Filomat 31:11 (2017), 3203–3207
https://doi.org/10.2298/FIL1711203P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fixed Point Theorem of Ćirić Type in Weak Partial Metric Spaces

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Abstract. In this paper we prove a general fixed point theorem of Ćirić type in weak partial metric space.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

In 1994, Matthews [10] introduced the concept of partial metric space as a part of the study of denotional semantics of dataflows networks and proved the Banach principle in such spaces. The partial metric spaces play an important role in constructing models in theory of computation.

Many authors studied the fixed points for mappings satisfying contractive conditions in complete partial metric spaces. Quite recently, in [1], [3], [9], are proved some fixed point theorems under various conditions in partial metric spaces.

In 1994, Heckmann [8] introduced the concept of weak partial metric space, which is a generalization of partial metric space. Some results are recently obtained in [2], [4], [5].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [11], [12] and in other papers. Recently, this method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, compact metric spaces, Hilbert spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying contractive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, G - metric spaces, and G_p - metric spaces. With this method, the proofs of some fixed point theorems are more simple. The study of fixed points using implicit relations in partial metric spaces is initiated in [6], [7], [13] - [15].

The purpose of this paper is to prove a general fixed point theorem for a mapping satisfying an implicit relation in weak partial metric spaces, different of the results from [2] and [5].

2010 *Mathematics Subject Classification*. Primary 54H25; Secondary 47H10
Keywords. weak partial metric space, fixed point, implicit relation, Ćirić type
Received: 24 November 2016; Accepted: 19 April 2017
Communicated by Vladimir Rakočević
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2. Preliminaries

Definition 2.1 ([10]). *A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$*

$(P_1) : x = y$ *if and only if* $p(x, x) = p(y, y) = p(x, y)$,

$(P_2) : p(x, x) \leq p(x, y)$,

$(P_3) : p(x, y) = p(y, x)$,

$(P_4) : p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space.

If $p(x, y) = 0$, then $x = y$, but the converse is not true.

Each partial metric space on a set X generates a T_0 - topology τ_p on X which has as base the family of open p - balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$.

A sequence $\{x_n\}$ in the partial metric space (X, p) converges with respect to τ_p to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.

If p is a partial metric on X, then

$$d_w(x, y) = p(x, y) - \min\{p(x, x), p(y, y)\}$$

is an ordinary metric on X.

Remark 2.2. *Let $\{x_n\}$ be a sequence in partial metric space (X, p) and $x \in X$. Then $\lim_{n\to\infty} d_w(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{1}$$

Definition 2.3 ([10]). *a) A sequence $\{x_n\}$ in (X, p) is a Cauchy sequence in (X, p) if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.*

b) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Definition 2.4 ([8]). *A weak partial metric on a nonempty set X is a function $p : X \times X$ such that for all $x, y, z \in X$*

$(wP_1) : x = y$ *if and only if* $p(x, x) = p(y, y) = p(x, y)$,

$(wP_2) : p(x, x) = p(y, x)$,

$(wP_3) : p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a weak partial metric space.

If $p(x, y) = 0$, then $x = y$.

Obviously, every partial metric space is a weak partial metric space, but the converse is not true. For example, if $X = [0, \infty)$ and $p(x, y) = \dfrac{x + y}{2}$, then (X, p) is a weak partial metric space and (X, p) is not a partial metric space.

Theorem 2.5 ([2]). *Let (X, p) be a weak partial metric space. Then $d_w(x, y) : X \times X \to [0, \infty)$ is a metric on X.*

Remark 2.6. *In a weak partial metric space, the convergent Cauchy sequence and the completness are defined as in partial metric space.*

Theorem 2.7 ([2]). *Let (X, p) be a weak partial metric space.*

a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if is a Cauchy sequence in (X, d_w).

b) (X, p) is complete if and only if (X, d_w) is complete.

Lemma 2.8. *Let (X, p) be a weak partial metric space and $\{x_n\}$ is a sequence in (X, p). If $\lim_{n\to\infty} x_n = x$ and $p(x, x) = 0$, then $\lim_{n\to\infty} p(x_n, y) = p(x, y)$, $\forall y \in X$.*

Proof. By (wP_3) we have

$$p(x,y) \leq p(x,x_n) + p(x_n,y),$$

hence

$$p(x,y) - p(x,x_n) \leq p(x_n,y) \leq p(x_n,x) + p(x,y).$$

Letting n tends to infinity we obtain

$$\lim_{n\to\infty} p(x_n,y) = p(x,y).$$

□

Remark 2.9. *Remark 2.2 is still true for weak partial metric spaces.*

3. Implicit Relations

Definition 3.1. *Let $\mathcal{F}_{5w}$ be the family of upper semi - continuous functions $F(t_1,...,t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ such that:*
(F_1) : *F is nonincreasing in variable t_5,*
(F_2) : *For all $u,v \geq 0$,there exists $h \in [0,1)$ such that $F(u,v,v,u,u+v) \leq 0$ implies $u \leq hv$,*
(F_3) : *$F(t,t,0,0,2t) > 0, \forall t > 0$.*

Example 3.2. $F(t_1,...,t_5) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}$, *where $k \in [0,1)$ and $0 \leq h = k < 1$.*

Example 3.3. $F(t_1,...,t_5) = t_1 - k\max\{t_2,t_3,t_4,t_5\}$, *where $k \in \left[0,\frac{1}{2}\right)$ and $0 \leq h = k < \frac{1}{2}$.*

Example 3.4. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, *where $a,b,c,d \geq 0$ and $0 < a+b+c+2d < 1$, with $0 \leq h = a+b+c+2d < 1$.*

Example 3.5. $F(t_1,...,t_5) = t_1 - a\max\{t_2,t_3,t_4\} - bt_5$, *where $a,b \geq 0$, $a+2b < 1$, with $0 \leq h = a+2b < 1$.*

Example 3.6. $F(t_1,...,t_5) = t_1^2 + \frac{t_1}{1+t_5} - \left(at_2^2 + bt_3^2 + ct_4^2\right)$, *where $a,b,c \geq 0$ and $a+b+c < 1$, with $0 \leq h = \sqrt{a+b+c} < 1$.*

Example 3.7. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - c\max\{2t_4,t_5\}$, *where $a,b,c \geq 0$ and $0 < a+b+2c < 1$, with $0 \leq h = a+b+2c < 1$.*

Example 3.8. $F(t_1,...,t_5) = t_1^2 + t_1(at_2 + bt_3 + ct_4) - dt_5$, *where $a,b,c,d \geq 0$ and $a+b+c+4d < 1$, with $0 \leq h = \sqrt{a+b+c+4d} < 1$.*

4. Main Result

Theorem 4.1. *Let (X,p) be a complete partial metric space and $f : (X,p) \to (X,p)$ such that for all $x,y \in X$*

$$F(p(fx,fy), p(x,y), p(x,fx), p(y,fy), p(x,fy) + p(y,fx)) \leq 0 \tag{2}$$

for some $F \in \mathcal{F}_{5w}$. Then f has a unique fixed point z with $p(z,z) = 0$.

Proof. Let x_0 be an arbitrary point in X. Define $\{x_n\}$ in X by $x_n = fx_{n-1}$, $n = 1, 2, \ldots$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \ldots$, then x_{n_0} is a fixed point of f. Suppose that for all $n \in \mathbb{N}$, $x_n \neq x_{n+1}$. Then by (2) we have

$$F(p(fx_{n-1}, fx_n), p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_n, fx_n), p(x_{n-1}, fx_n) + p(x_n, fx_{n-1})) \leq 0,$$

$$F(p(x_n, x_{n+1}), p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_{n+1}) + p(x_n, x_n)) \leq 0. \tag{3}$$

By (wP_3),

$$p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n).$$

By (3) and (F_1) we obtain

$$F(p(x_n, x_{n+1}), p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_n) + p(x_n, x_{n+1})) \leq 0.$$

By (F_2), there exists $h \in [0, 1)$ such that

$$p(x_n, x_{n+1}) \leq hp(x_n, x_{n-1})$$

which implies

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n) \leq \ldots \leq h^n p(x_0, x_1) \to 0 \text{ as } n \to \infty. \tag{4}$$

For $n, m \in \mathbb{N}$ with $n > m$, by (wP_3) we obtain

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m+1}) + p(x_{m+1}, x_{m+2}) + \ldots + p(x_{n-1}, x_n) \\ &\leq (h^m + h^{m+1} + \ldots + h^n) p(x_0, x_1) \\ &\leq \frac{h^n}{1-h} p(x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

By the definition of $d_w(x, y)$ we obtain

$$d_w(x_m, x_n) \leq p(x_m, x_n) \to 0 \text{ as } m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, d_w).

By Theorem 2.7, $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete, $\{x_n\}$ converges in (X, p) to a point $z \in X$ and $z = \lim_{n\to\infty} fx_n$. By Theorem 2.7 we obtain

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

By (2) we have

$$F(p(fx_n, fz), p(x_n, z), p(x_n, fx_n), p(z, fz), p(x_n, fz) + p(z, fx_n)) \leq 0,$$

$$F(p(x_{n+1}, fz), p(x_n, z), p(x_n, x_{n+1}), p(z, fz), p(x_n, fz) + p(z, x_{n+1})) \leq 0. \tag{5}$$

By (5), for $n \to \infty$ and Lemma 2.8, we obtain

$$F(p(z, fz), 0, 0, p(z, fz), p(z, fz) + 0) \leq 0.$$

By (F_2) we obtain $p(z, fz) = 0$. Hence $z = fz$ and z is a fixed point of f with $p(z, z) = 0$.

Suppose that z_1 is another fixed point of f with $p(z_1, z_1) = 0$. By (2) we obtain

$$F(p(fz, fz_1), p(z, z_1), p(z, fz), p(z_1, fz_1), p(z, fz_1) + p(z_1, fz)) \leq 0,$$

$$F(p(z, z_1), p(z, z_1), 0, 0, 2p(z, z_1)) \leq 0,$$

a contradiction of (F_3) if $p(z, z_1) > 0$. Hence, $p(z, z_1) = 0$ which implies $z = z_1$. □

By Theorem 4.1 and Example 3.2 we obtain a theorem of Ćirić type in complete weak partial metric spaces.

Theorem 4.2. *Let (X, p) be a complete weak partial metric space such that for all $x, y \in X$*

$$p(Tx, Ty) \leq k \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\},$$

where $k \in [0, 1)$. Then T has a unique fixed point z with $p(z, z) = 0$.

Remark 4.3. *By Theorem 4.1 and Examples 3.3 - 3.8 we obtain new results.*

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