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n-Ideals of Commutative Rings

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Abstract. In this paper, we present a new classes of ideals: called n-ideal. Let R be a commutative ring with nonzero identity. We define a proper ideal I of R as an n-ideal if whenever $ab \in I$ with $a \notin \sqrt{0}$, then $b \in I$ for every $a, b \in R$. We investigate some properties of n-ideals analogous with prime ideals. Also, we give many examples with regard to n-ideals.

1. INTRODUCTION

Throughout this study, all rings are assumed to be commutative with nonzero identity. Let R be a ring. If I is an ideal of R with $I \neq R$, then I is called a proper ideal. Suppose that I is an ideal of R. We denote the radical of I by $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$. In particular, we mean $\sqrt{0}$ by the set of all nilpotents in R, i.e, $\{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$. Let S be a nonempty subset of R. Then the ideal $\{a \in R : aS \subseteq I\}$, which contains I, will be designated by $(I : S)$.

The notion of prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied. See, for example, [4, 8]. Recall from [2], a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In [10], Mohamadian defined a proper ideal I of R as an r-ideal if whenever $a, b \in R$ with $ab \in I$ and $ann(a) = 0$ imply that $b \in I$, where $ann(a) = \{r \in R : ra = 0\}$. Motivated from this concept, in section 2, we give the notion of n-ideals and we investigate many properties of n-ideals with similar prime ideals. A proper ideal I of R is said to be an n-ideal if the condition $ab \in I$ with $a \notin \sqrt{0}$ implies $b \in I$ for every $a, b \in R$. Among many results in this paper, it is shown (in Theorem 2.7) that a proper ideal I of R is an n-ideal of R if and only if $I = (I : a)$ for every $a \notin \sqrt{0}$. In Proposition 2.5, we show that every n-ideal is also an r-ideal. Furthermore, in Theorem 2.14, we characterize the integral domains with n-ideal. Also, we show that (in Theorem 2.15) a ring R is a field if and only if R is von Neumann regular and 0 is an n-ideal. In Proposition 2.20 we show that if I is an n-ideal of R, then $S^{-1}I$ is an n-ideal of $S^{-1}R$, where S is a multiplicatively closed subset of R and $S^{-1}R$ is the ring of fraction on S. Moreover, in Proposition 2.25, we characterize the all rings in which every proper ideal is an n-ideal.

Let M be an R-module. Then the set $R(+)M = \{(r, m) : r \in R, m \in M\}$, which is called the idealization of M in R, is a commutative ring with coordinate-wise addition and the multiplication $(r_1, m_1)(r_2, m_2) =$

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$(r_1r_2, r_1m_2 + r_2m_1)$ for each $r_1, r_2 \in R$ and $m_1, m_2 \in M$ [11]. From Proposition 2.27 to Corollary 2.33 we study the n-ideals of $R(+)M$. Finally, in section 3, counter examples are given.

2. *n*-Ideals of Commutative Rings

Definition 2.1. *A proper ideal I of R is called an n-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$.*

Example 2.2. *(i) Suppose that (R, M) is a local ring with unique prime ideal. Then every ideal is an n-ideal.*
(ii) In any integral domain D, the zero ideal is an n-ideal.
(iii) Any ring R need not have an n-ideal. For instance, $\mathbb{Z}_6$ has not any n-ideal.

Proposition 2.3. *If I is an n-ideal of R, then $I \subseteq \sqrt{0}$.*

Proof. Assume that I is an n-ideal but $I \nsubseteq \sqrt{0}$. Then there exists an $a \in I$ such that $a \notin \sqrt{0}$. Since $a.1 = a \in I$ and I is an n-ideal, we conclude that $1 \in I$, so that $I = R$, a contradiction. Hence $I \subseteq \sqrt{0}$. □

Proposition 2.4. *Let $\{I_i\}_{i\in\Delta}$ be a nonmepty set of n-ideals of R, then $\bigcap_{i\in\Delta} I_i$ is an n-ideal of R.*

Proof. Let $ab \in \bigcap_{i\in\Delta} I_i$ with $a \notin \sqrt{0}$ for $a, b \in R$. Then $ab \in I_i$ for every $i \in \Delta$. Since I_i is an n-ideal of R, we get the result that $b \in I_i$ and so $b \in \bigcap_{i\in\Delta} I_i$. □

Recall that a proper ideal I of R is an r-ideal if the condition $ab \in I$ with $ann(a) = 0$ implies $b \in I$ for each $a, b \in R$. In the following proposition, we show that every n-ideal is also an r-ideal.

Proposition 2.5. *Let R be a ring. If I is an n-ideal of R, then it is an r-ideal.*

Proof. Suppose that I is an n-ideal of R and $ab \in I$ with $ann(a) = 0$ for $a, b \in R$. Since $a \notin \sqrt{0}$ and I is an n-ideal, we conclude that $b \in I$. Consequently, I is an r-ideal of R. □

Recall from [12], a proper ideal Q of R is a primary ideal if whenever $a, b \in R$ with $ab \in Q$, then $a \in Q$ or $b \in \sqrt{Q}$.

Remark 2.6. *It is well known that every nilpotent element is also a zero divisor. So zero divisors and nilpotent elements are equal in case $\langle 0 \rangle$ is a primary ideal of R. Thus the n-ideals and r-ideals are equivalent in any commutative ring whose zero ideal is primary.*

Remember that a proper ideal P of R is prime if and only if $P = (P : a)$ for every $a \notin P$. Now, we give a similar result for n-ideals.

Theorem 2.7. *Let R be a ring and I a proper ideal of R. Then the followings are equivalent:*
(i) I is an n-ideal of R.
(ii) $I = (I : a)$ for every $a \notin \sqrt{0}$.
(iii) For ideals J and K of R, $JK \subseteq I$ with $J \cap (R - \sqrt{0}) \neq \emptyset$ implies $K \subseteq I$.

Proof. $(i) \Rightarrow (ii)$: Assume that I is an n-ideal of R. For every $a \in R$, the inclusion $I \subseteq (I : a)$ always holds. Let $a \notin \sqrt{0}$ and $b \in (I : a)$. Then we have $ab \in I$. Since I is an n-ideal, we conclude that $b \in I$ and thus $I = (I : a)$.

$(ii) \Rightarrow (iii)$: Suppose that $JK \subseteq I$ with $J \cap (R - \sqrt{0}) \neq \emptyset$ for ideals J and K of R. Since $J \cap (R - \sqrt{0}) \neq \emptyset$, there exists an $a \in J$ such that $a \notin \sqrt{0}$. Then we have $aK \subseteq I$, and so $K \subseteq (I : a) = I$ by (ii).

$(iii) \Rightarrow (i)$: Let $ab \in I$ with $a \notin \sqrt{0}$ for $a, b \in R$. It is sufficient to take $J = aR$ and $K = bR$ to prove the result. □

Proposition 2.8. *For a prime ideal I of R, I is an n-ideal of R if and only if $I = \sqrt{0}$.*

Proof. Suppose that I is a prime ideal of R. It is clear that $\sqrt{0} \subseteq I$. If I is an n-ideal of R, then by Proposition 2.3, we have $I \subseteq \sqrt{0}$ and so $I = \sqrt{0}$. For the converse, assume that $I = \sqrt{0}$. Now we show that I is an n-ideal. Let $ab \in I$ and $a \notin \sqrt{0}$ for $a, b \in R$. Since I is a prime ideal and $a \notin \sqrt{0}$, we get $b \in I$ and so I is an n-ideal of R. □

Corollary 2.9. *(i) $\sqrt{0}$ is an n-ideal of R if and only if it is a prime ideal of R.*
(ii) Any reduced ring R, which is not integral domain, has no n-ideals.
(iii) Let R be a reduced ring. Then R is an integral domain if and only if 0 *is an n-ideal of R.*

Proof. (i) If $\sqrt{0}$ is a prime ideal of R, then $\sqrt{0}$ is an n-ideal of R by Proposition 2.8. Assume that $\sqrt{0}$ is an n-ideal of R. Let $ab \in \sqrt{0}$ and $a \notin \sqrt{0}$. Since $\sqrt{0}$ is an n-ideal of R, we conclude that $b \in \sqrt{0}$. Hence $\sqrt{0}$ is a prime ideal of R.

(ii) Let R be a reduced ring which is not integral domain. Then $\sqrt{0} = 0$ is not prime ideal of R and so by (i) it is not an n-ideal. On the other hand, if I is a nonzero n-ideal of R, then by Proposition 2.3 $I \subseteq \sqrt{0} = 0$ and so $I = 0$ which is a contradiction.

(iii) Suppose that R is a reduced ring. If R is an integral domain, then $0 = \sqrt{0}$ is a prime ideal, and so by (i) 0 is an n-ideal of R. For the converse if 0 is an n-ideal of R, then by (ii) R is an integral domain. □

Proposition 2.10. *Let R be a ring and S a nonempty subset of R. If I is an n-ideal of R with $S \nsubseteq I$, then $(I : S)$ is an n-ideal of R.*

Proof. It is easy to see that $(I : S) \neq R$. Let $ab \in (I : S)$ and $a \notin \sqrt{0}$. Then we have $abs \in I$ for every $s \in S$. Since I is an n-ideal of R, we conclude that $bs \in I$ and thus $b \in (I : S)$. □

Theorem 2.11. *If I is a maximal n-ideal of R, then $I = \sqrt{0}$.*

Proof. Let I be a maximal n-ideal of R. Now we show that I is a prime ideal of R. And so by Proposition 2.8, we have $I = \sqrt{0}$. Let $ab \in I$ and $a \notin I$ for $a, b \in R$. Since I is an n-ideal and $a \notin I$, $(I : a)$ is an n-ideal by Proposition 2.10. Thus $b \in (I : a) = I$ by the maximality of I. Hence I is a prime ideal of R. □

Theorem 2.12. *Let R be a ring. Then there exists an n-ideal of R if and only if $\sqrt{0}$ is a prime ideal of R.*

Proof. Suppose that I is an n-ideal of R and $\Omega = \{J : J \text{ is an } n\text{-ideal of } R\}$. Since $I \in \Omega$, $\Omega \neq \emptyset$. It is clear that Ω is a partially ordered set by the set inclusion. Suppose $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$ is a chain of Ω. Now, we show that $\bigcup_{n=1}^{\infty} I_i$ is an n-ideal of R. Let $ab \in \bigcup_{n=1}^{\infty} I_i$ with $a \notin \sqrt{0}$ for $a, b \in R$. Then we have $ab \in I_k$ for some $k \in \mathbb{N}$. Since I_k is an n-ideal, we conclude $b \in I_k \subseteq \bigcup_{n=1}^{\infty} I_i$. So $\bigcup_{n=1}^{\infty} I_i$ is a upper bound of the chain $\{I_i : i \in \mathbb{N}\}$. By Zorn's Lemma Ω has a maximal element K. Then by the previous theorem, we get the result that $K = \sqrt{0}$ is a prime ideal of R. For the converse, assume that $\sqrt{0}$ is a prime ideal of R. Then by Corollary 2.9(i), $\sqrt{0}$ is an n-ideal of R. □

We recall from [1] that an ideal I of R is called weakly primary if whenever $0 \neq ab \in I$ for some $a, b \in R$, then $a \in I$ or $b \in \sqrt{I}$. Also, we recall from [5] ([6]) that a proper ideal I of R is a 2-absorbing primary (weakly 2-absorbing primary) if whenever $abc \in I$ ($0 \neq abc \in I$) for some $a, b, c \in R$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ ($ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$). In view of Proposition 2.3 and Theorem 2.12, we have the following result. Since its proof is straightforward, we omit the proof.

Corollary 2.13. *Let I be an ideal of R such that $I \subseteq \sqrt{0}$.*
1) The following statements are equivalent:
(i) I is an n-ideal.
(ii) I is a primary ideal of R.
2) If I is an n-ideal of R, then I is a weakly primary (so weakly 2-absorbing primary) and 2-absorbing primary ideal. However the converse is not true (see Example 3.5 (ii)).
3) The followings are equivalent:
(i) I is a weakly 2-absorbing primary ideal of R and $\sqrt{0}$ is a prime ideal.
(ii) I is a 2-absorbing primary ideal of R and $\sqrt{0}$ is a prime ideal.
4) Suppose that R has at least one n-ideal. Then I is a weakly 2-absorbing primary ideal of R if and only if I is a 2-absorbing primary ideal.

Theorem 2.14. *For any ring R, the followings are equivalent.*
(i) R is an integral domain.
(ii) 0 is the only n-ideal of R.

Proof. $(i) \Rightarrow (ii)$: Suppose that R is an integral domain. Let I be an n-ideal of R. Then by Proposition 2.3, we have $I \subseteq \sqrt{0} = 0$ and so $I = 0$. Also, by Example 2.2 we know that 0 is an n-ideal.

$(ii) \Rightarrow (i)$: Assume that 0 is only n-ideal of R. Then by Theorem 2.12 and Corollary 2.9(i) $\sqrt{0}$ is both n-ideal and prime ideal. So by assumption $\sqrt{0} = 0$ is a prime ideal. Hence R is an integral domain. □

Recall from that a ring R is called von Neumann regular if for every $a \in R$, there exists an element x of R such that $a = a^2x$. Also a ring R is said to be a boolean ring if whenever $a = a^2$ for every $a \in R$. Notice that every boolean ring is also a von Neumann regular [2].

Theorem 2.15. *Let R be a ring. Then the followings hold:*
(i) R is a field if and only if R is von Neumann regular ring and 0 is an n-ideal.
(ii) Suppose that R is boolean ring. Then R is a field if and only if 0 is an n-ideal. In particular $R \cong \mathbb{Z}_2$.

Proof. (i) If R is a field, then it is clear that R is von Neumann regular. From Theorem 2.14, 0 is an n-ideal. For the converse, suppose that R is von Neumann regular ring and 0 is an n-ideal. Let $0 \neq a \in R$. Since R is von Neumann regular, $a = a^2x$ for some $x \in R$. Also it is easy to see that $\sqrt{0} = 0$. Since $a(1 - ax) = 0$ and 0 is an n-ideal of R, we conclude that $ax = 1$ and thus a is unit. Consequently, R is a field.

(ii) Suppose that R is boolean ring. Then R is a von Neumann regular ring. So by (i) it follows that R is a field if and only if 0 is an n-ideal. The rest is easily seen. □

Proposition 2.16. *Let R be a ring and K an ideal of R with $K \cap (R - \sqrt{0}) \neq \emptyset$. Then the followings hold:*
(i) If I_1, I_2 are n-ideals of R with $I_1K = I_2K$, then $I_1 = I_2$.
(ii) If IK is an n-ideal of R, then $IK = I$.

Proof. (i) Since I_1 is an n-ideal and $I_2K \subseteq I_1$, by Theorem 2.7 (iii), we get the result that $I_2 \subseteq I_1$. Likewise, we get $I_1 \subseteq I_2$.

(ii) Since IK is an n-ideal and $IK \subseteq IK$, we conclude that $I \subseteq IK$, so this completes the proof. □

Theorem 2.17. *Let $f : R \to S$ be a ring homomorphism. Then the followings hold:*
(i) If f is an epimorphism and I is an n-ideal of R containing $Ker(f)$, then $f(I)$ is an n-ideal of S.
(ii) If f is a monomorphism and J is an n-ideal of S, then $f^{-1}(J)$ is an n-ideal of R.

Proof. (i) Let $a'b' \in f(I)$ with $a' \notin \sqrt{0_S}$ for $a', b' \in S$. Since f is epimorphism, there exist $a, b \in R$ such that $a' = f(a)$ and $b' = f(b)$. Then $a'b' = f(ab) \in f(I)$. As $Ker(f) \subseteq I$, we conclude that $ab \in I$. Also, note that $a \notin \sqrt{0_R}$. Since I is an n-ideal of R, we get the result that $b \in I$ and so $f(b) = b' \in f(I)$ as it is needed.

(ii) Let $ab \in f^{-1}(J)$ and $a \notin \sqrt{0_R}$. Then $f(ab) = f(a)f(b) \in J$. Since $a \notin \sqrt{0_R}$ and f is a monomorphism, we get $f(a) \notin \sqrt{0_S}$. Since J is an n-ideal of S, $f(b) \in J$ and so $b \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is an n-ideal of R. □

Corollary 2.18. *Let R be a ring and $J \subseteq I$ be two ideals of R. Then the followings hold:*
(i) If I is an n-ideal of R, then I/J is an n-ideal of R/J.
(ii) If I/J is an n-ideal of R/J and $J \subseteq \sqrt{0}$, then I is an n-ideal of R.
(iii) If I/J is an n-ideal of R/J and J is an n-ideal of R, then I is an n-ideal of R.

Proof. (i) Assume that I is an n-ideal of R with $J \subseteq I$. Let $\pi : R \to R/J$ be the natural homomorphism. Note that $Ker(\pi) = J \subseteq I$, and so by Theorem 2.17(i) it follows that I/J is an n-ideal of R/J.

(ii) Let $ab \in I$ with $a \notin \sqrt{0}$ for $a, b \in R$. Then we have $(a + J)(b + J) = ab + J \in I/J$ and $a + J \notin \sqrt{0_{R/J}}$. Since I/J is an n-ideal of R/J, we conclude that $b + J \in I/J$ and so $b \in I$. Consequently, I is an n-ideal of R.

(iii) It follows from (ii) and Proposition 2.3. □

Corollary 2.19. *Let R be a ring and S a subring of R. If I is an n-ideal of R with $S \nsubseteq I$, then $I \cap S$ is an n-ideal of S.*

Proof. Suppose that S is a subring of R and I is an n-ideal of R with $S \nsubseteq I$. Consider the injection $i : S \to R$. And note that $i^{-1}(I) = I \cap S$, so by Proposition 2.17(ii), $I \cap S$ is an n-ideal of S. □

Recall that an element a of R is called regular if $ann(a) = 0$. Then we denote the set of all regular elements of R by $r(R)$. Further, it is easy to see that $r(R)$ is a multiplicatively closed subset of R.

Proposition 2.20. *Let R be a ring and S a multiplicatively closed subset of R. Then the followings hold:*
(i) If I is an n-ideal of R, then $S^{-1}I$ is an n-ideal of $S^{-1}R$.
(ii) If $S = r(R)$ and J is an n-ideal of $S^{-1}R$, then J^c is an n-ideal of R.

Proof. (i) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ with $\frac{a}{s} \notin \sqrt{0_{S^{-1}R}}$, where $a, b \in R$ and $s, t \in S$. Then we have $uab \in I$ for some $u \in S$. It is clear that $a \notin \sqrt{0}$. Since I is an n-ideal of R, we conclude that $ub \in I$ and so $\frac{b}{t} = \frac{ub}{ut} \in S^{-1}I$. Consequently, $S^{-1}I$ is an n-ideal of $S^{-1}R$.

(ii) Let $ab \in J^c$ and $a \notin \sqrt{0_R}$. Then we have $\frac{a}{1}\frac{b}{1} \in J$. Now we show that $\frac{a}{1} \notin \sqrt{0_{S^{-1}R}}$. Suppose $\frac{a}{1} \in \sqrt{0_{S^{-1}R}}$. There exists a positive integer k such that $(\frac{a}{1})^k = \frac{a^k}{1} = 0_{S^{-1}R}$. Then we get $ua^k = 0$ for some $u \in S$. Since $ann(u) = 0$, we conclude that $a \in \sqrt{0_R}$, a contradiction. Thus we have $\frac{a}{1} \notin \sqrt{0_{S^{-1}R}}$. Since J is an n-ideal of $S^{-1}R$, we get the result that $\frac{b}{1} \in J$ and so $b \in J^c$. □

Definition 2.21. *Let S be a nonempty subset of R with $R - \sqrt{0} \subseteq S$. Then S is called an n-multiplicatively closed subset of R if $xy \in S$ for all $x \in R - \sqrt{0}$ and all $y \in S$.*

Suppose that I is an n-ideal of R. Then by Proposition 2.3 we have $I \subseteq \sqrt{0}$ and so $R - \sqrt{0} \subseteq R - I$. Let $x \in R - \sqrt{0}$ and $y \in R - I$. Assume that $xy \in I$. Since $x \notin \sqrt{0}$ and I is an n-ideal, we conclude that $y \in I$, a contradiction. Thus we get the result that $xy \in R - I$, and so $R - I$ is an n-multiplicatively closed subset of R. For the converse, suppose that I is an ideal and $R - I$ is an n-multiplicatively closed subset of R. Now we show that I is an n-ideal. Let $ab \in I$ with $a \notin \sqrt{0}$ for $a, b \in R$. Then we have $b \in I$, or else we would have $ab \in R - I$ since $R - I$ is an n-multiplicatively closed subset of R. So it follows that I is an n-ideal of R. By the above observations we have the following result analogous with the relations between prime ideals and multiplicatively closed subsets.

Corollary 2.22. *For a proper ideal I of R, I is an n-ideal of R if and only if $R - I$ is an n-multiplicatively closed subset of R.*

We remind the reader that if I is an ideal which is disjiont from a multiplicatively closed subset S of R, then there exists a prime ideal P of R contaning I such that $P \cap S = \emptyset$. The following Theorem states that a similar result is true for n-ideals.

Theorem 2.23. *Let I be an ideal of R such that $I \cap S = \emptyset$, where S is an n-multiplicatively closed subset of R. Then there exists an n-ideal J containing I such that $J \cap S = \emptyset$.*

Proof. Consider the set $\Omega = \{I' : I'$ is an ideal of R with $I' \cap S = \emptyset\}$. Since $I \in \Omega$, we have $\Omega \neq \emptyset$. By using Zorn's lemma, we get a maximal element J of Ω. Now we show that J is an n-ideal of R. Suppose not. Then we have $ab \in J$ for some $a \notin \sqrt{0}$ and $b \notin J$. Thus we get $b \in (J : a)$ and $J \subsetneq (J : a)$. By the maximality of J, we have $(J : a) \cap S \neq \emptyset$ and thus there exists an $s \in S$ such that $s \in (J : a)$. So we have $as \in J$. Also $sa \in S$, because $a \in R - \sqrt{0}, s \in S$ and S is an n-multiplicatively closed subset of R.Thus we get $S \cap J \neq \emptyset$, and this contradicts by $J \in \Omega$. Hence J is an n-ideal of R. □

Proposition 2.24. *Suppose that $I \subseteq I_1 \cup I_2 \cup ... \cup I_n$, where $I, I_1, I_2, ..., I_n$ are ideals of R. If I_i is an n-ideal and the others have non-nilpotent elements with $I \nsubseteq \bigcup_{j \neq i} I_j$, then $I \subseteq I_i$.*

Proof. We may assume that $i = 1$. Since $I \nsubseteq I_2 \cup ... \cup I_n$, there exits an $x \in I - \bigcup_{j=2}^{n} I_j$. Thus we have $x \in I_1$. Let $y \in I \cap (I_2 \cap I_3 \cap ... \cap I_n)$. Since $x \notin I_k$ and $y \in I_k$ for every $2 \leq k \leq n$, we have $x + y \notin I_k$. Thus we have $x + y \in I - \bigcup_{j=2}^{n} I_j$ and so $x + y \in I_1$. As $x + y \in I_1$ and $x \in I_1$, it follows that $y \in I_1$ and so $I \cap \bigcap_{k=2}^{n} I_k \subseteq I_1$. By the way $\sqrt{0}$ is a prime ideal, because R has an n-ideal. So the product of non-nilpotent elements is also a non-nipotent element. Thus we have $(\prod_{k=2}^{n} I_k) \cap (R - \sqrt{0}) \neq \emptyset$. Since $I.(\prod_{k=2}^{n} I_k) \subseteq I_1$ and I_1 is an n-ideal of R, we have $I \subseteq I_1$ by Theorem 2.7. □

Recall from [7] a ring R is a UN-ring if every nonunit element a of R is a product a unit and a nilpotent element.

Proposition 2.25. *For any ring R, the followings are equivalent:*
(i) Every element of R is either nilpotent or unit.
(ii) Every proper principal ideal is an n-ideal.
(iii) Every proper ideal is an n-ideal.
(iv) R has a unique prime ideal which is $\sqrt{0}$.
(v) R is a UN-ring.
(vi) $R/\sqrt{0}$ is a field.

Proof. $(i) \Rightarrow (ii)$: Suppose that $\langle x \rangle \neq R$, where $x \in R$. Let $ab \in \langle x \rangle$ and $a \notin \sqrt{0}$. Since a is not nilpotent, by (i) a is a unit in R. Then we have $b = a^{-1}(ab) \in \langle x \rangle$ and so $\langle x \rangle$ is an n-ideal of R.

$(ii) \Rightarrow (iii)$: Let I be a proper ideal of R and $ab \in I$, where $a \notin \sqrt{0}$. Since $ab \in \langle ab \rangle$ and $\langle ab \rangle$ is an n-ideal of R, we conclude that $b \in \langle ab \rangle \subseteq I$. Hence I is an n-ideal of R.

$(iii) \Rightarrow (iv)$: Let P be a prime ideal of R. By (iii) and Proposition 2.8, we get the result that $P = \sqrt{0}$, as needed. Furthermore, $\sqrt{0}$ is a maximal ideal of R.

$(iv) \Leftrightarrow (v)$: It follows from [7, Proposition 2 (3)].

$(iv) \Rightarrow (vi)$: It is straightforward.

$(vi) \Rightarrow (i)$: Suppose that $R/\sqrt{0}$ is a field. Let $a \in R$ which is not nilpotent. Then we have $a \notin \sqrt{0}$ and $a + \sqrt{0}$ is nonzero element of the field $R/\sqrt{0}$. Thus we get the result that $ab - 1$ is nilpotent for some $b \in R$. Then we have $(ab - 1) + 1 = ab$ is unit. Hence a is unit, as needed. □

Suppose that R_1, R_2 are two commutative rings with nonzero identities and $R = R_1 \times R_2$. Then R becomes a commutative ring with coordinate-wise addition and multiplication. Also, every ideal I of R has the form $I = I_1 \times I_2$, where I_i is an ideal of R_i for $i = 1, 2$. Now, we give the following result.

Proposition 2.26. *Let R_1 and R_2 be two commutative rings. Then $R_1 \times R_2$ has no n-ideals.*

Proof. Assume that $I = I_1 \times I_2$ is an n-ideal of $R_1 \times R_2$, where I_i is an ideal of R_i for $i = 1, 2$. Since $(0, 1)(1, 0) \in I_1 \times I_2$, $(0, 1) \notin \sqrt{0_{R_1 \times R_2}}$ and $(1, 0) \notin \sqrt{0_{R_1 \times R_2}}$, we conclude that $(0, 1), (1, 0) \in I$ and so $I = R_1 \times R_2$, a contradiction. □

Let $R(+)M$ denote the idealization of M in R, where M is an R-module. Assume that I is an ideal of R and N is a submodule of M. Then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$, in that case $I(+)N$ is called a homogeneous ideal of $R(+)M$ [3]. In [3, 9], the nil radical of $R(+)M$ is characterized as follows:

$$\sqrt{0_{R(+)M}} = \sqrt{0}(+)M.$$

Notice that $(r, m) \notin \sqrt{0_{R(+)M}}$ if and only if $r \notin \sqrt{0}$.

Proposition 2.27. *Let I be an n-ideal of R. Then $I(+)M$ is an n-ideal of $R(+)M$.*

Proof. Let $(r_1, m_1)(r_2, m_2) \in I(+)M$ with $(r_1, m_1) \notin \sqrt{0_{R(+)M}}$. Then we have $r_1r_2 \in I$ and $r_1 \notin \sqrt{0}$. Since I is an n-ideal of R, we conclude that $r_2 \in I$ and so $(r_2, m_2) \in I(+)M$. Consequently, $I(+)M$ is an n-ideal of $R(+)M$. $\square$

Remark 2.28. *Let I be an n-ideal of R and N a submodule of M with $IM \subseteq N$, then $I(+)N$ need not be an n-ideal of $R(+)M$. For example 0 is an n-ideal of the ring of integers and $\overline{0}$ is a submodule of $\mathbb{Z}$-module $\mathbb{Z}_6$. But $0(+)\overline{0}$ is not an n-ideal, because $(2, \overline{0})(0, \overline{3}) \in 0(+)\overline{0}$ with $(2, \overline{0}) \notin \sqrt{0_{\mathbb{Z}(+)\mathbb{Z}_6}}$ but $(0, \overline{3}) \notin 0(+)\overline{0}$.*

Definition 2.29. *Let M be an R-module. Then we say that an element a of R is nilpotent in M if whenever $a^nM = 0_M$ for some positive integer n. Then the set of all nilpotents in M is denoted by $Nil(M)$. It is clear that $\sqrt{0} \subseteq Nil(M)$.*

Now we generalize the concept of n-ideals to modules in the following.

Definition 2.30. *Let M be an R-module. Then a proper submodule N of M is called an n-submodule if for $a \in R, m \in M$, $am \in N$ with $a \notin Nil(M)$, then $m \in N$.*

Theorem 2.31. *Let I be an ideal of R and N a proper submodule of M. If $I(+)N$ is an n-ideal of $R(+)M$, then I is an n-ideal of R and N is an n-submodule of M.*

Proof. Suppose that $I(+)N$ is an n-ideal of $R(+)M$. First, we show that I is an n-ideal of R. Let $ab \in I$ with $a \notin \sqrt{0}$. Then we have $(a, 0_M)(b, 0_M) = (ab, 0_M) \in I(+)N$ with $(a, 0_M) \notin \sqrt{0_{R(+)M}}$. Since $I(+)N$ is an n-ideal of $R(+)M$, we conclude that $(b, 0_M) \in I(+)N$ and so $b \in I$. Now, we show that N is an n-submodule of M. Let $am \in N$ with $a \notin Nil(M)$. Then we have $(a, 0_M)(0, m) = (0, am) \in I(+)N$ with $(a, 0_M) \notin \sqrt{0_{R(+)M}}$. Since $I(+)N$ is an n-ideal of $R(+)M$, we conclude that $(0, m) \in I(+)N$ and so $m \in N$, as needed. $\square$

Theorem 2.32. *Let M be an R-module with $Nil(M) \subseteq \sqrt{0}$. If I is an n-ideal of R and N is an n-submodule of M with $IM \subseteq N$, then $I(+)N$ is an n-ideal of $R(+)M$.*

Proof. Let $(r_1, m_1)(r_2, m_2) \in I(+)N$ with $(r_1, m_1) \notin \sqrt{0_{R(+)M}}$. Then $r_1r_2 \in I$ and $r_1 \notin \sqrt{0}$. Since I is an n-ideal of R, we conclude that $r_2 \in I$. Thus we have $r_2m_1 \in IM \subseteq N$, and so $r_1m_2 \in N$, because $r_1m_2 + r_2m_1 \in N$. Since N is an n-submodule of M and $r_1 \notin Nil(M) \subseteq \sqrt{0}$, we conclude $m_2 \in N$ so that $(r_2, m_2) \in I(+)N$ as it is needed. $\square$

Corollary 2.33. *Let M be an R-module with $Nil(M) \subseteq \sqrt{0}$. Suppose that I is an ideal of R and N is a proper submodule of M with $IM \subseteq N$. Then $I(+)N$ is an n-ideal of $R(+)M$ if and only if I is an n-ideal of R and N is an n-submodule of M.*

3. Examples

Proposition 3.1. *$\mathbb{Z}_n$ has an n-ideal if and only if $n = p^k$ for some $k \in \mathbb{Z}^+$, where p is prime number.*

Proof. If $n = p^k$ for some $k \in \mathbb{Z}^+$, then $\mathbb{Z}_n$ is a local ring with unique prime ideal and so by Example 2.2 every ideal is an n-ideal. Suppose that $n = p_1^{n_1}p_2^{n_2}...p_t^{n_t}$, where p_i's are distinct prime numbers with $t \geq 2$. First notice that $\sqrt{0} = \langle\overline{p_1p_2...p_t}\rangle$. Assume that I is an n-ideal of $\mathbb{Z}_n$. Then we get $I \subseteq \sqrt{0} = \langle\overline{p_1p_2...p_t}\rangle$. Hence $I = \langle\overline{p_1^{s_1}p_2^{s_2}...p_t^{s_t}}\rangle$ for some positive integers s_i with $s_i \leq n_i$ for $i = 1, 2, .., t$. It is easy to see that $\overline{p_2^{s_2}...p_t^{s_t}} \notin \sqrt{0} = \langle\overline{p_1p_2...p_t}\rangle$ and $\overline{p_1^{s_1}} \notin I = \langle\overline{p_1^{s_1}p_2^{s_2}...p_t^{s_t}}\rangle$ but $\overline{p_1^{s_1}}(\overline{p_2^{s_2}...p_t^{s_t}}) \in I$. So it follows that I is not an n-ideal, a contradiction. □

Now we give the following examples to compare with the notion of prime ideals, n-ideals and r-ideals.

Example 3.2. *(i) It is clear that $3\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. But it is not an n-ideal of $\mathbb{Z}$ by Example 2.2.*

(ii) In the ring $\mathbb{Z}_{27}$, $\langle\overline{9}\rangle$ is an n-ideal. But $\langle\overline{9}\rangle$ is not prime ideal, because $\overline{3}.\overline{3} \in \langle\overline{9}\rangle$ and $\overline{3} \notin \langle\overline{9}\rangle$.

(iii) $\langle\overline{3}\rangle$ is an r-ideal of $\mathbb{Z}_6$ but it is not an n-ideal by Proposition 3.1.

In the following example (i) we give an infinite ring having the n-ideals, and also in example (ii) we show the converse of Proposition 2.3 is not always correct.

Example 3.3. *(i) Consider the ring $\mathbb{Z}[X]$ and the prime ideal $P = \langle X\rangle$. Let $R = \mathbb{Z}[X]/P^n$ and $I = P^2/P^n$ for $n > 2$. First, note that $\sqrt{0} = P/P^n$. Let $(f + P^n)(g + P^n) \in I$ and $g + P^n \notin \sqrt{0}$. Then $fg \in \langle X\rangle^2$ and $g \notin \langle X\rangle$, so that X^2 divides fg but X can not divide g. Thus X^2 divides f and so $f + P^n \in I$. Hence I is an n-ideal of R.*

(ii) Let $R = \mathbb{Z}[X, Y]/\langle Y^4\rangle$ and $I = \langle xy, y^2\rangle$, where $x = X+\langle Y^4\rangle$ and $y = Y+\langle Y^4\rangle$. It is easy to see that $\sqrt{0_R} = \langle y\rangle$ is a prime ideal and so it is an n-ideal by Corollary 2.9(i). Furthermore, $I \subseteq \sqrt{0_R}$. Since $y(x + y) \in I$, $x + y \notin \sqrt{0_R}$ and $y \notin I$, it follows that I is not an n-ideal.

Example 3.4. *Consider the ring $\mathbb{Z}_9[x]$ and note that $\sqrt{0_{\mathbb{Z}_9[x]}} = \overline{3}\mathbb{Z}_9[x]$. Now, we show that $\sqrt{0_{\mathbb{Z}_9[x]}}$ is an n-ideal. Let us define a homomorphism as follows:*

$$\varphi : \mathbb{Z}_9[x] \to \mathbb{Z}_3[x],\ \varphi(\overline{a_0} + \overline{a_1}x + ... + \overline{a_n}x^n) = \overline{a_0} + \overline{a_1}x + ... + \overline{a_n}x^n.$$

It is clear that φ is an epimorphism and the $Ker(f) = \sqrt{0_{\mathbb{Z}_9[x]}}$. So we have $\mathbb{Z}_9[x]\ /\ \sqrt{0_{\mathbb{Z}_9[x]}} \cong \mathbb{Z}_3[x]$ is an integral domain and so $\sqrt{0_{\mathbb{Z}_9[x]}}$ is a prime ideal of $\mathbb{Z}_9[x]$. Then by Corollary 2.9(i), $\sqrt{0_{\mathbb{Z}_9[x]}}$ is an n-ideal of $\mathbb{Z}_9[x]$, which is nonzero.

The following examples show that the converses of Corollary 2.18(i) and Theorem 2.31 are not always true.

Example 3.5. *(i) Let $R = \mathbb{Z}$, $I = 3\mathbb{Z}$ and $J = 9\mathbb{Z}$. Then I/J is an n-ideal of R/J but I is not an n-ideal of R.*

(ii) Consider the $\mathbb{Z}$-module $\mathbb{Z}_9$. Note that 0 is an n-ideal of $\mathbb{Z}$ and $\overline{0}$ is an n-submodule of $\mathbb{Z}_9$. But $I = 0(+)\overline{0}$ is not an n-ideal of $\mathbb{Z}(+)\mathbb{Z}_9$, because $(3, \overline{0})(0, \overline{3}) = (0, \overline{0}) \in I$, $(3, 0) \notin \sqrt{0_{\mathbb{Z}(+)\mathbb{Z}_9}}$ and $(0, \overline{3}) \notin I$.

Remark 3.6. *Suppose that I is an n-ideal of R. Then it follows that $\sqrt{I} = \sqrt{0}$ is an n-ideal by Theorem 2.12 and Corollary 2.9 (i). Example 3.3 (ii) reserves that the converse is not true, that is, I may not be an n-ideal even if $\sqrt{I}$ is an n-ideal of R.*

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