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A Fixed Point Method to Solve Linear Operator Equations Involving Self-Adjoint Operators in Hilbert Space

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Abstract. In this paper we provide existence and uniqueness results for linear operator equations of the form $(I + A^m)\,x = f$, where A is a self-adjoint operator in Hilbert space. Some applications to the study of invertible matrices are also presented.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

Many problems in science and engineering have their mathematical formulation as an operator equation $Tx = f$, where T is a linear or nonlinear operator between certain function spaces. Therefore, we consider as interesting to present here, in a simple form, some existence and uniqueness results regarding linear operator equations involving self-adjoint operators from a Hilbert space into itself.

The methods of functional analysis including those from fixed point theory had a major contribution in the study of many linear or nonlinear equations. The papers [1-3] represent just a few of the many basic works regarding the study of operator equations using the methods of functional analysis.

Let H be a Hilbert space(real or complex) endowed with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm denoted by $\|\cdot\|$.

Let $\mathcal{B}(H)$ be the Banach algebra of all linear and bounded operators $T : H \to H$, endowed with the norm

$$\|T\|_{\mathcal{B}(H)} = \sup_{\|x\| \le 1} \|Tx\| \,.$$

We recall that if $T \in \mathcal{B}(H)$, then $\|Tx\| \le \|T\|_{\mathcal{B}(H)} \cdot \|x\|$ for all $x \in H$.

An operator $T : H \to H$ is said to be

i) self-adjoint if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$;

ii) positive if $\langle Tz, z \rangle \ge 0$ *for all* $z \in H$.

Clearly, if T is self-adjoint, then $I + T^p$ is self-adjoint for all natural number $p \ge 1$.

In this paper we give existence and uniqueness results for the equation $(I + A^m)\,x = f$, where $A : H \to H$ is a self-adjoint operator, $m \ge 1$ is a natural number and $f \in H$.

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2. Main Result

The proof of the following theorem uses an application of the Banach contraction principle in Hilbert spaces.

Theorem 2.1. *Let H be a real or complex Hilbert space and $A \in \mathcal{B}(H)$ be a self-adjoint operator.*

i) The equation $x + A^{2k}x = f$ has a unique solution in H for all $f \in H (k \geq 1)$.

ii) If in addition A is positive, then the equation $x + A^{2k+1}x = f$ has a unique solution in H for all $f \in H (k \geq 0)$.

Proof. i) The equation $x + A^{2k}x = f$ can be equivalently written as $(I + A^{2k})x = f$, where I is the identity of H.

We have

$$\left\|(I + A^{2k})x\right\| \leq \|x\| + \left\|A^{2k}x\right\| \leq \left(1 + \|A\|^{2k}_{\mathcal{B}(H)}\right)\|x\|$$

and

$$\left\langle (I + A^{2k})x, x\right\rangle = \langle x, x\rangle + \left\langle A^{2k}x, x\right\rangle = \|x\|^2 + \left\langle A^k x, A^k x\right\rangle = \|x\|^2 + \left\|A^k x\right\|^2 \geq \|x\|^2$$

for all $x \in H$.

We note:

$$\alpha = \|A\|_{\mathcal{B}(H)},$$

therefore

$$\left\|(I + A^{2k})x\right\| \leq \left(1 + \alpha^{2k}\right)\|x\| \text{ for all } x \in H.$$

For $\gamma > 0$ we consider now the operator $S_\gamma : H \to H$ defined by

$$S_\gamma x = x - \gamma\left((I + A^{2k})x - f\right).$$

We obtain

$$\left\|S_\gamma x - S_\gamma y\right\|^2 = \left\langle S_\gamma x - S_\gamma y, S_\gamma x - S_\gamma y\right\rangle$$

$$= \left\langle (x - y) - \gamma(I + A^{2k})(x - y), (x - y) - \gamma(I + A^{2k})(x - y)\right\rangle$$

$$= \left\|x - y\right\|^2 - 2\gamma\left\langle (I + A^{2k})(x - y), x - y\right\rangle + \gamma^2\left\|(I + A^{2k})(x - y)\right\|^2$$

$$\leq \left(1 - 2\gamma + \gamma^2\left(1 + \alpha^{2k}\right)^2\right)\left\|x - y\right\|^2,$$

so

$$\left\|S_\gamma x - S_\gamma y\right\| \leq \sqrt{1 - 2\gamma + \gamma^2\left(1 + \alpha^{2k}\right)^2}\left\|x - y\right\|$$

for all $x, y \in H$.

We remark that if $\gamma \in \left(0, \frac{2}{\left(1+\alpha^{2k}\right)^2}\right)$, then S_γ is a contraction and, due to the Banach fixed point theorem, it results that S_γ has a unique fixed point $u \in H$.

Now u is the unique solution of the equation $x + A^{2k}x = f$.

To prove ii) we use that

$$\left\|(I + A^{2k+1})x\right\| \leq \|x\| + \left\|A^{2k+1}x\right\| \leq \left(1 + \|A\|^{2k+1}_{\mathcal{B}(H)}\right)\|x\|$$

and

$$\left\langle (I + A^{2k+1})x, x \right\rangle = \langle x, x\rangle + \left\langle A^{2k+1}x, x \right\rangle = \|x\|^2 + \left\langle A(A^k x), A^k x \right\rangle \geq \|x\|^2 ,$$

due to the positivity of A.

Now the operator S_γ is defined as

$$S_\gamma x = x - \gamma \left((I + A^{2k+1})x - f \right),$$

the contraction constant is $\sqrt{1 - 2\gamma + \gamma^2 \left(1 + \alpha^{2k+1}\right)^2}$ with $\gamma \in \left(0, \frac{2}{\left(1+\alpha^{2k+1}\right)^2}\right)$, and we use the same type of reasoning as in the proof of i). So the proof of Theorem 2.1 is complete.□

In both cases the solution can be approximated using the Picard iteration associated to the contraction S_γ.

Theorem 2.1 has interesting consequences if we replace the self-adjoint operator A with a sum or a product of two self-adjoint operators:

Theorem 2.2. *Let H be a real or complex Hilbert space and $A, B \in \mathcal{B}(H)$ be two self-adjoint operators.*

i) The equation $x + (A + B)^{2k} x = f$ has a unique solution in H for all $f \in H (k \geq 1)$.

ii) If in addition $A + B$ is positive, then the equation $x + (A + B)^{2k+1} x = f$ has a unique solution in H for all $f \in H (k \geq 0)$.

Proof. Clearly $A + B$ is self-adjoint, and it is sufficient to apply Theorem2.1.□

Theorem 2.3. *Let H be a real or complex Hilbert space and $A, B \in \mathcal{B}(H)$ be two self-adjoint operators with $AB = BA$.*

i) The equation $x + (AB)^{2k} x = f$ has a unique solution in H for all $f \in H (k \geq 1)$.

ii) If in addition AB is positive, then the equation $x + (AB)^{2k+1} x = f$ has a unique solution in H for all $f \in H (k \geq 0)$.

Proof. According to Theorem 2.1 it is sufficient to prove that AB is a self-adjoint operator. Indeed we have

$$\langle ABx, y\rangle = \langle Bx, Ay\rangle = \langle x, BAy\rangle = \langle x, ABy\rangle$$

for all $x, y \in H$.□

3. Remark

Theorem 2.1 can be applied to obtain results regarding the invertibility of some matrices. Hence we obtain

Theorem 3.1. *Let A be a $n \times n$ complex matrix, A^* the adjoint matrix of A, $A = A^*$ and I_n the unit complex matrix.*

i) The matrix $I_n + A^{2k}$ is invertible for all $k \geq 0$.

ii) If the matrix A is positive(as linear operator, according to the above given definition) , then the matrix $I_n + A^{2k+1}$ is invertible for all $k \geq 0$,

and the appropriate consequences of Theorem 2.2 and Theorem 2.3, which we do not more mention.

Moreover, the method used in proving Theorem 2.1 shows that we can use the Picard iteration to approximate the solution of some linear algebraic systems proper to matrices of the form $I + A^m$, A satisfying the above given conditions.

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