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The Growth of Gradients of Solutions of Some Elliptic Equations and Bi-Lipschicity of QCH

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Abstract. In this paper, we study the growth of gradients of solutions of elliptic equations, including the Dirichlet eigenfunction solutions on bounded plane convex domain. Several results related to Bi-Lipschicity of quasiconformal harmonic (qch) mappings with respect to quasi-hyperbolic and euclidean metrics, are proved. In connection with the subject, we announce a few results concerning the so called interior estimate, including Proposition 1.1. In addition, a short review of the subject is given.

1. Introduction

There is a numerous literature related to the subject, see [16] and the literature cited there and in this paper. Here we give a short review of the subject, announce and prove a few new results. In particular, here our discussion is related to the following items:

(i) In Section 3, we outline a proof of result of Božin - Mateljević which gives an answer to an intriguing problem probably first posed by Kalaj and which states that QCH mappings between Lyapunov Jordan domains are co-Lipschitz.

(ii) In [19], Li Peijin, Jiaolong Chen, and Xiantao Wang proved the gradient of quasiconformal solutions of Poisson equations are bounded under some hypothesis. In Section 4 we announce some results related to the local version of the interior estimate (see for example Proposition 4.3), and discuss whether their result holds without the hypothesis that the radial derivative is bounded.

(iii) In Section 5, the author shows that the Dirichlet eigenfunction solutions on bounded plane convex domain have bounded gradients. Our considerations gives contribution to the problem posed in communication of the author with Yakov Sinai.

(iv) Bi-Lipschicity property of harmonic K-quasiconformal maps with respect to k-metrics (quasi-hyperbolic metrics) in space is subject of Section 6.

(v) In Section 7, we extend a result of Tam and Wan, [23], 1998. More precisely, we prove if f is K-qc hyperbolic harmonic mappings of $\mathbb{H}^n$ with respect to the hyperbolic metric with $K < 3^{n-1}$, then f is a quasi-isometry.

Concerning the items (i), (ii) and (iii) we only outline some proofs.

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Here we will only give a few comments related to item (ii) above[1)]. In [7], see also [8, 17, 26], we have initiated to study the growth of gradients of solutions of elliptic equations using Theorem 4' [6] (Heinz-Bernstein) stated here as Proposition 4.1. We call this result the interior estimate of Heinz-Bernstein type. It has turned out that the so called the interior estimates are very useful for our purposes. Note that in Section 4, we only announce the following improvement of Proposition 4.1[2)]:

Proposition 1.1 (Local version of Interior estimate). *Let $\chi : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

1. *χ is C^2 on $\mathbb{U}$.*
2. *There are positive constants a_0 and b_0 such that $|\Delta\chi| \le a_0|\nabla\chi|^2 + b_0$, on $V = V(r) = \mathbb{U} \cap B(w_0, r)$, $r > 0$, where $w_0 \in \mathbb{T}$, (the last inequality we will call Poisson-Laplace type inequality or the interior estimate inequality) and*
3. *$\chi_b(\theta) = \chi(e^{i\theta})$ is $C^{1,\alpha}$ on the interval $l = \overline{V(r)} \cap \mathbb{T}$.*

Then there is $0 < r_1 < r$ such that the function $|\nabla\chi|$ is bounded on $V(r_1) = \mathbb{U} \cap B(w_0, r_1)$.

Here $\mathbb{U}$ and $\mathbb{T}$ denote the unit disc and the unit circle respectively. See also Proposition 4.3 in Section 4 which is more complete statement.

In Section 2 we shortly consider the background of the subject and in Section 8, we collect some definitions.

2. Background

For a function h, we use notation $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient. We use the notation $\lambda_f(z) = |\partial f(z)| - |\bar\partial f(z)|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar\partial f(z)|$, if $\partial f(z)$ and $\bar\partial f(z)$ exist.

Throughout the paper we denote by Ω, G and D open subsets of $\mathbb{R}^n$, $n \ge 1$.

For $r > 0$ and $x \in \mathbb{R}^n$, let $B(x, r) = B^n(x, r) = \{z \in \mathbb{R}^n : |z - x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$ (abbreviated $S(x, r)$) and let $\mathbb{B}^n$, $\mathbb{S} = \mathbb{S}^{n-1}$ stand for the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. In particular, in dimension $n = 2$ frequently the notation $D(x, r)$ for planar disk is used instead of $B^2(x, r)$ and by $\mathbb{D}$ (or $\mathbb{U}$) we denote the unit disc $\mathbb{B}^2$ and $\mathbb{T} = \partial\mathbb{D}$ we denote the unit circle $\mathbb{S}^1$ in the complex plane. For a domain D in $\mathbb{R}^n$ with non-empty boundary, we define the distance function $d = d_D = \mathrm{dist}(D)$ by $d(x) = d(x; \partial D) = \mathrm{dist}(D)(x) = \inf\{|x - y| : y \in \partial D\}$; and if f maps D onto $D' \subset \mathbb{R}^n$, in some settings it is convenient to use short notation $d^* = d^*(x) = d_f(x)$ for $d(f(x); \partial D')$. It is clear that $d(x) = \mathrm{dist}(x, D^c)$, where D^c is the complement of D in $\mathbb{R}^n$. For a domain $G \subset \mathbb{R}^n$ let $\rho : G \to [0, \infty)$ be a continuous function. We say that ρ is a weight function or a metric density if for every locally rectifiable curve γ in G, the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x)ds$$

exists. In this case we call $l_\rho(\gamma)$ the ρ-length of γ. A metric density defines a metric $d_\rho : G \times G \to [0, \infty)$ as follows. For $a, b \in G$, let

$$d_\rho(a, b) = \inf_\gamma l_\rho(\gamma)$$

where the infimum is taken over all locally rectifiable curves in G joining a and b.

For the modern mapping theory, which also considers dimensions $n \ge 3$, we do not have a Riemann mapping theorem and therefore it is natural to look for counterparts of the hyperbolic metric. So called hyperbolic type metrics have been the subject of many recent papers. Perhaps the most important of these

[1)] for the other item see the corresponding sections

[2)] We believe that this result will find further application, in particular concerning the item (ii) above, see also [8, 17, 26].

metrics are the quasihyperbolic metric κ_G (shortly κ metric) and the distance ratio metric j_G of a domain $G \subset \mathbb{R}^n$ (see [4, 22]). The quasihyperbolic metric (shortly κ-metric) $\kappa = \kappa_G$ of G is a particular case of the metric d_ρ when $\rho(x) = \frac{1}{d(x,\partial G)}$ (see [4, 22]).

Some definitions are also given in Section 8. For example, for definitions of the outer dilatation $K_O(f)$ of f and K-quasiconformal maps (K-qc) see Section 8.

In this Section we mainly discuss the background related to Section 6. We first state a few results from our paper [12].

Proposition 2.1 (Proposition 5 [12]). *If h is a harmonic univalent orientation preserving K-qc mapping of domain D onto D' and $k = \frac{K-1}{K+1}$, then*

$$d(z)\Lambda_h(z) \le 16\, K\, d_h(z) \quad \text{and} \quad d(z)\lambda_h(z) \ge \frac{1-k}{4} d_h(z)\,. \tag{1}$$

Proposition 2.2 (Corollary 1, Proposition 5[12]). *Every Eucldean-harmonic quasi-conformal mapping of the unit disc (more generally of a strongly hyperbolic domain) is a quasi-isometry with respect to hyperbolic distances.*

From Proposition 2.1 directly follows next result (Proposition 2.3). [3)]

Proposition 2.3 ([10]). *Every Euclidean-harmonic quasi-conformal mapping of a domain different from $\mathbb{C}$ is a quasi-isometry with respect to quasi-hyperbolic distances.*

The next theorem concerns harmonic maps onto a convex domain. For the planar version of Theorem 2.4 cf. [11, 12], also [16], pp. 152-153. The space version was communicated on International Conference on Complex Analysis and Related Topics (Xth Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania), by Mateljević and stated in [12], cf. also [14].

Theorem 2.4 (Theorem 1.3, [12]). *Suppose that h is an Euclidean harmonic mapping from the unit ball $\mathbb{B}^n$ onto a bounded convex domain $D = h(\mathbb{B}^n)$, which contains the ball $h(0) + R_0\mathbb{B}^n$. Then for any $x \in \mathbb{B}^n$*

$$d(h(x), \partial D) \ge (1 - \|x\|)R_0/2^{n-1}.$$

Although the proofs of the above results are not difficult, it turns out that they have further impact on the subject. We will shortly discuss it in this paper.

We use a distortion property of quasiconformal maps to prove that for n-dimensional Euclidean harmonic quasiconformal mappings with $K_O(f) < 3^{n-1}$, Jacobian is never zero.

Theorem 2.5 ([1, 15]). *Suppose that $h : \Omega \mapsto \mathbb{R}^n$ is a harmonic quasiconformal map. If $K_O(h) < 3^{n-1}$, then its Jacobian has no zeros.*

Theorem 2.6 ([1, 15]). *Suppose h is a harmonic K-quasiconformal mapping from the unit ball $\mathbb{B}^n$ onto a bounded convex domain $D = h(\mathbb{B}^n)$, with $K < 3^{n-1}$. Then h is co-Lipschitz on $\mathbb{B}^n$ with respect Euclidean metrics.*

In particular, it is co-Lipschitz with respect to quasi-hyperbolic metrics (κ-metrics). We can generalize this result:

Theorem 2.7. *Suppose that $f : D_1 \longrightarrow D_2$, where $D_1, D_2 \subset \mathbb{R}^n$ and the complement D_1 has at least one point, is a harmonic K-quasiconformal mapping with $K_O(f) < 3^{n-1}$, (or and that f belongs to a non-zero Jacobian family of harmonic maps), then f is bi-Lipschitz with respect to κ-metrics.*

This theorem is stated as Theorem 6.1 in Section 6 and it is also proved by Shadia Shalandi [20].

In particular,

(A) f is Lipschitz with respect to κ-metrics.

Note that (A) holds more generally without the hypothesis that f belongs to a non-zero Jacobian family, cf. [18].

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3) In that time, the author did not realized that quasi-hyperbolic metrics have important applications and did not state this version which due to V. Manojlovic.

Theorem 2.8 ([18]). *Suppose that $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is K-qr and $\Omega' = f(\Omega)$. Let $\partial\Omega'$ be a continuum containing at least two distinct point.*
If f is a vector harmonic map, then f is Lipschitz with respect to quasi-hyperbolic metrics on Ω and Ω'.

3. Quasiconformal and QCH mappings between Lyapunov Jordan domains

Although the following two statements did not get attention immediately after their publication, it turns out surprisingly that they have an important role in the demonstration of Theorem 3.5 (co-Lip), [2].

Proposition 3.1 (Corollary 1, Proposition 5[12]; see also [10]). *Every euclidean-harmonic quasi-conformal mapping of the unit disc (more generally of a strongly hyperbolic domain) is a quasi-isometry with respect to hyperbolic distances.*

Theorem 3.2 ([11]). $(ii.1)$ *Suppose that $h = f + \overline{g}$ is a Euclidean orientation preserving harmonic mapping from $\mathbb{D}$ onto bounded convex domain $D = h(\mathbb{D})$, which contains a disc $B(h(0); R_0)$. Then $|f'| \geq R_0/4$ on $\mathbb{D}$.*
$(ii.2)$ *Suppose, in addition, that h is K-qc. Then $\lambda_h \geq (1-k)|f'| \geq (1-k)R_0/4$ on $\mathbb{D}$.*
$(ii.3)$ *In particular, h^{-1} is Lipschitz.*

Further Kalaj [24] proved that

Theorem 3.3. *Suppose $h : D_1 \to D_2$ is a qch homeomorphisam, where D_1 and D_2 are domains with $C^{1,\mu}$ boundary.*
(a)Then h is Lipschitz.
(b)If in addition D_2 is convex, then h is bi-Lipschitz.

Note that (b) is an immediate corollary of (a) and Theorem 3.2. But these results in mind the following question seems natural:
Question 1: Whether Quasiconformal and QCH mappings between Lyapunov Jordan domains is co-Lipschitz?

The proof of the part (a) of Theorem 3.3 in [24] is based on an application of Mori's theorem on quasiconformal mappings, which has also been used previously by Miroslav Pavlović in [25] in the case $D_1 = D_2 = \mathbb{U}$, and a geometric lemma related to Lyapunov domains. It seems that using local version of the interior estimate, Proposition 4.3, one can prove that the theorem holds if D_2 has C^2 boundary.

Note that our proof of Proposition 4.3 is not based on Mori's theorem on quasiconformal mappings, and a natural question arises:

Question 2. Whether a proof of Theorem 3.3(a) can be based on Proposition 4.3?

As an application of Gehring-Osgood inequality[4] concerning qc mappings and quasi-hyperbolic distances, in the particular case of punctured planes, we prove

Proposition 3.4. *Let f be a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$ (respectively $\theta^* \in [0, \pi]$) is the measure of convex angle between z_1, z_2 (respectively $f(z_1), f(z_2)$), then*

$$\theta^* \leq c \max\{\theta^\alpha, \theta\},$$

where $c = c(K)$. In particular, if $\theta \leq 1$, then $\theta^* \leq c\theta^\alpha$.

We shortly refer to this result as (GeOs-BM). Through the paper we frequently consider the setting ($\mathbb{U}_{qc}$): Let $h : \mathbb{U} \to D$ be K-qc map, where $\mathbb{U}$ is the unit disk and suppose that D is Lyapynov domain. Under this hypothesis, using (GeOs-BM), we prove that for every $a \in \mathbb{T} = \{|z| = 1\}$, there is a special Lyapunov domain U_a, of a fixed shape, in the unit disk $\mathbb{U}$ which touches a and a special, convex Lyapunov domain $lyp(D)^-_b$, of a fixed shape, in D such that $lyp(D)^-_b \subset h(U_a) \subset H_b$, where H_b is a half-plane H_b, which touches $b = h(a)$. We can regard this result as "good local approximation of qc mapping h by its restriction to a special Lyapunov domain so that codomain is locally convex". In addition if h is harmonic, using it, we prove that h is co-Lip on $\mathbb{U}$:

Theorem 3.5. *Suppose $h : \mathbb{U} \to D$ is a qch homeomorphisam, where D is a Lyapunov domain with $C^{1,\mu}$ boundary. Then h is co-Lipschitz.*

It settles an open intriguing problem in the subject and can be regarded as a version of Kellogg- Warschawski theorem for qch mappings.

4. Quasiconformal Solutions of Poisson Equations

In [19], Li Peijin, Jiaolong Chen, and Xiantao Wang proved that the gradient of quasiconformal solutions of Poisson equations are bounded under some hypothesis. Using local version of the interior estimate, Proposition 4.3, we outline an argument that their result holds without hypothesis (ii.2) (see below)[4].

We introduce the following hypothesis:

(i.1) Let g be a function from $\mathbb{D}$ to $\mathbb{C}$ with a continuous extension to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, and
(i.2) let $f : \mathbb{T} \to \mathbb{C}$ be a bounded integrable function on $\mathbb{T}$ and
(i.3) let Ω be a Jordan domain with C^2 boundary.

Further, let

(ii.1) $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$ denote the family of solutions $w : \mathbb{D} \to \Omega$ of the Poisson equation $\Delta w = g$, where $w|\mathbb{T} = f \in L^1(\mathbb{T})$ and each w is a sense-preserving diffeomorphism. In [19], it is proved that if Ω is C^2 domain,
(ii.2) $|\partial u/\partial r|$ is bounded on $\mathbb{D}$, where $u = P[f]$, and
(ii.3) w belongs to $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$,
then w is Lip on $\mathbb{D}$.

For example, by results related to the interior estimate in mind, it seems natural to ask the following:

Question 3. Whether we can weaken (or remove) the hypothesis that $|\partial u/\partial r|$ is bounded and whether we can weaken the hypothesis that g is continuous on the closure of the unit disk?

We suggest a procedure to drop the hypothesis (ii.2).

Recall, roughly speaking, in our previous research we have initiated a study of the extent to which the conformal theory can be extended to harmonic qc mappings. It has turned out that the following result is very useful for our purposes.

Proposition 4.1 (Interior estimate). (Heinz-Bernstein, see Theorem 4' [6]). *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

1. *s is C^2 on $\mathbb{U}$,*
2. *$s_b(\theta) = s(e^{i\theta})$ is C^2 and*
3. *$|\Delta s| \le a_0|\nabla s|^2 + b_0$, on $\mathbb{U}$ for some constants a_0 and b_0 (the last inequality we will call Poisson-Laplace type inequality or the interior estimate inequality).*

Then the function $|\nabla s|$ is bounded on $\mathbb{U}$.

We call Theorem 4' [6], the interior estimate of Heinz-Bernstein type.

Proposition 4.2. *If w belongs $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$, then $|\nabla w|$ is bounded on $\mathbb{D}$.*

By hypotheses (i.1) and (ii.1), $|\Delta w|$ is bounded on $\mathbb{D}$, and w satisfies the Poisson type inequality on $\mathbb{D}$. It seems the idea[5] behind the proof is to use local coordinates ψ to make the part of boundary of the image to lay on $\mathbb{R}$ (a hyperplane if we work in space) whose 2-th coordinate is 0 and then to apply inner estimate on 2-th coordinate of function $\psi \circ u$, which is 0 on the the part of boundary of the unit disk $\mathbb{D}$. An application of Proposition 4.1 (the interior estimate of Heinz-Bernstein) (more precisely the local version of Interior estimate, Proposition 4.3 below) yields the proof.

[4] At this point it may seem that we use a heuristic approach, but we hope to fill details in a forth-coming paper.
[5] We discussed this shortly as a new idea at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, Dec. 10-19, 2009, see Course-materials [26].

4.1. Further results related to the interior estimate

Recall, we only announce here that one can refine the methods of the proof of Theorem 4′ in [6] to derive:

Proposition 4.3 (Local version of Interior estimate). *Let $\chi : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

1. *χ is C^2 on $\mathbb{U}$,*
2. *There are constants a_0 and b_0 such that $|\Delta\chi| \le a_0|\nabla\chi|^2 + b_0$, on $V = V(r) = \mathbb{U} \cap B(w_0, r)$, $r > 0$, where $w_0 \in \mathbb{T}$, and*
3. *$\chi \in L^1(0, 2\pi)$, and $\chi' \in L^\infty(l)$ and the Hilbert transform of $H(\chi') \in L^\infty(l)$, where $l = l(w_0)$ is the interval $l = \overline{V(r)} \cap \mathbb{T}$.*

(I) *Then there is $0 < r_1 < r$ such that the function $|\nabla\chi|$ is bounded on $V(r_1) = \mathbb{U} \cap B(w_0, r_1)$.*

Introduce the hypothesis

4. $\chi_b(\theta) = \chi(e^{i\theta})$ is C^2 on the interval $l = \overline{V(r)} \cap \mathbb{T}$.

5. χ is $C^{1,\alpha}$ on l.

Note that, the hypothesis 4. implies 5., and 5. implies 3. In particular, the the corresponding versions of this result hold if we suppose 4. or 5. instead of 3.

The proof of this result will appear elsewere. Using Proposition 4.3, one can prove:

Theorem 4.4. *Let f be a quasiconformal C^2 diffeomorphism from the plane domain Ω onto the plane domain G. Let $\gamma_\Omega \subset \partial\Omega$ and $\gamma_G = f(\gamma_\Omega) \subset \partial G$ be $C^{1,\alpha}$ respectively C^2 Jordan arcs. If for some $\tau \in \gamma_\Omega$ there exist positive constants r, a and b such that*

$$|\Delta f| \le a|\nabla f|^2 + b\,, \quad z \in \Omega \cap D(\tau, r), \tag{2}$$

then f has bounded partial derivatives in $\Omega \cap D(\tau, r_\tau)$ for some $r_\tau < r$. In particular it is a Lipschitz mapping in $\Omega \cap D(\tau, r_\tau)$.

Under the stronger hypothesis that γ_G is $C^{2,\alpha}$ this is proved in [8](and it has been used there as the main tool).

5. The Boundary Regularity of Dirichlet Eigenfunctions

By Ω we denote a domain in $\mathbb{R}^n$. In communication with Yakov Sinai[6)] (April 2016, Princeton) the following question appeared:

S-M Question. What can we say about the boundary regularity of Dirichlet Eigenfunctions on bounded domains which are C^2 except at a finite number of corners[7)].

We have discussed the subject with Pier Lamberti who informed about numerous literature related to this subject and in particular about items 1)-3).

1) the eigenfunctions of the Dirichlet Laplacian are always bounded, not matter what the boundary regularity is.

2) the gradient of the eigenfunctions may not be bounded. The typical situation in the plane is as follows. If you have a corner with angle β, then the gradient is bounded around it if $\beta \le \pi$ and unbounded if $\beta > \pi$.

3) An example: if Ω is a circular sector in the plane with central angle β, then for all $n \in \mathbb{N}$, $\nabla\varphi_n \in L^\infty(\Omega)$ if $0 < \beta \le \pi$; if $\pi < \beta < 2\pi$ then for all $n \in \mathbb{N}$, $\nabla\varphi_n \in L^p(\Omega)$ for all $1 \le p < \frac{2\beta}{\beta-\pi}$ and there exists an infinite number of eigenfunctions φ_n such that $\nabla\varphi_n \notin L^p(\Omega)$ if $p \ge \frac{2\beta}{\beta-\pi}$. This example is discussed in Example 6.2.5

6) Abel prize Laureate 2014

7) we address this question as Y. Sinai's question or shortly S-M question

in E.B. Davies, Spectral theory and differential operators, Cambridge University Press, Cambridge, 1995. For example, if Ω is C^2, using so called Interior estimate one can show that the Dirichlet eigenfunctions are Lipschitz.

Let $1 \leq p \leq \infty$. If a function f possibly after modifying on a set of measure zero is ACL on Ω (the restriction to the intersection of almost every line parallel to the coordinate directions in $\mathbb{R}^n$ with Ω is absolutely continuous), f and $|\nabla f|$ are both in $L^p(\Omega)$, then we say that f belongs to the Sobolev space $W^{1,p}(\Omega)$.

The Sobolev space $W^{1,2}(\Omega)$ is also denoted by $H^1(\Omega)$. It is a Hilbert space, with an important subspace $H^1_0(\Omega)$ (the notation $W^{1,2}_0(\Omega)$ is also used) defined to be the closure in $H^1(\Omega)$ of the infinitely differentiable functions compactly supported in Ω.

In standard spectral theory for differential operators, the eigenvalue problem for the Dirichlet Laplacian is defined as follows:

Find $u \in W^{1,2}_0(\Omega)$ (eigenfunction) and $\lambda \in \mathbb{R}$ (eigenvalue) such that

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega u\varphi dx$$

for all functions $\varphi \in W^{1,2}_0(\Omega)$.

We will call eigenfunctions in the above sense eigenfunctions for the Dirichlet Laplacian (in standard spectral theory meaning; shortly in SSTM) if there is possibility of misunderstanding.

By w we denote a unique solution to the Dirichlet problem $L(\partial_x)w = f,\ w \in W^{m,2}_0(\Omega)$, where $f \in W^{1-m,q}(\Omega)$ with $q \in (2,\infty)$. Here $W^{l,p}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in the Sobolev space $W^{l,p}(\Omega)$, $1 < p < \infty$, and $W^{-l,p'}(\Omega)$ with $p' = p/(p-1)$ is the dual of $W^{l,p}_0(\Omega)$. The operator $L(\partial_x)$ is strongly elliptic and given by $L(\partial_x) = \sum_{0\leq k\leq 2m} a_k \partial_1^k \partial_2^{2m-k}$.

It turns out that, in connection to **S-M Question**, we can use a Kozlov-Mazya result:

Theorem 5.1 (Theorem 2, [9]). *Let u be a solution of the Dirichlet problem for elliptic equations of order $2m$ with constant coefficients in an arbitrary bounded plane convex domain G. Then m-th order derivatives of u are bounded if the coefficients of the equation are real.*

Note that Laplacian is elliptic equation of order 2. An application of Theorem 5.1 yields

Theorem 5.2. *Suppose that Ω is bounded plane convex domain and $w \in W^{1,2}_0(\Omega)$ is the Dirichlet eigenfunction solution on Ω. Then the gradient of w is bounded.*

6. Bi-Lipschicity of Quasiconformal Harmonic Mappings in n-dimensional Space with Respect to Quasi-Hyperbolic Metrics

(II) Suppose that G and G' are domains in $\mathbb{R}^n$ and the complement of G has at least one point and in addition let $f : G \xrightarrow{onto} G'$ be K-quasiconformal mapping. Then, by the distortion property of qc mappings (see [3], p. 383, [21], p. 63), there are the constants C_* and c_* depend on n and K only, such that

$$B(f(x), c_*d_*) \subset f(B_x) \subset B(f(x), C_*d_*), \quad x \in G, \tag{3}$$

where $d_*(x) := d(f(x)) = d\big(f(x),\ \partial G'\big)$ and $d(x) := d(x,\ \partial G)$.

For definition of a non-zero Jacobian family see Definition 8.3 in Section 8 below.

Using our considerations in [1, 13, 15, 18], we can give a short proof of the following result:

Theorem 6.1. *Suppose that $f : D_1 \xrightarrow{onto} D_2$, where $D_1, D_2 \subset \mathbb{R}^n$ and the complement of D_1 has at least one point, is a harmonic K-quasiconformal mapping with $K_O(f) < 3^{n-1}$, (or and that f belongs to a non-zero Jacobian family of harmonic maps), then f is bi-Lipschitz with respect k-metrics.*

Hence, in particular,
(A) f is Lipschitz with respect to k-metrics.
Note that (A) holds more generally without the hypothesis that f belongs to a non-zero Jacobian family:

Theorem 6.2 ([18]). *Suppose that $\Omega \subset \mathbb{R}^n$, $f : \Omega \to f(\Omega)$ is harmonic and K-qc.*
Then h is pseudo-isometry w.r. to quasi-hyperbolic metrics on Ω and $\Omega' = f(\Omega)$. In particular, it is Lipschitz with respect to k-metrics.

Now we give a simple proof of Theorem 6.1.

Proof. Set $r = r(z) = d(z, \partial D_1)$ and $R = R(fz) = d(f(z), \partial D_2)$. Then, by (3) (see also [13]), there is a constant c such that
$f(B(z, r(z)) \supset B(fz, cR(z))$, $z \in D_1$. There there is a constant c_0 such that $r(z)\lambda_f(z) \geq cR(z)$, $z \in D_1$, and hence
(B) f is co-Lipschitz with respect to quasi-hyperbolic metrics.
This together with (A) completes the proof. □

By $J(z) = J_f(z) = J(f, z)$ we denote the Jacobian determinant of f at z.
The above consideration also shows that
(C) $r(z)J^{\frac{1}{n}}(z) \approx R(z)$, $z \in D_1$.
After writing a version of this manuscript we received information about Shadia Shalandi work, see [20]. She also proved Theorem 6.1. Note that in her formulation the hypothesis that the complement of D_1 has at least one point is missing.

7. On Harmonic K-Quasiconformal Map on $\mathbb{H}^n$

Given Riemannian manifolds (M, g), (N, h) and a map $\phi : M \to N$, the energy density of $e(\phi)$ at a point x in M is defined as

$$e(\phi) = \frac{1}{2}\|d\phi\|^2 .$$

The energy density can be written more explicitly as

$$e(\phi) = \frac{1}{2}\,\mathrm{trace}_g\,\phi^* h.$$

The energy of ϕ on a compact subset K of M is

$$E_K(\phi) = \int_K e(\phi)\,dv_g = \frac{1}{2}\int_M \|d\phi\|^2\,dv_g,$$

where dv_g denotes the measure on M induced by its metric.
Using the Einstein summation convention, if the metrics g and h are given in local coordinates by $g = \sum g_{ij}dx^i dx^j$ and $h = \sum h_{\alpha\beta}du^\alpha du^\beta$, the right hand side of this equality reads

$$e(\phi) = \frac{1}{2} g^{ij} h_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j}.$$

We define the tension field $\tau(u)$ of u by coordinates

$$\tau(u)^\nu = \Delta_g u^\nu + g^{ij}\Gamma'^{\nu}_{\alpha\beta} \circ u\, u_i^\alpha u_j^\beta , \tag{4}$$

where Δ_g is the Laplace-Beltrami operator on M and $\Gamma'^{\nu}_{\alpha\beta}$ are the Christoffel symbols on N. The Euler-Lagrange equation for this energy functional is the condition for the vanishing of the tension, which is, in local coordinates given by (4), $\tau(u)^\nu = 0$.

If $\mathbb{H}^m$ is identified as $\{(x^1,...,x^m) : x^m > 0\}$ with the metric:

$$\frac{1}{(x^m)^2}((dx^1)^2 + \cdots (dx^m)^2)$$

then the tension field of u is given by

$$\tau(u)^\nu = (x^m)^2\Big(\Delta_0 y^\nu - \frac{m-2}{(x^m)^2} y^\nu_m - \frac{2}{(y^m)^2}\langle\nabla_0 y^\nu, \nabla_0 y^m\rangle\Big)$$

for $1 \le \nu \le m-1$, and

$$\tau(u)^m = (x^m)^2\Big(\Delta_0 y^m - \frac{m-2}{(x^m)^2} y^m_m + \frac{1}{(y^m)^2}(\sum_{\nu=1}^{m-1}|\nabla_0 y^\nu|^2 - |\nabla_0 y^m|^2)\Big),$$

where ∇_0 is the Euclidean gradient and Δ_0 is the Euclidean Laplacian.

In [15], we proved:

Proposition 7.1. *Suppose that f has continuous partial derivatives up to the order 3 at the origin 0 and that $f : U(0) \to \mathbb{R}^n$ is K-qc, where $U(0)$ is a neighborhood of 0 in $\mathbb{R}^n$. If $K_O(f) < 3^{n-1}$, then $J(f,0) \neq 0$.*
In particular, if g is analytic (more generally $C^{(3)}(U(0))$ or g only has partial derivatives up to the order 3), and if g is K-qc with $K_O(g) < 3^{n-1}$, then $J(g,0) \neq 0$.

The proof of the next proposition is based on Proposition 7.1.

Proposition 7.2. *Let F be K-qc hyperbolic harmonic mappings of $\mathbb{H}^n$ with respect to the hyperbolic metric. If $K < 3^{n-1}$, then f is a quasi-isometry.*

Tam and Wan, [23], 1998, proved the result if $K < 2^{n-1}$.

Proof. We follow their argument. Suppose that there is a sequence of points $x_n \in \mathbb{H}^n$, such that $e_n(F)(x_n) \to 0$ as $n \to \infty$. Let $o \in \mathbb{H}^n$ be a fixed point and A_n and B_n be isometry such that $A_n(o) = x_n$ and $B_n(F(x_n)) = o$. Then $u_n = B_n \circ F \circ A_n$ are harmonic maps such that $e(u_n)(o) \to 0$ as $n \to \infty$. A subsequence of u_n converges uniformly to a K-qc hyperbolic harmonic mappings u with $u(o) = o$ and $e(u)(o) = 0$. This contradicts the statement of a version of Proposition 7.1 for C^3 mapping, cf. also [14]. □

8. Appendix

8.1. Some definitions and results

Let $\Omega \in \mathbb{R}^n$ and $\mathbb{R}^+ = [0,\ \infty)$ and $f,\ g : \Omega \to \mathbb{R}^+$. If there is a positive constant c such that $f(x) \le c\, g(x)\,,\ x \in \Omega\,$, we write $f \preceq g$ on Ω. If there is a positive constant c such that

$$\frac{1}{c}\, g(x) \le f(x) \le c\, g(x)\,, \quad x \in \Omega\,,$$

we write $f \approx g$ (or $f \asymp g$) on Ω.

Let $G \subset \mathbb{R}^2$ be a domain and let $f : G \to \mathbb{R}^2,\ f = (f_1,\ f_2)$, be a harmonic mapping. This means that f is a map from G into $\mathbb{R}^2$ and both f_1 and f_2 are harmonic functions, i. e. solutions of the two-dimensional Laplace equation

$$\Delta u = 0\,. \tag{5}$$

The above definition of a harmonic mapping extends in a natural way to the case of vector-valued mappings $f : G \to \mathbb{R}^n,\ f = (f_1,\ \ldots,\ f_n)$, defined on a domain $G \subset \mathbb{R}^n,\ n \ge 2$. Let h be a harmonic univalent orientation preserving mapping on a domain D, $D' = h(D)$ and $d_h(z) = d\,(h(z), \partial D')$. If $h = f + \overline{g}$ has the form, where f and g are analytic, we define $\lambda_h(z) = D^-(z) = |f'(z)| - |g'(z)|$, and $\Lambda_h(z) = D^+(z) = |f'(z)| + |g'(z)|$.

Let Ω be a domain in R^n and $f : \Omega \to R^n$ be continuous. We say that f is quasiregular (abbreviated qr) if
(1) f belongs to Sobolev space $W^n_{1,loc}(\Omega)$
(2) there exists K, $1 \le K < \infty$, such that

$$|f'(x)|^n \le K\, J_f(x) \;\; a.e. \tag{6}$$

The smallest K in (6) is called the outer dilatation $K_O(f)$. A qr mapping is a qc if and only if it is a homeomorphism.

Theorem 8.1. *Let (f_j), $f_j : \Omega \mapsto \overline{\mathbb{R}^n}$, be a sequence of K-quasiconformal maps, which converges pointwise to a mapping $f : \Omega \mapsto \overline{\mathbb{R}^n}$. Then there are three possibilities:*
a. f is a homeomorphism and the convergence is uniform on compact sets.
b. f assumes exactly two values, one of which at exactly one point; covergence is not uniform on compact sets in that case.
c. f is constant.

Definition 8.2. *We say that a family $\mathcal{F}$ of maps from domains in $\mathbb{R}^n$ to $\mathbb{R}^n$ is RHTC-closed if the following holds:*

- *(Restrictions) If $f : \Omega \mapsto \mathbb{R}^n$ is in $\mathcal{F}$, $\Omega' \subset \Omega$ is open, connected and nonempty, then $f|_{\Omega'} \in \mathcal{F}$.*
- *(Homothety) If $f : \Omega \mapsto \mathbb{R}^n$ is in $\mathcal{F}$, $a \in \mathbb{R}$, $a > 0$ then $g : \Omega \mapsto \mathbb{R}^n$ and $h : a\Omega \mapsto \mathbb{R}^n$ are in $\mathcal{F}$, where $g(x) = af(x)$ and $h(x) = f(x/a)$.*
- *(Translations) If $f : \Omega \mapsto \mathbb{R}^n$ is in $\mathcal{F}$, $t \in \mathbb{R}^n$, then $g : \Omega \mapsto \mathbb{R}^n$ and $h : t + \Omega \mapsto \mathbb{R}^n$ are in $\mathcal{F}$, where $g(x) = t + f(x)$ and $h(x) = f(x - t)$.*
- *(Completeness) If $f_j : \Omega \mapsto \mathbb{R}^n$, $j \in \mathbb{N}$ are in $\mathcal{F}$, (f_j) converges uniformly on compact sets to $g : \Omega \mapsto \mathbb{R}^n$, where g is non-constant, then $g \in \mathcal{F}$.*

For instance, families of harmonic maps and of gradients of harmonic functions are RTHC-closed. Also, due to Theorem 8.1, for any given $K \ge 1$, a subfamily of K-quasiconformal members of a RTHC-closed family is also RTHC-closed.

Definition 8.3. *We say that a family $\mathcal{F}$ of harmonic maps from domains in $\mathbb{R}^n$ to $\mathbb{R}^n$ is non-zero Jacobian closed, if it is RHTC-closed and Jacobians of all maps in the family have no zeros.*

Note that uniform convergence on compact sets in the case of harmonic maps implies convergence of higher order derivatives, via Hölder and Schauder apriori estimates (see [5], pp. 60, 90). This is related to elliptic regularity and holds for more general elliptic operators, and not just Laplacian, so that this method applies in that more general setting too.

8.2. Remarks on the background

During the visiting position at Wayne State University, Detroit, 1988/89, the author[8)] started considering qch mappings. In particular, the author observed that the following results hold (see Proposition 2.1 and 2.2 below). When I returned to Belgrade, I used to talk on the seminar permanently and asked several open questions related to the subject. Many research papers are based on these communications. Since I had not published all of them it happens that some researchers discovered them later. In Section 2 we only discussed a few results from Revue Roum. Math. Pures Appl. Vol. **51** (2006) 5–6, 711–722 [9)]

[8)]the author refers to me
[9)]It seems that there is some problems concerning visibility of this journal.

8.3. Recent lectures and communications with colleagues

The author gave a few lectures related to the subject.

(a) two plenary lectures on VII Symposium of Mathematics and Applications, 4-5 Nov, 2016, Belgrade and two plenary lecture on XIX Geometrical Seminar, Zlatibor, Serbia, August 28-September 4, 2016(http://tesla.pmf.ni.ac.rs/people/geometrijskiseminarxix/presentation.php#ps, [17]).

(b) The lecture at Seminar za kompleksno analizo, Miodrag Mateljević: Interior estimates for Poisson type inequality and quasiconformal hyperbolic harmonic mappings at University of Ljubljana, Faculty of Mathematics and Physics, Institute of Mathematics, Physics and Mechanics, Ljubljana, 24. 11. 2016.

(c) In April 2016, the author delivered lectures at Cincinnati University, Fordham University NY and University Texas at Dallas.

At Cincinnati University the author has initiated some considerations with David Minda (Cincinnati University) and at CUNY (New York) with Fred Gardiner, Dragomir Saric, and Melkana Brakalova (Fordham University NY).

(d) At Princenton University the author discussed some questions with Yakov Sinai and gave important contribution to **S-M Question**, the problem related to the boundary behavior of gradient of Dirichlet eigenvalues functions. Massimo Lanza de Cristoforis forwarded a version of this question to Pier Domenico Lamberti.

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(ii) https://www.researchgate.net/publication/275153938, uploaded by Miodrag Mateljevic on Apr 19, 2015.