Filomat 31:10 (2017), 2943–2950
https://doi.org/10.2298/FIL1710943K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On 2-Absorbing Quasi Primary Submodules

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Abstract. Let R be a commutative ring with nonzero identity, and let M be a nonzero unital R-module. In this article, we introduce the concept of 2-absorbing quasi primary submodules which is a generalization of prime submodules. We define 2-absorbing quasi primary submodule as a proper submodule N of M having the property that $abm \in N$, then $ab \in \sqrt{(N :_R M)}$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$. Various results and examples concerning 2-absorbing quasi primary submodules are given.

1. Introduction

It is well known that prime submodules play an important role in the theory of modules over commutative rings. So far there has been a lot of research on this issue. For various studies one can look [2-3,7-8]. One of the main interest of many researchers is to generalize the notion of prime submodule by using different ways. For instance, 2-absorbing submodule which is a generalization of prime submodules was firstly introduced and studied in [9], after that another generalization, which is called 2-absorbing primary submodule was studied in [15].

Throughout this paper all rings under consideration are commutative with nonzero identity and all modules are nonzero unital. In addition, R always denotes such a ring and M denotes such an R-module. Suppose that I is an ideal of R and N is a submodule of M. Then the radical of I, denoted by $\sqrt{I}$, is defined as intersection of all prime ideals containing I and equally consists of all elements a of R whose some power in I, i.e, $\{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$. Also, the ideal $(N :_R M)$ is defined as $\{a \in R : aM \subseteq N\}$, and for every $a \in R$, the submodule $(N :_M a)$ is defined to be $\{m \in M : am \in N\}$. Similar to radical of an ideal, radical of a submodule of a given R-module M can be identified. If there is any prime submodule P of M that contains N, then the intersection of all prime submodules containing N is denoted by $rad_M(N)$. Otherwise, that is if there is no prime submodule containing N, say $rad_M(N) = M$. Recall that a submodule N of M is a prime submodule if whenever $N \neq M$ and $am \in N$, then either $a \in (N :_R M)$ or $m \in N$. A proper submodule N of M is defined as 2-absorbing submodule if for every $a, b \in R, m \in M$ and whenever $abm \in N$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Also recall that a proper submodule N of M is said to be a 2-absorbing primary submodule if the condition $abm \in N$ implies either $ab \in (N :_R M)$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$.

2010 *Mathematics Subject Classification*. Primary 13A15
Keywords. multiplication module, prime submodule, 2-absorbing submodule
Received: 29 August 2016; Accepted: 26 February 2017
Communicated by Miroslav Ćirić
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This paper is based on introducing a new class of submodules, which is called 2-absorbing quasi primary submodules, and studying its properties. We define a proper submodule N of M a 2-absorbing quasi primary submodule if whenever $abm \in N$, then either $ab \in \sqrt{(N :_R M)}$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$ for each $a, b \in R$ and $m \in M$. Among many other results in this paper, we show in Example 2.2 a 2-absorbing quasi primary submodule is not necessarily 2-absorbing submodule and 2-absorbing primary submodule. In Theorem 2.4, we characterize all homogeneous 2-absorbing quasi primary ideals of idealization of a module. We remind the reader that an R-module M is a multiplication if every submodule N of M has the form $N = IM$ for some ideal I of R [6]. In addition, it is easy to see that $N = (N :_R M)M$ in case $N = IM$ for some ideal I of R. Suppose that M is multiplication R-module, $N = IM$ and $K = JM$ for ideals I, J of R, then product of submodules N and K of M, designated by NK, is defined to be $(IJ)M$. In [3], it is proved that a proper submodule N of a multiplication R-module M is prime if and only if $KL \subseteq N$ implies either $K \subseteq N$ or $L \subseteq N$ for submodules K, L of M. In Corollary 2.8, for finitely generated multiplication modules, we show that a proper submodule N of M is a 2-absorbing quasi primary if and only if $N_1N_2N_3 \subseteq N$ implies either $N_1N_2 \subseteq rad_M(N)$ or $N_1N_3 \subseteq rad_M(N)$ or $N_2N_3 \subseteq rad_M(N)$ for submodules N_1, N_2 and N_3 of M. In [6], Z, El Bast and P. Smith showed that the followings are eqivalent for a proper submodule N of a multiplication module M :

(*i*) N is a prime submodule.

(*ii*) $(N :_R M)$ is a prime ideal.

(*iii*) $N = PM$ for some prime ideal P of R such that $Ann(M) \subseteq P$, where $Ann(M) = (0 :_R M)$.

In Theorem 2.12, we prove that similar result is true for 2-absorbing quasi primary submodules in finitely generated multiplication modules. Also in Corollary 2.11, we give various characterizations of 2-absorbing quasi primary submodules of finitely generated multiplication modules. In Theorem 2.14, we study the 2-absorbing quasi primary submodules of fractional modules. Moreover, in Thoerem 2.18, we investigate the behaviour of 2-absorbing quasi primary submodules under the homorphism of modules. Finally, in Theorem 2.23, all 2-absorbing quasi primary submodules of cartesian product of finitely generated multiplication modules are determined.
The reader may consult [5],[10] and [12] for general background and terminology.

2. 2-Abdorbing Quasi Primary Submodules

Definition 2.1. *A proper submodule N of an R-module M is said to be a 2-absorbing quasi primary submodule (weakly 2-absorbing quasi primary submodule) if the condition $abm \in N$ $(0 \neq abm \in N)$ implies either $ab \in \sqrt{(N :_R M)}$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$ for every $a, b \in R$ and $m \in M$.*

In [17], a 2-absorbing quasi primary ideal is defined as a proper ideal I of R whose the radical is a 2-absorbing ideal. The authors (in Proposition 2.5) showed that a proper ideal I of R is a 2-absorbing quasi primary ideal if and only if whenever $abc \in I$, then $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ for each $a, b, c \in R$. From this aspect, we can see the 2-absorbing quasi primary submodules of an R-module R are all 2-absorbing quasi primary ideals of R. In addition, by the definition 2.1, it is clear that every 2-absorbing submodule and 2-absorbing primary submodule are also a 2-absorbing quasi primary submodule. However, we give an example showing the converse fails as follows:

Example 2.2. *Let $R_0 = \{a_0 + a_1X + a_2X^2 + ... + a_nX^n : a_1$ is a multiple of $3\} \subseteq \mathbb{Z}[X]$ and $R = R_0 \times R_0$. Now, consider the R-module $R = M$ and the submodule $N = Q \times Q$, where $Q = \langle 9X^2, 3X^3, X^4, X^5, X^6 \rangle$. First note that $rad_M(N) = \sqrt{(N :_R M)} = \sqrt{Q} \times \sqrt{Q}$, where $\sqrt{Q} = \langle 3X, X^2, X^3 \rangle$. Since $(3, X^2)(X^2, 3)(3, 3) = (9X^2, 9X^2) \in N$ but $(3, X^2)(X^2, 3) = (3X^2, 3X^2) \notin (N :_R M) = N$ and $(3, X^2)(3, 3) \notin rad_M(N)$ and $(X^2, 3)(3, 3) \notin rad_M(N)$, it follows that N is not a 2-absorbing primary submodule of M. Also, one can easily see that N is a 2-absorbing quasi primary submodule of M.*

Theorem 2.3. *For a proper submodule N of M, the following statements are equivalent:*
(i) N is a 2-absorbing quasi primary submodule of M.

(ii) For every $a, b \in R$, $(N :_M a^k b^k) = M$ for some $k \in \mathbb{Z}^+$ or $(N :_M ab) \subseteq (rad_M(N) :_M a) \cup (rad_M(N) :_M b)$.

(iii) For every $a, b \in R$, $(N :_M a^k b^k) = M$ for some $k \in \mathbb{Z}^+$ or $(N :_M ab) \subseteq (rad_M(N) :_M a)$ or $(N :_M ab) \subseteq (rad_M(N) :_M b)$.

Proof. $(i) \Rightarrow (ii)$: Suppose that N is a 2-absorbing quasi primary submodule of M. Let $a, b \in R$. If $ab \in \sqrt{(N :_R M)}$, then $(ab)^k = a^k b^k \in (N :_R M)$ for some $k \in \mathbb{Z}^+$ and so $(N :_M a^k b^k) = M$. Now, assume $ab \notin \sqrt{(N :_R M)}$. Let $m \in (N :_M ab)$. Then we have $abm \in N$, and thus $am \in rad_M(N)$ or $bm \in rad_M(N)$ since N is a 2-absorbing quasi primary submodule. Hence we get the result that $(N :_M ab) \subseteq (rad_M(N) :_M a) \cup (rad_M(N) :_M b)$

$(ii) \Rightarrow (iii)$: It is well known that if a submodule is contained in two submodules, then it is contained in at least one of them.

$(iii) \Rightarrow (i)$: Let $abm \in N$ with $ab \notin \sqrt{(N :_R M)}$ for $a, b \in R$ and $m \in M$. Then we have $(N :_M a^k b^k) \neq M$ for every $k \in \mathbb{Z}^+$. Thus by (iii) we get the result that $m \in (N :_M ab) \subseteq (rad_M(N) :_M a)$ or $m \in (rad_M(N) :_M b)$, so we have $am \in rad_M(N)$ or $bm \in rad_M(N)$ as it is needed. □

Let M be an R-module. In [16], Nagata introduced the idealization of a module. Recall that the idealization $R(+)M = \{(r, m) : r \in R, m \in M\}$ is a commutative ring with the following addition and multiplication:

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$$
$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$$

for every $r_1, r_2 \in R$; $m_1, m_2 \in M$. Suppose that I is an ideal of R and N is a submodule of M. Then $I(+)N = \{(i, n) : i \in I, n \in N\}$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. In this case, $I(+)N$ is called a homogeneous ideal. Anderson (in [4]) characterizes the radical of homogeneous ideals as the following:

$$\sqrt{I(+)N} = \sqrt{I}(+)M.$$

Theorem 2.4. *Let M be an R-module. For a proper ideal I of R and submodule N of M with $IM \subseteq N$, $I(+)N$ is a 2-absorbing quasi primary ideal of $R(+)M$ if and only if I is a 2-absorbing quasi primary ideal of R.*

Proof. Suppose that I is a 2-absorbing quasi primary ideal of R. Let $(r_1, m_1)(r_2, m_2)(r_3, m_3) = (r_1 r_2 r_3, r_1 r_2 m_3 + r_1 r_3 m_2 + r_2 r_3 m_1) \in I(+)N$, where $r_i \in R$ and $m_i \in M$ for $i = 1, 2, 3$. Then we have $r_1 r_2 r_3 \in I$. Since I is a 2-absorbing quasi primary ideal of R, we conclude either $r_1 r_2 \in \sqrt{I}$ or $r_1 r_3 \in \sqrt{I}$ or $r_2 r_3 \in \sqrt{I}$. Thus we have $(r_1, m_1)(r_2, m_2) \in \sqrt{I}(+)M = \sqrt{I(+)N}$ or $(r_1, m_1)(r_3, m_3) \in \sqrt{I(+)N}$ or $(r_2, m_2)(r_3, m_3) \in \sqrt{I(+)N}$. Hence $I(+)N$ is a 2-absorbing quasi primary ideal of $R(+)M$. For the converse, assume that $I(+)N$ is a 2-absorbing quasi primary ideal of $R(+)M$. Let $abc \in I$ for $a, b, c \in R$. Then we have $(a, 0_M)(b, 0_M)(c, 0_M) = (abc, 0_M) \in I(+)N$. Since $I(+)N$ is a 2-absorbing quasi primary ideal of $R(+)M$, we conclude either $(a, 0_M)(b, 0_M) \in \sqrt{I}(+)M$ or $(a, 0_M)(c, 0_M) \in \sqrt{I}(+)M$ or $(b, 0_M)(c, 0_M) \in \sqrt{I}(+)M$. Thus we have $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, this completes the proof. □

Lemma 2.5. *Let M be an R-module. Suppose that N is a 2-absorbing quasi primary submodule of M and $abK \subseteq N$ for $a, b \in R$ and submodule K of M. If $ab \notin \sqrt{(N :_R M)}$, then $aK \subseteq rad_M(N)$ or $bK \subseteq rad_M(N)$.*

Proof. Since $K \subseteq (N :_M ab)$ and $(N :_M a^k b^k) \neq M$ for every $k \in \mathbb{Z}^+$, by Theorem 2.3 we have $K \subseteq (N :_M ab) \subseteq (rad_M(N) :_M a)$ or $K \subseteq (N :_M ab) \subseteq (rad_M(N) :_M b)$. Hence we get the result that $aK \subseteq rad_M(N)$ or $bK \subseteq rad_M(N)$. □

Theorem 2.6. *For a proper submodule N of M, the followings are equivalent:*

(i) N is a 2-absorbing quasi primary submodule.

(ii) For $a \in R$, an ideal I_2 of R and submodule K of M with $aI_2K \subseteq N$, then either $aI_2 \subseteq \sqrt{(N :_R M)}$ or $aK \subseteq rad_M(N)$ or $I_2K \subseteq rad_M(N)$.

(iii) For ideals I_1, I_2 of R and submodule K of M with $I_1I_2K \subseteq N$, then either $I_1I_2 \subseteq \sqrt{(N :_R M)}$ or $I_1K \subseteq rad_M(N)$ or $I_2K \subseteq rad_M(N)$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $aI_2K \subseteq N$ with $aI_2 \nsubseteq \sqrt{(N :_R M)}$ and $I_2K \nsubseteq rad_M(N)$. Then there exist $b_2, b_2' \in I_2$ such that $ab_2 \notin \sqrt{(N :_R M)}$ and $b_2'K \nsubseteq rad_M(N)$. Now, we show that $aK \subseteq rad_M(N)$. Assume that $aK \nsubseteq rad_M(N)$. Since $ab_2K \subseteq N$, by previous lemma we conclude that $b_2K \subseteq rad_M(N)$ and so $(b_2 + b_2')K \nsubseteq rad_M(N)$. By using previous lemma we have $a(b_2 + b_2') = ab_2 + ab_2' \in \sqrt{(N :_R M)}$, because $a(b_2 + b_2')K \subseteq N$. Since $ab_2 + ab_2' \in \sqrt{(N :_R M)}$ and $ab_2 \notin \sqrt{(N :_R M)}$, we get $ab_2' \notin \sqrt{(N :_R M)}$. As $ab_2'K \subseteq N$, by previous lemma we get the result that $b_2'K \subseteq rad_M(N)$ or $aK \subseteq rad_M(N)$, which is a contradiction.

$(ii) \Rightarrow (iii)$: Assume that $I_1I_2K \subseteq N$ with $I_1I_2 \nsubseteq \sqrt{(N :_R M)}$ for ideals I_1, I_2 of R and submodule K of M. Then we have $aI_2 \nsubseteq \sqrt{(N :_R M)}$ for some $a \in I_1$. Now, we show that $I_1K \subseteq rad_M(N)$ or $I_2K \subseteq rad_M(N)$. Suppose not. Since $aI_2K \subseteq N$, by (ii) we get the result that $aK \subseteq rad_M(N)$. Also there exists an element a_1 of I_1 such that $a_1K \nsubseteq rad_M(N)$ because of the assumption $I_1K \nsubseteq rad_M(N)$. As $a_1I_2K \subseteq N$, we get the result that $a_1I_2 \subseteq \sqrt{(N :_R M)}$ and so $(a + a_1)I_2 \nsubseteq \sqrt{(N :_R M)}$. Since $(a + a_1)I_2K \subseteq N$, we have $(a + a_1)K \subseteq rad_M(N)$ and hence $a_1K \subseteq rad_M(N)$, which is a contradiction.

$(iii) \Rightarrow (i)$: Let $abm \in N$ for $a, b \in R$ and $m \in M$. Put $I_1 = aR$, $I_2 = bR$ and $K = Rm$, the rest is easy. □

Lemma 2.7. *Let M be a finitely generated multiplication R-module and N a submodule of M. Then $(rad_M(N) : M) = \sqrt{(N :_R M)}$.*

Proof. It follows from [15, Lemma 2.4]. □

Corollary 2.8. *Let M be a finitely generated multiplication R-module and N a proper submodule of M. Then the followings are equivalent:*

(i) N is a 2-absorbing quasi primary submodule.

(ii) $N_1N_2N_3 \subseteq N$ implies either $N_1N_2 \subseteq rad_M(N)$ or $N_1N_3 \subseteq rad_M(N)$ or $N_2N_3 \subseteq rad_M(N)$ for submodules N_1, N_2 and N_3 of M.

Proof. $(i) \Rightarrow (ii)$: Suppose that N is a 2-absorbing quasi primary submodule and $N_1N_2N_3 \subseteq N$ for submodules N_1, N_2 and N_3 of M. Since M is multiplication, $N_i = I_iM$ for ideals I_i of R_i and $1 \le i \le 3$. Then we have $N_1N_2N_3 = I_1I_2(I_3M) \subseteq N$. By Theorem 2.6, we get $I_1I_2 \subseteq \sqrt{(N :_R M)} = (rad_M(N) : M)$ or $I_1I_3M \subseteq rad_M(N)$ or $I_2I_3M \subseteq rad_M(N)$. Thus we have $N_1N_2 \subseteq rad_M(N)$ or $N_1N_3 \subseteq rad_M(N)$ or $N_2N_3 \subseteq rad_M(N)$.

$(ii) \Rightarrow (i)$: Suppose that $I_1I_2K \subseteq N$ for ideals I_1, I_2 of R and submodule K of M. Put $N_1 = I_1M$, $N_2 = I_2M$ and $N_3 = K$. Then we have $N_1N_2N_3 \subseteq N$. By (ii), we get the result that $N_1N_2 = I_1I_2M \subseteq rad_M(N)$ or $N_1N_3 = I_1K \subseteq rad_M(N)$ or $N_2N_3 = I_2K \subseteq rad_M(N)$. Hence we have $I_1I_2 \subseteq \sqrt{(N :_R M)}$ or $I_1K \subseteq rad_M(N)$ or $I_2K \subseteq rad_M(N)$, as needed. □

Theorem 2.9. *Let M an R-module and N a submodule of M. Then the followings are satisfied:*

(i) If M is a multiplication module and $(N :_R M)$ is a 2-absorbing quasi primary ideal of R, then N is a 2-absorbing quasi primary submodule of M.

(ii) If M is a finitely generated multiplication module and N is a 2-absorbing quasi primary submodule of M, then $(N :_R M)$ is a 2-absorbing quasi primary ideal of R.

Proof. (i) Suppose that M is a multiplication module, $(N :_R M)$ is a 2-absorbing quasi primary ideal of R and $I_1I_2K \subseteq N$ for ideals I_1, I_2 of R and submodule K of M. We have $K = I_3M$ for some ideal I_3 of R since M is multiplication. Then we get $I_1I_2K = I_1I_2I_3M \subseteq N$ and so $I_1I_2I_3 \subseteq (N :_R M)$. As $(N :_R M)$ is a 2-absorbing quasi primary ideal of R, by [17, Theorem 2.21] we conclude that $I_1I_2 \subseteq \sqrt{(N :_R M)}$ or $I_1I_3 \subseteq \sqrt{(N :_R M)} \subseteq (rad_M(N) : M)$ or $I_2I_3 \subseteq \sqrt{(N :_R M)} \subseteq (rad_M(N) : M)$. Thus we have $I_1I_2 \subseteq \sqrt{(N :_R M)}$ or $I_1K \subseteq rad_M(N)$ or $I_2K \subseteq rad_M(N)$. By Theorem 2.6, it follows that N is a 2-absorbing quasi primary submodule of M.

(ii) Suppose that N is a 2-absorbing quasi primary submodule of a finitely generated multiplication R-module M. Let $a, b, c \in R$ such that $abc \in (N :_R M)$ with $ab \notin \sqrt{(N :_R M)}$. Then we have $ab(cm) \in N$ for every $m \in M$. Since N is a 2-absorbing quasi primary submodule of M and $ab \notin \sqrt{(N :_R M)}$, we conclude that $acm \in rad_M(N)$ or $bcm \in rad_M(N)$ for all $m \in M$. Thus we get the result that $(rad_M(N) :_M ac) \cup (rad_M(N) :_M$

$bc) = M$ and so $(rad_M(N) :_M ac) = M$ or $(rad_M(N) :_M bc) = M$. Hence we get $ac \in (rad_M(N) : M) = \sqrt{(N :_R M)}$ or $bc \in \sqrt{(N :_R M)}$. □

Theorem 2.10. *Let M be a finitely generated multiplication R-module. For any submodule N of M, the followings are equivalent:*
(i) N is a 2-absorbing quasi primary submodule of M.
(ii) $rad_M(N)$ is a 2-absorbing submodule of M.

Proof. $(ii) \Rightarrow (i)$: Suppose that $rad_M(N)$ is a 2-absorbing submodule of M. Let $abm \in N$ for $a, b \in R$ and $m \in M$. Then we have $abm \in rad_M(N)$, because $N \subseteq rad_M(N)$. Since $rad_M(N)$ is a 2-absorbing submodule of M, we conclude that $ab \in (rad_M(N) : M) = \sqrt{(N :_R M)}$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$, and so N is a 2-absorbing quasi primary submodule of M.

$(i) \Rightarrow (ii)$: Suppose that N is a 2-absorbing quasi primary submodule of M. Then by previous theorem and [17, Theorem 2.15], we conclude that $\sqrt{(N :_R M)} = P$ is a prime ideal of R or $\sqrt{(N :_R M)} = P_1 \cap P_2$, where P_1, P_2 are distinct prime ideals minimal over $(N :_R M)$. If $\sqrt{(N :_R M)} = P$, then $rad_M(N) = PM$ is a prime submodule by [6, Corollary 2.11] and so it is a 2-absorbing submodule of M. In other case, we have $rad_M(N) =(P_1 \cap P_2)M$. Also it is easy to see that $Ann(M) \subseteq P_1, P_2$. Thus we have $rad_M(N) = ((P_1 + Ann(M) \cap (P_2 + Ann(M))M = P_1M \cap P_2M$, which is the intersection of two prime submodule, is also a 2-absorbing submodule of M. □

In view of Theorem 2.9 and 2.10, we have the following useful corollary to determine the 2-absorbing quasi primary submodules of a finitely generated multiplication module.

Corollary 2.11. *For any submodule N of a finitely generated multiplication R-module M, the followings are equivalent:*
(i) N is a 2-absorbing quasi primary submodule of M;
(ii) $rad_M(N)$ is a 2-absorbing submodule of M;
(iii) $rad_M(N)$ is a 2-absorbing primary submodule of M;
(iv) $rad_M(N)$ is a 2-absorbing quasi primary submodule of M;
(v) $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of R;
(vi) $\sqrt{(N :_R M)}$ is a 2-absorbing primary ideal of R;
(vii) $\sqrt{(N :_R M)}$ is a 2-absorbing quasi primary ideal of R;
(viii) $(N :_R M)$ is a 2-absorbing quasi primary ideal of R.

Theorem 2.12. *Let M be a finitely generated multiplication R-module. For a proper submodule N of M, the followings are equivalent:*
(i) N is a 2-absorbing quasi primary submodule of M.
(ii) $(N :_R M)$ is a 2-absorbing quasi primary ideal of R.
(iii) $N = IM$ for some 2-absorbing quasi primary ideal of R with $Ann(M) \subseteq I$.

Proof. $(i) \Rightarrow (ii)$: It follows from Corollary 2.11.

$(ii) \Rightarrow (iii)$: It is clear.

$(iii) \Rightarrow (i)$: Suppose that $N = IM$ for some 2-absorbing quasi primary ideal I of R with $Ann(M) \subseteq I$. Then we have $\sqrt{(N :_R M)} = \sqrt{(IM :_R M)} = (rad_M(IM) :_R M) = (rad_M(\sqrt{I}M) :_R M)$. By [17, Theorem 2.15] and [13, Result 2], we conclude that either $\sqrt{(N :_R M)} = (rad_M(\sqrt{I}M) :_R M) = (PM :_R M) = P$ is a 2-absorbing quasi primary ideal of R or $\sqrt{(N :_R M)} = ((P_1 \cap P_2)M :_R M) = (P_1M \cap P_2M :_R M) = (P_1M :_R M) \cap (P_2M :_R M) = P_1 \cap P_2$ is a 2-absorbing quasi primary ideal of R. Accordingly, by Corollary 2.11, N is a 2-absorbing quasi primary submodule of M. □

Remark 2.13. *In Theorem 2.12 (iii) if we release the assumption $Ann(M) \subseteq I$, then (iii) does not imply (i). To illustrate this, consider the finitely generated multiplication $\mathbb{Z}$-module $\mathbb{Z}_{180}$. Note that $I = \langle 0 \rangle$ is a 2-absorbing quasi primary ideal of the ring of integers and $Ann(\mathbb{Z}_{180}) = 180\mathbb{Z} \nsubseteq I$. Let $N = \langle 0 \rangle \mathbb{Z}_{180} = \langle \overline{0} \rangle$. Then by Corollary 2.11,*

N is not a 2-absorbing quasi primary submodule because $\sqrt{(N :_R M)} = 30\mathbb{Z}$ is not a 2-absorbing quasi primary ideal of $\mathbb{Z}$.

Theorem 2.14. *Let S be a multiplicatively closed subset of R and M an R-module. If N is a 2-absorbing quasi primary submodule of M with $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 2-absorbing quasi primary submodule of $S^{-1}M$.*

Proof. Assume that N is a 2-absorbing quasi primary submodule of M with $S^{-1}N \neq S^{-1}M$. Let $\frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}N$ for $a, b \in R$; $s_i \in S$ and $m \in M$. Then we have $ab(um) \in N$ for some $u \in S$. Since N is a 2-absorbing quasi primary submodule of M, we get either $ab \in \sqrt{(N :_R M)}$ or $uam \in rad_M(N)$ or $ubm \in rad_M(N)$. Thus we have $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}(\sqrt{(N :_R M)}) \subseteq \sqrt{(S^{-1}N :_{S^{-1}R} S^{-1}M)}$ or $\frac{a}{s_1}\frac{m}{s_3} = \frac{uam}{us_1s_3} \in S^{-1}(rad_M(N)) \subseteq rad_{S^{-1}M}(S^{-1}N)$ or $\frac{b}{s_2}\frac{m}{s_3} = \frac{ubm}{us_2s_3} \in S^{-1}(rad_M(N)) \subseteq rad_{S^{-1}M}(S^{-1}N)$. Hence, it follows that $S^{-1}N$ is a 2-absorbing quasi primary submodule of $S^{-1}M$. □

Lemma 2.15. *Let M be a multiplication R-module and L, K be submodules of M. Then $rad_M(L \cap K) = rad_M(L) \cap rad_M(K)$.*

Proof. See [15, Proposition 2.14]. □

Theorem 2.16. *Let M be a multiplication R-module. Suppose that $N_1, N_2, ..., N_n$ are 2-absorbing quasi primary submodules of M with $rad_M(N_i) = rad_M(N_j)$ for every $1 \leq i, j \leq n$. Then $N = \bigcap_{i=1}^{n} N_i$ is a 2-absorbing quasi primary submodule of M.*

Proof. Suppose that $N_1, N_2, ..., N_n$ are 2-absorbing quasi primary submodule of M with $rad_M(N_i) = rad_M(N_j)$ for every $1 \leq i, j \leq n$. By the previous lemma, we have $rad_M(N) = rad_M(N_j)$ for $1 \leq j \leq n$. Let $abm \in N$ for $a, b \in R$ and $m \in M$. If $ab \in \sqrt{(N :_R M)}$, we are done. Now, assume that $ab \notin \sqrt{(N :_R M)} = \bigcap_{i=1}^{n} \sqrt{(N_i :_R M)}$. Then we have $ab \notin \sqrt{(N_j :_R M)}$ for some $1 \leq j \leq n$. Since N_j is a 2-absorbing quasi primary submodule and $abm \in N_j$, we conclude either $am \in rad_M(N_j) = rad_M(N)$ or $bm \in rad_M(N_j) = rad_M(N)$. Hence N is a 2-absorbing quasi primary submodule of M. □

Lemma 2.17. *Let $f : M \to M'$ be an R-module epimorphism. If N is a submodule of M with $Ker(f) \subseteq N$, then $f(rad_M(N)) = rad_{M'}(f(N))$.*

Proof. See [14, Corollary 1.3]. □

Theorem 2.18. *Let $f : M \to M'$ be a homomorphism of R-modules. Then the following statements hold:*

(i) If N' is a 2-absorbing quasi primary submodule of M' with $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a 2-absorbing quasi primary submodule of M.

(ii) If f is epimorphism and N is a 2-absorbing quasi primary submodule of M with $Ker(f) \subseteq N$, then $f(N)$ is a 2-absorbing quasi primary submodule of M'.

Proof. (i) Suppose that N' is a 2-absorbing quasi primary submodule of M' with $f^{-1}(N') \neq M$. Let $abm \in f^{-1}(N')$ for $a, b \in R$ and $m \in M$. Then we have $f(abm) = abf(m) \in N'$. Since N' is a 2-absorbing quasi primary submodule of M', we conclude either $ab \in \sqrt{(N' :_R M')} \subseteq \sqrt{(f^{-1}(N') :_R M)}$ or $af(m) = f(am) \in rad_{M'}(N')$ or $bf(m) = f(bm) \in rad_{M'}(N')$. Since $f^{-1}(rad_{M'}(N')) \subseteq rad_M(f^{-1}(N'))$, we get the result that $ab \in \sqrt{(f^{-1}(N') :_R M)}$ or $am \in rad_M(f^{-1}(N'))$ or $bm \in rad_M(f^{-1}(N'))$. Hence $f^{-1}(N')$ is a 2-absorbing quasi primary submodule of M.

(ii) Let $abm' \in f(N)$ for $a, b \in R$ and $m' \in M'$. Since f is epimorphism, there exists $m \in M$ such that $f(m) = m'$ and so $abm' = abf(m) = f(abm) \in f(N)$. As $Ker(f) \subseteq N$, we have $abm \in N$. Then we get the result that $ab \in \sqrt{(N :_R M)} \subseteq \sqrt{(f(N) :_R M')}$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$, because N is a 2-absorbing quasi primary submodule of M. By Lemma 2.17, we get $ab \in \sqrt{(f(N) :_R M')}$ or $am' \in f(rad_M(N)) = rad_{M'}(f(N))$ or $bm' \in rad_{M'}(f(N))$ as required. □

As an immediate consequences of previous theorem, we have the following result.

Corollary 2.19. *Let M be an R-module and L a submodule of M. Then the followings hold:*

(i) If N is a 2-absorbing quasi primary submodule of M with $L \nsubseteq N$, then $L \cap N$ is a 2-absorbing quasi primary submodule of L.

(ii) If N is a 2-absorbing quasi primary submodule of M with $L \subseteq N$, then N/L is a 2-absorbing quasi primary submodule of M/L.

Theorem 2.20. *Suppose that L, N are submodules of M with $L \subseteq N$. If L is a 2-absorbing quasi primary submodule of M and N/L is a weakly 2-absorbing quasi primary submodule of M/L, then N is a 2-absorbing quasi primary submodule of M.*

Proof. Let $abm \in N$ for $a, b \in R$ and $m \in M$. If $abm \in L$, then $ab \in \sqrt{(L :_R M)} \subseteq \sqrt{(N :_R M)}$ or $am \in rad_M(L) \subseteq rad_M(N)$ or $bm \in rad_M(L) \subseteq rad_M(N)$. Now assume that $abm \notin L$. Then we have $0 \neq ab(m + L) \in N/L$. Since N/L is a weakly 2-absorbing quasi primary submodule of M/L, we conclude that $ab \in \sqrt{(N/L : M/L)}$ or $a(m + L) \in rad_{M/L}(N/L) = \frac{rad_M(N)}{L}$ or $b(m + L) \in rad_{M/L}(N/L) = \frac{rad_M(N)}{L}$. Thus we get the result that $ab \in \sqrt{(N :_R M)}$ or $am \in rad_M(N)$ or $bm \in rad_M(N)$, this completes the proof. □

Recall from [11] a proper ideal Q of R is a quasi primary ideal if whenever $\sqrt{Q}$ is a prime ideal of R. Also a proper submodule N of M is called a quasi primary submodule preciesly when $(N :_R M)$ is a quasi primary ideal of R [1].

Lemma 2.21. *Let M be a multiplication R-module. Suppose that N_1, N_2 are quasi primary submodules of M. Then $N_1 \cap N_2$ are 2-absorbing quasi primary submodule of M.*

Proof. Suppose that N_1, N_2 are quasi primary submodules of M. Then we have $(N_1 : M)$ and $(N_2 : M)$ are quasi primary ideal of R. Thus we get $(N_1 : M) \cap (N_2 : M) = (N_1 \cap N_2 : M)$ are 2-absorbing quasi primary ideal by [17, Theorem 2.17]. Therefore, by Theorem 2.9, $N_1 \cap N_2$ is a 2-absorbing quasi primary submodule of M. □

Let M_1 be an R_1-module and M_2 be an R_2-module. Then the set $M = M_1 \times M_2$ becomes an $R = R_1 \times R_2$-module with component-wise addition and multiplication. Also, all submodules of M has the form $N_1 \times N_2$, where N_1 is a submodule of M_1 and N_2 is a submodule of M_2. Further, If M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module, then M is a multiplication R-module. In addition, $rad_M(N_1 \times N_2) = rad_{M_1}(N_1) \times rad_{M_2}(N_2)$ holds for every submodule N_1 of M_1 and N_2 of M_2 .

Theorem 2.22. *Suppose that M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then the followings hold:*

(i) $N = K_1 \times M_2$ is a 2-absorbing quasi primary submodule of $M = M_1 \times M_2$ if and only if K_1 is a 2-absorbing quasi primary submodule of M_1.

(ii) $N = M_1 \times K_2$ is a 2-absorbing quasi primary submodule of $M = M_1 \times M_2$ if and only if K_2 is a 2-absorbing quasi primary submodule of M_2.

(iii) If K_1 is a quasi primary submodule of M_1 and K_2 is a quasi primary submodule of M_2, then $N = K_1 \times K_2$ is a 2-absorbing quasi primary submodule of M.

Proof. (i) Suppose that K_1 is a 2-absorbing quasi primary submodule of M_1. Let $(a_1, a_2)(b_1, b_2)(m_1, m_2) = (a_1b_1m_1, a_2b_2m_2) \in K_1 \times M_2$, where $a_i, b_i \in R_i$ and $m_i \in M_i$ for $i = 1, 2$. Then we have $a_1b_1m_1 \in K_1$ and so $a_1b_1 \in \sqrt{(K_1 :_{R_1} M_1)}$ or $a_1m_1 \in rad_{M_1}(K_1)$ or $b_1m_1 \in rad_{M_1}(K_1)$. Thus we get the result that $(a_1, a_2)(b_1, b_2) \in \sqrt{(N :_R M)}$ or $(a_1, a_2)(m_1, m_2) \in rad_M(N)$ or $(b_1, b_2)(m_1, m_2) \in rad_M(N)$. For the converse, assume that $K_1 \times M_2$ is a 2-absorbing quasi primary submodule of M. Let $abm \in K_1$ for $a, b \in R_1$ and $m \in M_1$. Then we have $(a, 0)(b, 0)(m, 0_M) \in K_1 \times M_2$ and so $(a, 0)(b, 0) = (ab, 0) \in \sqrt{(K_1 \times M_2 :_R M_1 \times M_2)} = \sqrt{(K_1 :_{R_1} M_1)} \times R_2$ or $(b, 0)(m, 0_M) = (bm, 0_M) \in rad_{M_1}(K_1) \times M_2$ or $(a, 0)(m, 0_M) = (am, 0_M) \in rad_{M_1}(K_1) \times M_2$. Thus we get the result that $ab \in \sqrt{(K_1 :_{R_1} M_1)}$or $am \in rad_{M_1}(K_1)$ or $bm \in rad_{M_1}(K_1)$, as needed.

(ii) The proof is similar to (i)

(iii) Suppose that K_1, K_2 are quasi primary submodules of M_1 and M_2, respectively. Then $N_1 = K_1 \times M_2$ and $N_2 = M_1 \times K_2$ are quasi primary submodules of M and so $N = N_1 \cap N_2 = K_1 \times K_2$ is a 2-absorbing quasi primary submodule of M by Lemma 2.21. □

Theorem 2.23. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be a finitely generated multiplication R-module, where M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module. If $N = N_1 \times N_2$ is a proper submodule of M, then the followings are equivalent:*

(i) N is a 2-absorbing quasi primary submodule of M.

(ii) $N_1 = M_1$ and N_2 is a 2-absorbing quasi primary submodule of M_2 or $N_2 = M_2$ and N_1 is a 2-absorbing quasi primary submodule of M_1 or N_1, N_2 are quasi primary submodules of M_1 and M_2, respectively.

Proof. $(i) \Rightarrow (ii)$: Suppose that $N = N_1 \times N_2$ is a 2-absorbing quasi primary submodule of M. Then $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ is a 2-absorbing quasi primary ideal of R. By [17, Theorem 2.23], we have $(N_1 :_{R_1} M_1) = R_1$ and $(N_2 :_{R_2} M_2)$ is a 2-absorbing quasi primary ideal of R_2 or $(N_2 :_{R_2} M_2) = R_2$ and $(N_1 :_{R_1} M_1)$ is a 2-absorbing quasi primary ideal of R_1 or $(N_1 :_{R_1} M_1), (N_2 :_{R_2} M_2)$ are quasi primary ideals of R_1 and R_2, respectively. Assume that $(N_1 :_{R_1} M_1) = R_1$ and $(N_2 :_{R_2} M_2)$ is a 2-absorbing quasi primary ideal of R_2. Then $N_1 = M_1$ and N_2 is a 2-absorbing quasi primary submodule of M_2 by Theorem 2.9. If $(N_2 :_{R_2} M_2) = R_2$ and $(N_1 :_{R_1} M_1)$ is a 2-absorbing quasi primary ideal of R_1, similarly we have $N_2 = M_2$ and N_1 is a 2-absorbing quasi primary submodule of M_1. Now, assume that $(N_1 :_{R_1} M_1), (N_2 :_{R_2} M_2)$ are quasi primary ideals of R_1 and R_2, respectively. By the definition of quasi primary submodule, N_1 and N_2 are quasi primary submodules of N_1 and N_2, respectively.

$(ii) \Rightarrow (i)$: It follows from previous theorem. □

Acknowledgement.

We would like to thank the refree his/her great effort in profreading the manuscript.

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