Filomat 31:10 (2017), 2901–2915
https://doi.org/10.2298/FIL1710901D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Studying the Krull Dimension of Finite Lattices Under the Prism of Matrices

**T. Dube[a], D.N. Georgiou[b], A.C. Megaritis[c], F. Sereti[b]**

[a]*Department of Mathematical Sciences, University of South Africa, P.O. Box 392, Pretoria, 0003, South Africa*
[b]*University of Patras, Department of Mathematics, 265 00 Patras, Greece*
[c]*Technological Educational Institute of Western Greece, Department of Accounting and Finance, 302 00 Messolonghi, Greece*

**Abstract.** The Krull dimension of a finite lattice $(X, \leqslant)$ is equal to the height of the poset of join prime elements of $X$ minus 1. To every partially ordered set we assign an order-matrix, and we use these order-matrices to characterize the join prime elements of finite lattices. In addition, we present a reduction algorithm for the computation of the height of a finite poset. The algorithm is based on the concept of the incidence matrix. Our main objective, ultimately, is to use these processes to calculate the Krull dimension of any given finite lattice.

## Introduction

The main goal in this paper is to use methods of matrix theory to calculate the Krull dimension of any finite lattice. This requires developing certain algorithms, starting with one for calculating the height of a finite partially ordered set using matrix theory. The matrices that come to play are what are called order-matrices.

We then characterize join prime elements of a finite lattice in terms of its order-matrix (Theorem 2.5). This theorem enables us to develop an algorithm for computing the set of join prime elements. The algorithm terminates in eight steps. This is the contents of Section 2.

In Section 3 we consider the height of a finite poset, and in this case the matrices that form the basic tools are the incidence matrices. The main step here in developing the algorithm for calculating the height of a finite poset with $n$ elements is Corollary 3.4 which, for any positive integer $k \leq n$, characterizes when the height of the poset is $k$, in terms of columns of the incidence matrix of the poset.

The results outlined above are used in Section 4 to calculate the Krull dimension of a finite lattice. We end the paper with a few questions associated with this topic.

## 1. Preliminaries

In this section we recall some definitions and notations (see, for example, [1–4]) that are needed in the sequel.

---

2010 *Mathematics Subject Classification*. Primary 06A07
*Keywords*. Krull dimension, finite poset, finite lattice, incidence matrix, order-matrix, join prime elements, prime filters, height
Received: 06 July 2016; Accepted: 07 December 2016
Communicated by Ljubiša D.R. Kočinac
*Email addresses:* dubeta@unisa.ac.za (T. Dube), georgiou@math.upatras.gr (D.N. Georgiou), thanasismeg13@gmail.com (A.C. Megaritis), seretifot@gmail.com (F. Sereti)

Let $X$ be a partially ordered set and $M \subseteq L$. The set of all upper bounds of $M$ is denoted by $\mathcal{UB}(M)$. A supremum (respectively, infimum) of $M$, if it exists, is denoted by $\vee M$ (respectively, $\wedge M$) and often referred to as the *join* (respectively, the *meet*) of $M$. We will also use the symbols $x \vee y$ for $\vee\{x, y\}$ and $x \wedge y$ for $\wedge\{x, y\}$. Henceforth, we shall write, as usual, "poset" for "partially ordered set".

The *height* of a finite poset $X$, denoted by $h(X)$, is the maximum of cardinalities of maximal chains in $X$. When we say a lattice is finite, we shall mean that its underlying set is a finite set. A finite lattice has a least element and a greatest element, which are denoted by 0 and 1, respectively.

Let $(X, \leqslant)$ be a finite lattice.

(1) A non-empty subset $F$ of $X$ is called a *filter* if $F$ has the following properties:

  (i) $F \neq X$.

  (ii) If $x \in F$ and $x \leqslant y$, then $y \in F$.

  (iii) If $x, y \in F$, then $x \wedge y \in F$.

  Thus, by a filter we mean a proper filter. The set of all filters of $X$ is denoted by $\mathcal{F}(X)$. For every $x \in X$, the subset $\uparrow x = \{y \in X : x \leqslant y\}$ of $X$ is called the *principal filter* generated by $x$.

(2) A filter $F$ is called *prime* if for every $x, y \in X$ with $x \vee y \in F$, we have $x \in F$ or $y \in F$. The set of all prime filters of $X$ is denoted by $\mathcal{PF}(X)$.

(3) If $\mathcal{PF}(X) \neq \emptyset$, then the *Krull dimension* (see, for example, [5] and [6]) of $(X, \leqslant)$ is defined as follows:

$$\mathrm{Kdim}(X) = \max\{k : \text{there exist prime filters } F_0 \subset F_1 \subset \cdots \subset F_k\}.$$

  It is known that a finite lattice $(X, \leqslant)$ has Krull dimension zero if and only if it is a Boolean algebra (see for example [7]).

(4) An element $x$ of $(X, \leqslant)$ is said to be *join prime* if it is nonzero and the inequality $x \leqslant a \vee b$ implies $x \leqslant a$ or $x \leqslant b$, for all $a, b \in X$. For distributive lattices, these are exactly the join-irreducible elements.

Let $(X, \leqslant)$ be a finite poset, where $X = \{x_1, \ldots, x_n\}$. The $n \times n$ matrix $\mathbf{A}_X^{\leqslant} = (\alpha_{ij})$, where

$$\alpha_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } x_i < x_j \\ -2, & \text{if } x_j < x_i \\ 0, & \text{if } x_i \parallel x_j \end{cases}$$

is called the *order-matrix* of $X$. For example, let $(X, \leqslant)$ be the poset represented by the diagram of Figure 1.

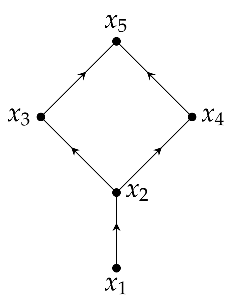


Figure 1: The poset $(X, \leqslant)$

The order-matrix of $X$ is the following $5 \times 5$ matrix:

$$\mathbf{A}_X^{\leqslant} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ -2 & 1 & 2 & 2 & 2 \\ -2 & -2 & 1 & 0 & 2 \\ -2 & -2 & 0 & 1 & 2 \\ -2 & -2 & -2 & -2 & 1 \end{pmatrix}.$$

In what follows, we denote by $r_1(A_X^{\leqslant}), \ldots, r_n(A_X^{\leqslant})$ the $n$ rows of the matrix $A_X^{\leqslant}$. Also, if $\alpha$ is an entry in the matrix $A_X^{\leqslant}$, then, for the sake of simplicity, we write $\alpha \in A_X^{\leqslant}$.

## 2. Join Prime Elements and Order-Matrices

Throughout this section, $(X, \leqslant)$ denotes a finite lattice, where $X = \{x_1, \ldots, x_n\}$, and $A_X^{\leqslant} = (\alpha_{ij})$ denotes the $n \times n$ order-matrix of $X$. We write $|S|$ for the cardinality of a set $S$.

**Proposition 2.1.** *Let $i_1, i_2$ be two distinct elements of $\{1, 2, \ldots, n\}$ such that*

$$r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}) = \begin{pmatrix} \alpha_1^{i_1 i_2} & \alpha_2^{i_1 i_2} & \ldots & \alpha_n^{i_1 i_2} \end{pmatrix}.$$

*The following statements are true:*

*(1) The set $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\}$ is one of the following sets: $\emptyset$, $\{i_1, i_2\}$.*

*(2) $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\} = \{i_1, i_2\}$ if and only if $x_{i_1} \parallel x_{i_2}$.*

*(3) $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\} \neq \emptyset$ if and only if $x_{i_1} \parallel x_{i_2}$.*

**Proof.** (1) We have $\alpha_{i_1 i_1} = \alpha_{i_2 i_2} = 1$. Let $i \in \{1, \ldots, n\}$ such that $\alpha_i^{i_1 i_2} = 1$. Then, $i = i_1$ or $i = i_2$.

(a) Let $\alpha_{i_1}^{i_1 i_2} = 1$.

In this case $\alpha_{i_2 i_1} = 0$ and, therefore, $\alpha_{i_1 i_2} = 0$ which means that $\alpha_{i_2}^{i_1 i_2} = 1$. Thus, $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\} = \{i_1, i_2\}$.

(b) Let $\alpha_{i_2}^{i_1 i_2} = 1$.

Similar to (a), $\alpha_{i_1 i_2} = 0$ and, therefore, $\alpha_{i_2 i_1} = 0$ which means that $\alpha_{i_1}^{i_1 i_2} = 1$. Thus, $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\} = \{i_1, i_2\}$.

(2) If $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\} = \{i_1, i_2\}$, then $\alpha_{i_1}^{i_1 i_2} = \alpha_{i_2}^{i_1 i_2} = 1$. Therefore, $\alpha_{i_1 i_2} = \alpha_{i_2 i_1} = 0$ which means that $x_{i_1} \parallel x_{i_2}$.

Conversely, let $x_{i_1} \parallel x_{i_2}$. Then, $\alpha_{i_1 i_2} = \alpha_{i_2 i_1} = 0$. Moreover, $\alpha_{i_1 i_1} = \alpha_{i_2 i_2} = 1$. Therefore, $\alpha_{i_1}^{i_1 i_2} = \alpha_{i_2}^{i_1 i_2} = 1$ which means that $\{i \in \{1, \ldots, n\} : \alpha_i^{i_1 i_2} = 1\} = \{i_1, i_2\}$.

(3) This statement is a consequence of statements (1) and (2). □

**Proposition 2.2.** *Let $i_0, i_1, i_2$ be three distinct elements of $\{1, 2, \ldots, n\}$ such that $x_{i_1} \parallel x_{i_2}$ and*

$$r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}) = \begin{pmatrix} \alpha_1^{i_0 i_1 i_2} & \alpha_2^{i_0 i_1 i_2} & \ldots & \alpha_n^{i_0 i_1 i_2} \end{pmatrix}.$$

*The following statements are true:*

*(1) The set $\{i \in \{1, \ldots, n\} : \alpha_i^{i_0 i_1 i_2} = 1\}$ is one of the following sets: $\emptyset$, $\{i_1\}$, $\{i_2\}$, $\{i_0, i_1, i_2\}$.*

*(2) If* $\left|\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\}\right| \neq 3$ *and*

$$3 \notin (-1)\cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}), \tag{2.1}$$

*then* $\alpha_{i_0i_1} = 2$ *or* $\alpha_{i_0i_2} = 2$.

**Proof.** (1) Since $x_{i_1} \parallel x_{i_2}$, we have $\alpha_{i_1i_2} = \alpha_{i_2i_1} = 0$. Also, we have $\alpha_{i_0i_0} = \alpha_{i_1i_1} = \alpha_{i_2i_2} = 1$. Let $i \in \{1,\dots,n\}$ such that $\alpha_i^{i_0i_1i_2} = 1$. Then, $i = i_0$ or $i = i_1$ or $i = i_2$.

(a) Let $\alpha_{i_0}^{i_0i_1i_2} = 1$.

In this case $\alpha_{i_1i_0} = \alpha_{i_2i_0} = 0$ and, therefore, $\alpha_{i_0i_1} = \alpha_{i_0i_2} = 0$ which means that $a_{i_1}^{i_0i_1i_2} = a_{i_2}^{i_0i_1i_2} = 1$. Thus, $\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\} = \{i_0, i_1, i_2\}$.

(b) Let $\alpha_{i_0}^{i_0i_1i_2} \neq 1$.

In this case $\alpha_{i_1i_0} \neq 0$ or $\alpha_{i_2i_0} \neq 0$.
(i) If $\alpha_{i_1i_0} \neq 0$, then $\alpha_{i_0i_1} \neq 0$ and, therefore, $\alpha_{i_1}^{i_0i_1i_2} \neq 1$. Thus, the set $\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\}$ is one of the following sets: $\emptyset$, $\{i_2\}$.
(ii) If $\alpha_{i_2i_0} \neq 0$, then $\alpha_{i_0i_2} \neq 0$ and, therefore, $\alpha_{i_2}^{i_0i_1i_2} \neq 1$. Thus, the set $\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\}$ is one of the following sets: $\emptyset$, $\{i_1\}$.

(2) By statement (1) we have the following cases:

(a) $\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\} = \{i_1\}$

In this case $\alpha_{i_0}^{i_0i_1i_2} \neq 1$, $\alpha_{i_0i_1} = 0$, and $\alpha_{i_0i_2} \in \{-2, 2\}$. By (2.1) we have $\alpha_{i_0i_2} = 2$.

(b) $\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\} = \{i_2\}$

In this case $\alpha_{i_0}^{i_0i_1i_2} \neq 1$, $\alpha_{i_0i_2} = 0$, and $\alpha_{i_0i_1} \in \{-2, 2\}$. By (2.1) we have $\alpha_{i_0i_1} = 2$.

(c) $\{i \in \{1,\dots,n\} : \alpha_i^{i_0i_1i_2} = 1\} = \emptyset$

In this case $\alpha_{i_0i_1}, \alpha_{i_0i_2} \in \{-2, 2\}$. If $\alpha_{i_0i_1} = -2$ or $\alpha_{i_0i_2} = -2$, then $3 \in (-1)\cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant})$, which is a contradiction. Thus, $\alpha_{i_0i_1} = \alpha_{i_0i_2} = 2$. □

**Proposition 2.3.** *Let* $j, i_1, i_2$ *be three distinct elements of* $\{1, 2, \dots, n\}$ *such that* $x_{i_1} \parallel x_{i_2}$. *The following statements are true:*

*(1)* $x_j \in \mathcal{UB}(\{x_{i_1}, x_{i_2}\})$ *if and only if*

$$5 \in r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}) + r_j(\mathbf{A}_X^{\leqslant}). \tag{2.2}$$

*(2) Let* $x_j \in \mathcal{UB}(\{x_{i_1}, x_{i_2}\})$. *Then,* $x_j \neq x_{i_1} \vee x_{i_2}$ *if and only if*

$$6 \in r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}) + (-1)\cdot r_j(\mathbf{A}_X^{\leqslant}). \tag{2.3}$$

**Proof.** (1) Let $x_j \in \mathcal{UB}(\{x_{i_1}, x_{i_2}\})$. Then, $\alpha_{i_1j} = \alpha_{i_2j} = 2$. Moreover, $\alpha_{jj} = 1$. Therefore, the condition (2.2) is satisfied.

Conversely, we suppose that the condition (2.2) is satisfied. Since $\alpha_{i_1i_1} = \alpha_{i_2i_2} = \alpha_{jj} = 1$ and $\alpha_{i_1i_2} = \alpha_{i_2i_1} = 0$, we have $\alpha_{i_1j} = \alpha_{i_2j} = 2$. Therefore, $x_j \in \mathcal{UB}(\{x_{i_1}, x_{i_2}\})$.

(2) We suppose that $x_j \neq x_{i_1} \vee x_{i_2}$. Then, there exists $k \in \{1,\dots,n\}$ such that $x_{i_1} < x_k$, $x_{i_2} < x_k$, and $x_k < x_j$. Therefore, $\alpha_{i_1k} = \alpha_{i_2k} = 2$ and $\alpha_{jk} = -2$. Thus, the condition (2.3) is satisfied.

Conversely, we suppose that the condition (2.3) is satisfied. Then, there exists $k \in \{1,\dots,n\}$ such that $\alpha_{i_1k} = \alpha_{i_2k} = 2$ and $\alpha_{jk} = -2$. Therefore, $x_{i_1} < x_k$, $x_{i_2} < x_k$, and $x_k < x_j$. This mean that $x_j \neq x_{i_1} \vee x_{i_2}$. □

**Proposition 2.4.** *Let $i_0, i_1, i_2$ be three distinct elements of $\{1, 2, \dots, n\}$ such that $x_{i_1} \parallel x_{i_2}$. Then, $x_{i_0} < x_{i_1} \vee x_{i_2}$ if and only if*

$$-6 \notin r_{i_0}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_1}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_2}(\mathbf{A}_X^{\leqslant}) \tag{2.4}$$

*and*

$$4 \notin 5 \cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}). \tag{2.5}$$

**Proof.** Let $x_{i_0} < x_{i_1} \vee x_{i_2}$. We suppose that the condition (2.4) is not satisfied. Then, there exists $k \in \{1, 2, \dots, n\}$ such that $\alpha_{i_0k} = -2$, $\alpha_{i_1k} = 2$, and $\alpha_{i_2k} = 2$. Thus, $x_{i_1} \vee x_{i_2} \leqslant x_k < x_{i_0}$, which is a contradiction.

Now, we suppose that the condition (2.5) is not satisfied. Then, there exists $k$ such that $\alpha_{i_0k} = 0$, $\alpha_{i_1k} = 2$, and $\alpha_{i_2k} = 2$. Therefore, $x_{i_1} < x_k$ and $x_{i_2} < x_k$ which means that $x_k \in \mathcal{UB}(\{x_{i_1}, x_{i_2}\})$. Hence, $x_{i_1} \vee x_{i_2} \leqslant x_k$. By assumption, $x_{i_0} < x_{i_1} \vee x_{i_2} \leqslant x_k$, which is a contradiction since $x_{i_0} \parallel x_k$.

Conversely, we suppose that the conditions (2.4) and (2.5) are satisfied. We prove that $x_{i_0} < x_{i_1} \vee x_{i_2}$. Let $x_j = x_{i_1} \vee x_{i_2}$. Then, $x_{i_1} < x_j$ and $x_{i_2} < x_j$. Therefore, $\alpha_{i_1j} = \alpha_{i_2j} = 2$. It suffices to prove that $x_{i_0} < x_j$. We have the following cases:

(a) Let $x_{i_0} \parallel x_j$.

In this case $\alpha_{i_0j} = 0$ and, therefore, $4 \in 5 \cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant})$, which is a contradiction.

(b) Let $x_j < x_{i_0}$.

In this case $\alpha_{i_0j} = -2$. Hence, $-6 \in r_{i_0}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_1}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_2}(\mathbf{A}_X^{\leqslant})$, which is a contradiction.

Thus, $x_{i_0} < x_j$. □

**Theorem 2.5.** *Let $i_0 \in \{1, 2, \dots, n\}$. The element $x_{i_0}$ is join prime if and only if $0 \in r_{i_0}(A_X^{\leqslant})$ or $-2 \in r_{i_0}(A_X^{\leqslant})$ and for every $i_1, i_2 \in \{1, \dots, n\} \setminus \{i_0\}$, with $x_{i_1} \parallel x_{i_2}$, the following conditions are satisfied:*

*(C1) If*

$$5 \in r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + r_{i_0}(A_X^{\leqslant}), \tag{2.6}$$

*then*

$$6 \in r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_0}(\mathbf{A}_X^{\leqslant}). \tag{2.7}$$

*(C2) If*

$$-6 \notin r_{i_0}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_1}(\mathbf{A}_X^{\leqslant}) + (-1) \cdot r_{i_2}(\mathbf{A}_X^{\leqslant}), \tag{2.8}$$

$$4 \notin 5 \cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}), \tag{2.9}$$

*then*

$$\left|\{i \in \{1, \dots, n\} : \alpha_i^{i_0i_1i_2} = 1\}\right| \neq 3 \tag{2.10}$$

*and*

$$3 \notin (-1) \cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}). \tag{2.11}$$

**Proof.** Let $x_{i_0} \in X$. We suppose that the conditions of the theorem are satisfied. We shall prove that the element $x_{i_0}$ is join prime. Since $0 \in r_{i_0}(A_X^{\leqslant})$ or $-2 \in r_{i_0}(A_X^{\leqslant})$, the element $x_{i_0}$ of $X$ is nonzero. Let $x_{i_1}, x_{i_2} \in X$, where $i_1, i_2 \in \{1, 2, \dots, n\} \setminus \{i_0\}$, such that $x_{i_1} \parallel x_{i_2}$ and $x_{i_0} \leqslant x_{i_1} \vee x_{i_2}$. It suffices to prove that $x_{i_0} \leqslant x_{i_1}$ or $x_{i_0} \leqslant x_{i_2}$. First we observe that $x_{i_0} < x_{1_1} \vee x_{i_2}$. Indeed, if $x_{i_0} = x_{i_1} \vee x_{i_2}$, then by Proposition 2.3 (1), the equation (2.6) is satisfied. Thus, $6 \in r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + (-1)r_{i_0}(A_X^{\leqslant})$ or equivalently, by Proposition 2.3 (2), $x_{i_0} \neq x_{i_1} \vee x_{i_2}$, which is a contradiction. Thus, $x_{i_0} < x_{1_1} \vee x_{i_2}$. Since $x_{i_1} \parallel x_{i_2}$, by Proposition 2.4, (2.8) and (2.9) are satisfied. Therefore, by assumption, (2.10) and (2.11) are satisfied. Thus, by Proposition 2.2 (2), $\alpha_{i_0 i_1} = 2$ or $\alpha_{i_0 i_2} = 2$. Equivalently, $x_{i_0} \leqslant x_{i_1}$ or $x_{i_0} \leqslant x_{i_2}$. Thus, $x_{i_0}$ is join prime element of X.

Conversely, let $x_{i_0}$ be a join prime element of $X$. We shall prove the conditions of the theorem. Since the element $x_{i_0}$ of $X$ is nonzero, we have $0 \in r_{i_0}(A_X^{\leqslant})$ or $-2 \in r_{i_0}(A_X^{\leqslant})$. We consider $i_1, i_2 \in \{1, \ldots, n\} \setminus \{i_0\}$ with $x_{i_1} \parallel x_{i_2}$. We suppose that the equation (2.6) is satisfied. We shall prove the equation (2.7). We suppose that $6 \notin r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + (-1) \cdot r_{i_0}(A_X^{\leqslant})$. By Proposition 2.3 (2), $x_{i_0} = x_{i_1} \vee x_{i_2}$, which is a contradiction since $x_{i_0}$ is join prime. We suppose that the conditions (2.8) and (2.9) are satisfied. Then, by Proposition 2.4, $x_{i_0} < x_{i_1} \vee x_{i_2}$. We prove the relations (2.10) and (2.11). We suppose that

$$\left|\{i \in \{1, \ldots, n\} : \alpha_i^{i_0 i_1 i_2} = 1\}\right| = 3. \tag{2.12}$$

By (2.12) we have $\alpha_{i_0 i_1} = \alpha_{i_0 i_2} = \alpha_{i_1 i_2} = 0$ and, therefore, $x_{i_0} \nless x_{i_1}$ and $x_{i_0} \nless x_{i_2}$ which is a contradiction (since $x_{i_0}$ is join prime). Now we suppose that

$$3 \in (-1) \cdot r_{i_0}(\mathbf{A}_X^{\leqslant}) + r_{i_1}(\mathbf{A}_X^{\leqslant}) + r_{i_2}(\mathbf{A}_X^{\leqslant}). \tag{2.13}$$

By (2.13) we have $\alpha_{i_0 i_1} = -2$ or $\alpha_{i_0 i_2} = -2$. Moreover, since $x_{i_0} < x_{i_1} \vee x_{i_2}$ and $x_{i_0}$ is a join prime element of $X$, we have $\alpha_{i_0 i_1} = 2$ or $\alpha_{i_0 i_2} = 2$. Therefore, we have the following cases:

(1) $\alpha_{i_0 i_1} = -2$ or $\alpha_{i_0 i_2} = 2$

In this case $x_{i_1} < x_{i_2}$, which is a contradiction since $x_{i_1} \parallel x_{i_2}$.

(2) $\alpha_{i_0 i_1} = 2$ or $\alpha_{i_0 i_2} = -2$

In this case $x_{i_2} < x_{i_1}$, which is a contradiction since $x_{i_1} \parallel x_{i_2}$. □

Let $X = \{x_1, \ldots, x_n\}$ be a finite lattice. Using Theorem 2.5 we can compute the set of join prime elements of $X$ in the following manner.

**Algorithm 2.6.** Our intended algorithm consists of the following steps:

**Step 1:** Find the $n$ rows $r_1(A_X^{\leqslant}), r_2(A_X^{\leqslant}), \ldots, r_n(A_X^{\leqslant})$ of the order-matrix $A_X^{\leqslant}$ of $X$.

**Step 2:** Set $k = 0$ and $j = 1$.

**Step 3:** If $0 \in r_j(A_X^{\leqslant})$ or $-2 \in r_j(A_X^{\leqslant})$, then go to Step 4. Otherwise, go to Step 7.

**Step 4:** Find the $1 \times n$ matrices

$$\begin{aligned}
&r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + r_j(A_X^{\leqslant})\\
&r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + (-1) \cdot r_j(A_X^{\leqslant})\\
&r_j(A_X^{\leqslant}) + (-1) \cdot r_{i_1}(A_X^{\leqslant}) + (-1) \cdot r_{i_2}(A_X^{\leqslant})\\
&5 \cdot r_j(A_X^{\leqslant}) + r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant})\\
&(-1) \cdot r_j(A_X^{\leqslant}) + r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}),
\end{aligned}$$

and the set

$$\{i \in \{1, \ldots, n\} : a_i^{j i_1 i_2} = 1\}$$

for each $i_1, i_2 \in \{1, 2, \ldots, n\} \setminus \{j\}$ with $\alpha_{i_1 i_2} = 0$.

**Step 5:** Check the following:

- If

$$5 \in r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + r_j(A_X^{\leqslant}),$$

then

$$6 \in r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}) + (-1) \cdot r_j(A_X^{\leqslant}).$$

- If
$$-6 \notin r_j(A_X^{\leqslant}) + (-1)\cdot r_{i_1}(A_X^{\leqslant}) + (-1)\cdot r_{i_2}(A_X^{\leqslant}),$$
$$4 \notin 5\cdot r_j(A_X^{\leqslant}) + r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}),$$
then
$$\left|\{i \in \{1,\ldots,n\} : a_i^{ji_1i_2} = 1\}\right| \neq 3$$
and
$$3 \notin (-1)\cdot r_j(A_X^{\leqslant}) + r_{i_1}(A_X^{\leqslant}) + r_{i_2}(A_X^{\leqslant}).$$

If the above are true, then put $x_{i_k} = x_j$ and go to Step 6. Otherwise, go to Step 7.

**Step 6:** Put $k \leftarrow k+1$ (meaning the number $k+1$ replaces the old $k$) and go to Step 7.

**Step 7:** If $j < n$, then put $j \leftarrow j+1$ (meaning the number $j+1$ replaces the old $j$) and go to Step 3. Otherwise, go to Step 8.

**Step 8:** Create the set $\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$.

**Remark 2.7.** Let $m$ be the number of 2-element antichains. Algorithm 2.6 iterates at most $n \cdot \binom{m}{2}$ times to calculate 5 matrices and the set mentioned above.

**Example 2.8.** Let $(X, \leqslant)$ be the lattice represented by the diagram of Figure 2.

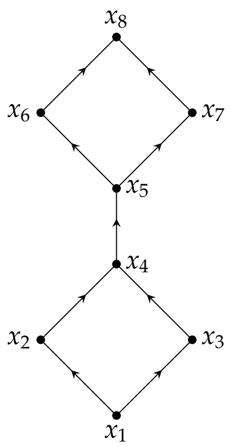


Figure 2: The lattice $(X, \leqslant)$

The order-matrix of $X$ is the following $8 \times 8$ matrix:

$$\mathbf{A}_X^{\leqslant} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ -2 & -2 & -2 & 1 & 2 & 2 & 2 & 2 \\ -2 & -2 & -2 & -2 & 1 & 2 & 2 & 2 \\ -2 & -2 & -2 & -2 & -2 & 1 & 0 & 2 \\ -2 & -2 & -2 & -2 & -2 & 0 & 1 & 2 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 \end{pmatrix}.$$

Using Theorem 2.5, we show that $x_4$ is not join prime, but $x_5$ is. Later on in Example 4.6 we shall exhibit all join primes of this lattice. Consider the element $x_5$. For the elements $x_6$ and $x_7$, for which $x_6 \parallel x_7$, we have

$$5 \notin r_6(A_X^{\leqslant}) + r_7(A_X^{\leqslant}) + r_5(A_X^{\leqslant}) = \begin{pmatrix} -6 & -6 & -6 & -6 & -3 & 3 & 3 & 6 \end{pmatrix}.$$

Moreover,

$$-6 \notin r_5(A_X^{\leqslant}) + (-1)\cdot r_6(A_X^{\leqslant}) + (-1)\cdot r_7(A_X^{\leqslant}) = \begin{pmatrix} 2 & 2 & 2 & 2 & 5 & 1 & 1 & -2 \end{pmatrix},$$

$$4 \notin 5\cdot r_5(A_X^{\leqslant}) + r_6(A_X^{\leqslant}) + r_7(A_X^{\leqslant}) = \begin{pmatrix} -14 & -14 & -14 & -14 & 1 & 11 & 11 & 14 \end{pmatrix},$$

$$\{i \in \{1,2,\ldots,8\} : a_i^{567} = 1\} = \emptyset$$

and

$$3 \notin (-1)\cdot r_5(A_X^{\leqslant}) + r_6(A_X^{\leqslant}) + r_7(A_X^{\leqslant}) = \begin{pmatrix} -2 & -2 & -2 & -2 & -5 & -1 & -1 & 2 \end{pmatrix}.$$

In a similar way we see that for the elements $x_2$ and $x_3$, for which $x_2 \parallel x_3$, the conditions of the theorem are satisfied. It follows directly that the element $x_5$ is join prime.

The element $x_4$ is not join prime since

$$5 \in r_2(A_X^{\leqslant}) + r_3(A_X^{\leqslant}) + r_4(A_X^{\leqslant}) = \begin{pmatrix} -6 & -1 & -1 & 5 & 6 & 6 & 6 & 6 \end{pmatrix}$$

and

$$6 \notin r_2(A_X^{\leqslant}) + r_3(A_X^{\leqslant}) + (-1)\cdot r_4(A_X^{\leqslant}) = \begin{pmatrix} -2 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

## 3. An Algorithm for Calculating the Height of a Finite Poset

We start by reminding the reader how the incidence matrix is defined.

**Definition 3.1.** (see [1]) Let $(X, \leqslant)$ be a poset, where $X = \{x_1, \ldots, x_n\}$. The $n \times n$ matrix $T_X^{\leqslant} = (t_{ij})$, where

$$t_{ij} = \begin{cases} 1, & \text{if } x_i \leqslant x_j \\ 0, & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of $X$.

**Notation 3.2.** (1) The $n$ columns of the incidence matrix $T_X^{\leqslant}$ are denoted by $C_1(T_X^{\leqslant}), \ldots, C_n(T_X^{\leqslant})$. Let

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \text{ and } C' = \begin{pmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{pmatrix}$$

be two $n \times 1$ matrices. Then, by $C + C'$ we denote the $n \times 1$ matrix

$$C + C' = \begin{pmatrix} c_1 + c'_1 \\ c_2 + c'_2 \\ \vdots \\ c_n + c'_n \end{pmatrix}.$$

Also, we write $C \le C'$ if only if $c_i \le c'_i$ for each $i = 1, \ldots, n$.

(2) We denote by $\mathbf{1}_{i_1}$ and $\mathbf{2}_{i_1}$, where $i_1 \in \{1, \ldots, n\}$, respectively, the $n \times 1$ matrices

$$\begin{pmatrix} \alpha_{i_1}^1 \\ \alpha_{i_1}^2 \\ \vdots \\ \alpha_{i_1}^n \end{pmatrix}, \quad \alpha_{i_1}^i = \begin{cases} 1, & \text{if } i = i_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\begin{pmatrix} \beta^1_{i_1} \\ \beta^2_{i_1} \\ \vdots \\ \beta^n_{i_1} \end{pmatrix}, \quad \beta^i_{i_1} = \begin{cases} 2, & \text{if } i = i_1 \\ 0, & \text{otherwise.} \end{cases}$$

Also, we denote by $\mathbf{2}_{i_1 i_2 \dots i_m}$, where $i_1, i_2, \dots, i_m$ are distinct elements of $\{1, \dots, n\}$ and $m \leq n$, the $n \times 1$ matrix

$$\begin{pmatrix} \gamma^1_{i_1 i_2 \dots i_m} \\ \gamma^2_{i_1 i_2 \dots i_m} \\ \vdots \\ \gamma^n_{i_1 i_2 \dots i_m} \end{pmatrix}, \quad \gamma^i_{i_1 i_2 \dots i_m} = \begin{cases} 2, & \text{if } i \in \{i_1, i_2, \dots, i_m\} \\ 0, & \text{otherwise.} \end{cases}$$

(3) Let $T^{\leqslant}_X = (t_{ij})$ be the incidence matrix of $X$. We denote by $C_{i_2 i_1}(T^{\leqslant}_X)$, where $i_1, i_2$ are distinct elements of $\{1, \dots, n\}$, the $n \times 1$ matrix

$$\begin{pmatrix} c^1_{i_2 i_1} \\ c^2_{i_2 i_1} \\ \vdots \\ c^n_{i_2 i_1} \end{pmatrix}, \quad c^i_{i_2 i_1} = \begin{cases} t_{i_2 i_1}, & \text{if } i = i_1 \\ 1, & \text{if } i = i_2 \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 3.3.** *Let $(X, \leqslant)$ be a poset, $X = \{x_1, \dots, x_n\}$ and let $i_1, i_2, \dots, i_k$, where $k \leq n$, be distinct elements of $\{1, \dots, n\}$. Then,*

$$\mathbf{1}_{i_1} + C_{i_2 i_1}(T^{\leqslant}_X) + C_{i_3 i_2}(T^{\leqslant}_X) + \ldots + C_{i_k i_{k-1}}(T^{\leqslant}_X) \geq \mathbf{2}_{i_1 i_2 i_3 \dots i_{k-1}} \tag{3.1}$$

*if and only if*

$$x_{i_1} > x_{i_2} > x_{i_3} > \ldots > x_{i_k}. \tag{3.2}$$

**Proof.** Let the condition (3.1) be satisfied. We prove that $x_{i_1} > x_{i_2} > x_{i_3} > \ldots > x_{i_k}$. We suppose that there exists $a \in \{1, 2, \dots, k-1\}$ such that $x_{i_{a+1}} \nless x_{i_a}$ or equivalently $t_{i_{a+1} i_a} = 0$. By the definition of the matrix $C_{i_{a+1} i_a}(T^{\leqslant}_X)$, we have $c^{i_a}_{i_{a+1} i_a} = t_{i_{a+1} i_a} = 0$.

Let $a = 1$. Then, $c^{i_1}_{i_2 i_1} = 0$. Since the elements $i_1, i_2, \dots, i_k$ of $\{1, \dots, n\}$ are distinct, by the definition of the matrix $C_{i_{l+1} i_l}(T^{\leqslant}_X)$ we have $c^{i_1}_{i_{l+1} i_l} = 0$, for each $l \in \{2, 3, \dots, k-1\}$. Thus,

$$\alpha^{i_1}_{i_1} + c^{i_1}_{i_2 i_1} + c^{i_1}_{i_3 i_2} + \ldots + c^{i_1}_{i_k i_{k-1}} = 1 + 0 + 0 + \ldots + 0 = 1 < 2 = \gamma^{i_1}_{i_1 i_2 i_3 \dots i_{k-1}}$$

and, therefore, $\mathbf{1}_{i_1} + C_{i_2 i_1}(T^{\leqslant}_X) + C_{i_3 i_2}(T^{\leqslant}_X) + \ldots + C_{i_k i_{k-1}}(T^{\leqslant}_X) \ngeq \mathbf{2}_{i_1 i_2 i_3 \dots i_{k-1}}$, which is a contradiction.

Let $a > 1$. By the definition of the matrix $C_{i_a i_{a-1}}(T^{\leqslant}_X)$, $c^{i_a}_{i_a i_{a-1}} = 1$. Since the elements $i_1, i_2, \dots, i_k$ of $\{1, \dots, n\}$ are distinct, by the definition of the matrix $C_{i_{l+1} i_l}(T^{\leqslant}_X)$ we have $c^{i_a}_{i_{l+1} i_l} = 0$, for each $l \in \{1, 2, \dots, k-1\} \setminus \{a-1, a\}$. Hence,

$$\alpha^{i_a}_{i_1} + c^{i_a}_{i_2 i_1} + \ldots + c^{i_a}_{i_a i_{a-1}} + c^{i_a}_{i_{a+1} i_a} + \ldots + c^{i_a}_{i_k i_{k-1}} = 0 + 0 + \ldots + 1 + 0 + \ldots + 0 = 1 < 2 = \gamma^{i_a}_{i_1 i_2 i_3 \dots i_{k-1}}$$

and, therefore, $\mathbf{1}_{i_1} + C_{i_2 i_1}(T^{\leqslant}_X) + C_{i_3 i_2}(T^{\leqslant}_X) + \ldots + C_{i_k i_{k-1}}(T^{\leqslant}_X) \ngeq \mathbf{2}_{i_1 i_2 i_3 \dots i_{k-1}}$, which is a contradiction. Thus, in each case the condition (3.2) is satisfied.

Conversely, let the condition (3.2) be satisfied. We prove that

$$\mathbf{1}_{i_1} + C_{i_2 i_1}(T^{\leqslant}_X) + C_{i_3 i_2}(T^{\leqslant}_X) + \ldots + C_{i_k i_{k-1}}(T^{\leqslant}_X) \geq \mathbf{2}_{i_1 i_2 i_3 \dots i_{k-1}}.$$

We suppose that $\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant}) + C_{i_3i_2}(T_X^{\leqslant}) + \ldots + C_{i_ki_{k-1}}(T_X^{\leqslant}) \ngeq \mathbf{2}_{i_1i_2i_3\ldots i_{k-1}}$. There exists $a \in \{1, 2, \ldots, k-1\}$ such that $\alpha_{i_1}^{i_a} + c_{i_2i_1}^{i_a} + \ldots + c_{i_ki_{k-1}}^{i_a} \in \{0, 1\}$.

Let $a = 1$. Then, $\alpha_{i_1}^{i_1} = 1$. Hence, $\alpha_{i_1}^{i_1} + c_{i_2i_1}^{i_1} + \ldots + c_{i_ki_{k-1}}^{i_1} = 1$ and $c_{i_2i_1}^{i_1} = 0$. Therefore, by the definition of the matrix $C_{i_2i_1}(T_X^{\leqslant})$, we have $t_{i_2i_1} = 0$ or equivalently $x_{i_2} \nless x_{i_1}$, which is a contradiction.

Let $a > 1$. By the definition of the matrix $C_{i_ai_{a-1}}(T_X^{\leqslant})$, $c_{i_ai_{a-1}}^{i_a} = 1$. So we have

$$\alpha_{i_1}^{i_a} + c_{i_2i_1}^{i_a} + \ldots + c_{i_ai_{a-1}}^{i_a} + c_{i_{a+1}i_a}^{i_a} + \ldots + c_{i_ki_{k-1}}^{i_k} = 1.$$

Thus, $c_{i_{a+1}i_a}^{i_a} = 0$ and, therefore, by the definition of the matrix $C_{i_{a+1}i_a}(T_X^{\leqslant})$, we have $t_{i_{a+1}i_a} = 0$ or equivalently $x_{i_{a+1}} \nless x_{i_a}$, which is a contradiction. Thus, the condition (3.1) is satisfied. □

**Corollary 3.4.** *Let $(X, \leqslant)$ be a poset, where $X = \{x_1, \ldots, x_n\}$. Then, $h(X) = k$, $1 \leq k \leq n$, if and only if there exist distinct elements $i_1^k, i_2^k, \ldots, i_k^k$ of $\{1, \ldots, n\}$ such that*

*(1)* $\mathbf{1}_{i_1^k} + C_{i_2^ki_1^k}(T_X^{\leqslant}) + \ldots + C_{i_k^ki_{k-1}^k}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^ki_2^k\ldots i_{k-1}^k}$

*(2)* $\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant}) + \ldots + C_{i_{k+1}i_k}(T_X^{\leqslant}) \ngeq \mathbf{2}_{i_1i_2\ldots i_k}$ *for every distinct elements $i_1, i_2, \ldots, i_{k+1}$ of $\{1, \ldots, n\}$.*

**Proof.** Let $h(X) = k$, where $1 \leq k \leq n$. Then, there exist distinct elements $i_1^k, i_2^k, \ldots, i_k^k$ of $\{1, \ldots, n\}$ such that $x_{i_1^k} < x_{i_2^k} < \ldots < x_{i_k^k}$. By Theorem 3.3 we have

$$\mathbf{1}_{i_1^k} + C_{i_2^ki_1^k}(T_X^{\leqslant}) + \ldots + C_{i_k^ki_{k-1}^k}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^ki_2^k\ldots i_{k-1}^k}.$$

Also, any subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}\}$ of $X$ is not a chain or equivalently, by Theorem 3.3,

$$\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant}) + \ldots + C_{i_{k+1}i_k}(T_X^{\leqslant}) \ngeq \mathbf{2}_{i_1i_2\ldots i_k}.$$

Conversely, suppose that there exist distinct elements $i_1^k, i_2^k, \ldots, i_k^k$ of $\{1, \ldots, n\}$ such that

$$\mathbf{1}_{i_1^k} + C_{i_2^ki_1^k}(T_X^{\leqslant}) + \ldots + C_{i_k^ki_{k-1}^k}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^ki_2^k\ldots i_{k-1}^k}$$

and

$$\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant}) + \ldots + C_{i_{k+1}i_k}(T_X^{\leqslant}) \ngeq \mathbf{2}_{i_1i_2\ldots i_k}$$

for every distinct elements $i_1, i_2, \ldots, i_{k+1}$ of $\{1, \ldots, n\}$. Then, by Theorem 3.3, we have $x_{i_1^k} < x_{i_2^k} < \ldots < x_{i_k^k}$. Also, any subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}\}$ of $X$ is not a chain. Thus, $h(X) = k$. □

Using Corollary 3.4 we give an algorithm for computing the height of a finite poset.

**Algorithm 3.5.** Let $(X, \leqslant)$ be a poset, where $X = \{x_1, \ldots, x_n\}$. Our intended algorithm consists of the following $n-1$ steps:

**Step 1.**
Find the sums $\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant})$ for each distinct elements $i_1, i_2$ of $\{1, \ldots, n\}$. If there exist distinct elements $i_1^1, i_2^1$ of $\{1, \ldots, n\}$ such that $\mathbf{1}_{i_1^1} + C_{i_2^1i_1^1}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^1}$, then go to the Step 2. Otherwise print $h(X) = 1$.

**Step 2.**
Find the sums $\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant}) + C_{i_3i_2}(T_X^{\leqslant})$ for each distinct elements $i_1, i_2, i_3$ of $\{1, \ldots, n\}$ with $\mathbf{1}_{i_1} + C_{i_2i_1}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1}$. If there exist distinct elements $i_1^2, i_2^2, i_3^2$ of $\{1, \ldots, n\}$ such that $\mathbf{1}_{i_1^2} + C_{i_2^2i_1^2}(T_X^{\leqslant}) + C_{i_3^2i_2^2}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^2i_2^2}$, then go to the Step 3. Otherwise print $h(X) = 2$.

.........
.........
.........

**Step $n-2$.**
Find the sums $\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + \ldots + C_{i_{n-1} i_{n-2}}(T_X^{\leqslant})$ for each distinct elements $i_1, i_2, \ldots, i_{n-1}$ of $\{1, \ldots, n\}$ with

$$\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + \ldots + C_{i_{n-2} i_{n-3}}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1 i_2 \ldots i_{n-3}}.$$

If there exist distinct elements $i_1^{n-2}, i_2^{n-2}, \ldots, i_{n-1}^{n-2}$ of $\{1, \ldots, n\}$ such that

$$\mathbf{1}_{i_1^{n-2}} + C_{i_2^{n-2} i_1^{n-2}}(T_X^{\leqslant}) + \ldots + C_{i_{n-1}^{n-2} i_{n-2}^{n-2}}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^{n-2} i_2^{n-2} \ldots i_{n-2}^{n-2}},$$

then go to the Step $n-1$. Otherwise print $h(X) = n-2$.

**Step $n-1$.**
Find the sums $\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + \ldots + C_{i_n i_{n-1}}(T_X^{\leqslant})$ for each distinct elements $i_1, i_2, \ldots, i_n$ of $\{1, \ldots, n\}$ with

$$\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + \ldots + C_{i_{n-1} i_{n-2}}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1 i_2 \ldots i_{n-2}}.$$

If there exist distinct elements $i_1^{n-1}, i_2^{n-1}, \ldots, i_n^{n-1}$ of $\{1, \ldots, n\}$ such that

$$\mathbf{1}_{i_1^{n-1}} + C_{i_2^{n-1} i_1^{n-1}}(T_X^{\leqslant}) + \ldots + C_{i_n^{n-1} i_{n-1}^{n-1}}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1^{n-1} i_2^{n-1} \ldots i_{n-1}^{n-1}},$$

then print $h(X) = n$. Otherwise print $h(X) = n-1$.

**Proposition 3.6.** *An upper bound on the number of iterations of the Algorithm 3.5 is the number*

$$n(n-1)\Big(1 + (n-2) + (n-2)(n-3) + \ldots + (n-2)!\Big).$$

**Proof.** We observe that the number of iterations Algorithm 3.5 performs in Steps $1,\ 2,\ 3,\ \ldots,\ n-1$ is

$$n(n-1),\ n(n-1)(n-2),\ n(n-1)(n-2)(n-3),\ \ldots,\ n!,$$

respectively. Thus, the number of iterations the algorithm performs is

$$\begin{aligned} &n(n-1) + n(n-1)(n-2) + n(n-1)(n-2)(n-3) + \ldots + n! = \\ &n(n-1)\Big(1 + (n-2) + (n-2)(n-3) + \ldots + (n-2)!\Big). \ \square \end{aligned}$$

**Example 3.7.** Let $(X, \leqslant)$ be the poset represented by the diagram of Figure 3.

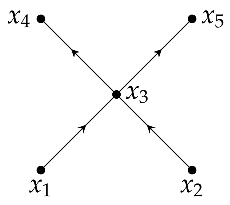


Figure 3: The poset $(X, \leqslant)$

The incidence matrix of $X$ is

$$T_X^{\leqslant} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to find $h(X)$ we follow the following steps (see Algorithm 3.5):

**Step 1.** Find the sums $\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant})$ for every $i_1, i_2 \in \{1, 2, \ldots, 5\}$.

We observe that there exist $5, 3 \in \{1, 2, \ldots, 5\}$ such that

$$\mathbf{1}_5 + C_{35}(T_X^{\leqslant}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \mathbf{2}_5.$$

Thus, $h(X) > 1$.

**Step 2.** Find the sums $\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + C_{i_3 i_2}(T_X^{\leqslant})$ for every $i_1, i_2, i_3 \in \{1, 2, \ldots, 5\}$ with $\mathbf{1}_{i_1} + C_{i_2 i_1} \geq \mathbf{2}_{i_1}$.

We observe that there exist $5, 3, 2 \in \{1, 2, \ldots, 5\}$ such that

$$\mathbf{1}_5 + C_{35}(T_X^{\leqslant}) + C_{23}(T_X^{\leqslant}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \mathbf{2}_{53}.$$

Thus, $h(X) > 2$.

**Step 3.** Find the sums

$$\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + C_{i_3 i_2}(T_X^{\leqslant}) + C_{i_4 i_3}(T_X^{\leqslant})$$

for every $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, 5\}$ with $\mathbf{1}_{i_1} + C_{i_2 i_1}(T_X^{\leqslant}) + C_{i_3 i_2}(T_X^{\leqslant}) \geq \mathbf{2}_{i_1 i_2}$.

We have only the following cases:

$$\mathbf{1}_4 + C_{34}(T_X^{\leqslant}) + C_{23}(T_X^{\leqslant}) + C_{12}(T_X^{\leqslant}) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} \ngeq \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \mathbf{2}_{432},$$

$$\mathbf{1}_5 + C_{35}(T_X^{\leqslant}) + C_{23}(T_X^{\leqslant}) + C_{12}(T_X^{\leqslant}) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} \ngeq \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \mathbf{2}_{532}.$$

Thus, $h(X) = 3$.

## 4. Krull Dimension of Finite Lattices through the Order Matrices

Throughout this section $(X, \leqslant)$ denotes a lattice, where $X = \{x_1, \ldots, x_n\}$ and $A_X^{\leqslant} = (\alpha_{ij})$ denotes the $n \times n$ order-matrix of $X$.

**Proposition 4.1.** *The following statements are true:*

*(1) For every $i \in \{1, \ldots, n\}$ we have $\uparrow x_i = \left\{x_j : \alpha_{ij} \in \{1, 2\}\right\}$.*

*(2) $\mathcal{F}(X) = \left\{\{x_j : \alpha_{ij} \in \{1, 2\}\} : i = 1, 2, \ldots, n\right\} \setminus \{X\}$.*

*(3)* $\uparrow x_{i_0} \subset \uparrow x_{i_1}$ *if and only if* $x_{i_1} < x_{i_0}$.

*(4)* $\uparrow x_{i_0} \subset \uparrow x_{i_1}$ *if and only if* $\alpha_{i_0 i_1} = -2$.

**Proof.** (1) Follows from the definitions of the set $\uparrow x_i$ and the order-matrix.

(2) Since $X$ is finite, we have $\mathcal{F}(X) = \{\uparrow x_i : i = 1, 2, \ldots, n\} \setminus \{X\}$. Therefore, by statement (1) we have that $\mathcal{F}(X)$ is the desired set.

(3) Let $\uparrow x_{i_0} \subset \uparrow x_{i_1}$. Then, $x_{i_0} \in \uparrow x_{i_0} \subset \uparrow x_{i_1}$ and, therefore, $x_{i_1} < x_{i_0}$. Conversely, let $x_{i_1} < x_{i_0}$. We prove that $\uparrow x_{i_0} \subset \uparrow x_{i_1}$. Indeed, let $x_j \in \uparrow x_{i_0}$. Then, $x_{i_0} \leqslant x_j$ and, therefore, $x_{i_1} < x_j$. Thus, $x_j \in \uparrow x_{i_1}$.

(4) Follows from the definition of the order-matrix and the statement (3). □

**Proposition 4.2.** *Let* $x_{i_0}, x_{i_1}, \ldots, x_{i_k} \in X$. *The following statements are equivalent:*

*(1)* $\uparrow x_{i_0} \subset \uparrow x_{i_1} \subset \cdots \subset \uparrow x_{i_k}$.

*(2)* $x_{i_0} > x_{i_1} > \cdots > x_{i_k}$.

*(3)* $\alpha_{i_0 i_1} = \alpha_{i_1 i_2} = \cdots = \alpha_{i_{k-1} i_k} = -2$.

**Proof.** The proof is a straightforward verification from Proposition 4.1 (3) and (4). Indeed, for $l = 1, 2, \ldots, k$ we have $\uparrow x_{i_{l-1}} \subset \uparrow x_{i_l} \Leftrightarrow x_{i_l} < x_{i_{l-1}} \Leftrightarrow \alpha_{i_{l-1} i_l} = -2$. □

**Proposition 4.3.** *Let* $x_{i_0}, x_{i_1} \in X$. *Then,* $\uparrow x_{i_0} \subset \uparrow x_{i_1}$ *if and only if the following condition is satisfied:*

$$1 \notin r_{i_0}(\mathbf{A}_X^{\leqslant}) - r_{i_1}(\mathbf{A}_X^{\leqslant}).$$

**Proof.** Let $\uparrow x_{i_0} \subset \uparrow x_{i_1}$. We suppose that $1 \in r_{i_0}(\mathbf{A}_X^{\leqslant}) - r_{i_1}(\mathbf{A}_X^{\leqslant})$. We have the following cases:

(1) $\alpha_{i_1 i_1} = 1$ and $\alpha_{i_0 i_1} = 2$.

(2) $\alpha_{i_0 i_0} = 1$ and $\alpha_{i_0 i_1} = 0$.

Therefore, in each case $\alpha_{i_0 i_1} \neq -2$, which contradicts Proposition 4.1 (4). Thus, $1 \notin r_{i_0}(\mathbf{A}_X^{\leqslant}) - r_{i_1}(\mathbf{A}_X^{\leqslant})$.

Conversely, we suppose that the condition of the proposition is satisfied. We prove that $\uparrow x_{i_0} \subset \uparrow x_{i_1}$. By Proposition 4.1 (4) it suffices to prove that $\alpha_{i_0 i_1} = -2$. We have the following cases:

(1) Let $\alpha_{i_0 i_1} = 0$.

Since $\alpha_{i_0 i_0} = 1$ and $\alpha_{i_1 i_0} = 0$, we have $1 \in r_{i_0}(\mathbf{A}_X^{\leqslant}) - r_{i_1}(\mathbf{A}_X^{\leqslant})$, which is a contradiction.

(2) Let $\alpha_{i_0 i_1} = 2$.

Since $\alpha_{i_0 i_1} = 2$ and $\alpha_{i_1 i_1} = 1$, we have $1 \in r_{i_0}(\mathbf{A}_X^{\leqslant}) - r_{i_1}(\mathbf{A}_X^{\leqslant})$, which is a contradiction. □

**Theorem 4.4.** *Let* $x_{i_0}, x_{i_1}, \ldots, x_{i_k} \in X$. *Then,* $\uparrow x_{i_0} \subset \uparrow x_{i_1} \subset \cdots \subset \uparrow x_{i_k}$ *if and only if the following condition is satisfied:*

$$1 \notin r_{i_{l-1}}(\mathbf{A}_X^{\leqslant}) - r_{i_l}(\mathbf{A}_X^{\leqslant}) \ \textit{for} \ \ l = 1, 2, \ldots, k.$$

**Proof.** The proof is a straightforward verification from Proposition 4.3. Indeed, for $l = 1, 2, \ldots, k$ we have $\uparrow x_{i_{l-1}} \subset \uparrow x_{i_l}$ if and only if $1 \notin r_{i_{l-1}}(\mathbf{A}_X^{\leqslant}) - r_{i_l}(\mathbf{A}_X^{\leqslant})$. □

**Proposition 4.5.** *If* $\mathcal{PF}(X) \neq \emptyset$, *then the following statements hold:*

*(1)* $\mathcal{PF}(X) = \{\uparrow x : x$ *is join prime*$\}$.

*(2)* $\mathcal{PF}(X) = \Big\{\{x_j : a_{ij} \in \{1, 2\}\} : x_i$ *is join prime*$\Big\}$.

*(3)* $\mathrm{Kdim}(X) = \max\{k : \uparrow x_{i_0} \subset \cdots \subset \uparrow x_{i_k},\ x_{i_0}, \ldots, x_{i_k}$ *are join primes*$\}$.

*(4)* $\mathrm{Kdim}(X) = \max\{k : x_{i_0} > \cdots > x_{i_k},\ x_{i_0}, \ldots, x_{i_k}$ *are join primes*$\}$.

*(5)* $\mathrm{Kdim}(X) = h(Y) - 1$, *where* $Y$ *is the poset of join primes elements of* $X$.

**Proof.** (1) Let $F \in \mathcal{PF}(X)$. Since $X$ is finite, $\wedge F \in X$. Let $x = \wedge F$. We observe that $F = \uparrow x$. We prove that $x$ is join prime. Let $a, b \in X$ such that $x \leqslant a \vee b$. Then, $a \vee b \in F$. Hence, $a \in F$ or $b \in F$ and, therefore, $x \leqslant a$ or $x \leqslant b$. Thus, $x$ is join prime.

Conversely, let $x$ be a join prime element of $X$. We set $F = \uparrow x$ and prove that $F \in \mathcal{PF}(X)$. Let $a, b \in X$ such that $a \vee b \in F$. Then, $x \leqslant a \vee b$. Hence, $x \leqslant a$ or $x \leqslant b$ and, therefore, $a \in F$ or $b \in F$. Thus, $F$ is prime.

(2) It follows from Proposition 4.1 (1) and from the (1) of this proposition.

(3) It is known that

$$\mathrm{Kdim}(X) = \max\{k : \text{there exist prime filters } F_0 \subset F_1 \subset \ldots \subset F_k\}.$$

Therefore, the result follows from the (1).

(4) Follows from Proposition 4.2 and the statement (3).

(5) Follows from the definition of the height and the statement (4). □

**Example 4.6.** Consider the finite lattice $(X, \leqslant)$ which is represented in the Example 2.8. By Algorithm 2.6 we find that the poset of join primes elements of $X$ is $Y = \{x_2, x_3, x_5, x_6, x_7\}$ and represented by the diagram of Figure 4.

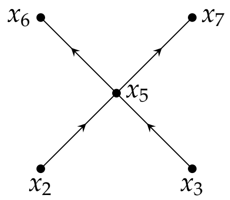


Figure 4: The poset $(Y, \leqslant)$

By Example 3.7 we have $h(Y) = 3$. Hence, by Proposition 4.5 (5),

$$\mathrm{Kdim}(X) = h(Y) - 1 = 2.$$

## 5. Questions

An element $x$ of a finite lattice $(X, \leqslant)$ is said to be *meet prime* (see, for example, [4]) if $x \neq 1$ and the inequality $a \wedge b \leqslant x$ implies $a \leqslant x$ or $b \leqslant x$, for all $a, b \in X$.

**Question 1.** Are there some characterizations of the meet-prime elements of a finite lattice, using order matrices?

**Question 2.** We know that for an arbitrary finite lattice $(X, \leqslant)$ with $\mathcal{PF}(X) \neq \emptyset$ the Krull dimension can be defined, also, as follows:

$$\mathrm{Kdim}(X) = \max\{k : \text{there exist prime ideals } P_0 \subset P_1 \subset \cdots \subset P_k\}.$$

Are there some characterizations of the prime ideals and, therefore, of the Krull dimension, using matrices?

**Acknowledgements.** The authors would like to thank the referees for their careful reading of this paper and helpful comments.

## References

[1] G. Birkhoff, Lattice theory, American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, R.I., 1979, vi+418 pp.

[2] D. N. Georgiou, A. C. Megaritis and F. Sereti, A study of the order dimension of a poset through the matrix theory, accepted for publication in Quaestiones Mathematicae (2015).

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. S. Scott, Continuous Lattices and Domains, Encyclopedia of Mathematics and its Applications, 93. Cambridge University Press, Cambridge, 2003, xxxvi+591 pp.

[4] S. Roman, Lattices and ordered sets, Springer, New York, 2008, xvi+305 pp.

[5] Juan B. Sancho de Salas, M. Teresa Sancho de Salas, Dimension of distributive lattices and universal spaces, Topology Appl. 42 (1991), no. 1, 25–36.

[6] V. G. Vinokurov, A lattice method of defining dimension, Soviet Math. Dokl., Vol. 168, No. 3, (1966), 663–66.

[7] Thierry Coquand, Henri Lombardi, Hidden constructions in abstract algebra (3): Krull dimension of distributive lattices and commutative rings, Lecture notes in Pure and Applied Mathematics, vol. 231, 2002, pp. 477–499.