) $S_{\theta'}(\mathcal{I}) \subseteq S_{\theta}(\mathcal{I})$

If (2) holds then,

- (i) $S_{\theta}^{\alpha}(I) \subseteq S_{\theta'}^{\alpha}(\mathcal{I})$ for each $\alpha \in (0, 1]$,
- (ii) $S_{\theta}^{\alpha}(\mathcal{I}) \subseteq S_{\theta'}(\mathcal{I})$ for each $\alpha \in (0, 1]$,
- (iii) $S_{\theta}(I) \subseteq S_{\theta'}(\mathcal{I})$.

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