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Schwarz Lemma and Kobayashi Metrics for Holomorphic Functions

Miodrag Mateljević[a]

[a] *University of Belgrade, Faculty of Mathematics*

Abstract. In this paper, we mainly consider Schwarz lemma for holomorphic functions and contraction properties of holomorphic functions with respect to Kobayashi distances. We also review some results related to the subject using some novelty and announce a few new results.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The Schwarz lemma is one of the most influential results in complex analysis and it has a great impact to the development of several research fields, such as geometric function theory, hyperbolic geometry, complex dynamical systems, and theory of quasi-conformal mappings. There is a numerous literature related to the subject Schwarz lemma (see for example L. Ahlfors [1],Yau [25] ,Royden [23],H. Boas [3], R. Osserman [22] and the literature cited there) and we apology if we did not mention some important papers. For the Carathéodory and Kobayashi metrics see for example S. Kobayashi[10, 11] and S.G. Krantz [12, 14]. In this paper we mainly consider Schwarz lemma for holomorphic functions and contraction properties of holomorphic functions with respect to Kobayashi distances including complex Banach spaces and hyperbolic Riemann surfaces.

2. Schwarz Lemma in the Unit Ball

If f is a function on a set X and $x \in X$ sometimes we write fx instead of $f(x)$. We write $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$.

On $\mathbb{C}^n$ we define the standard Hermitian inner product by

$< z, w >= \sum_{k=1}^{n} z_k \overline{w_k}$ for $z, w \in \mathbb{C}^n$ and by $|z| = \sqrt{< z, z >}$ we denote the norm of vector z. We also use notation (z, w) instead of $< z, w >$ on some places. By $\mathbb{B}_n$ we denote the unit ball in $\mathbb{C}^n$. In particular we use also notation $\mathbb{U}$ and $\mathbb{D}$ for the unit disk in complex plane.

For planar domains G and D we denote by $\mathrm{Hol}(G, D)$ the class of all holomorphic mapping from G into D. For complex Banach manifold X and Y we denote by $\mathcal{O}(X, Y)$ the class of all holomorphic mapping from X into Y.

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Email address: miodrag@matf.bg.ac.rs (Miodrag Mateljević)

We need some properties of bi -holomorphic automorphisms of unit ball (see [24] for more details). For a fixed z, $B_z = \{w : (w-z, z) = 0, |w| < 1\}$ and denote by $R(z)$ radius of ball B_z. Denote by $P_a(z)$ the orthogonal projection onto the subspace $[a]$ generated by a and let $Q_a = I - P_a$ be the projection on the orthogonal complement. For $z, a \in \mathbb{B}^n$ we define

$$\tilde{z} = \varphi_a(z) = \frac{a - Pz - s_a Qz}{1 - (z, a)}, \tag{1}$$

where $P_a(z) = \dfrac{< z, a >}{< a, a >} a$ and $s_a = (1 - |a|^2)^{1/2}$. Set $U^a = [a] \cap \mathbb{B}$, $Q^b = b + [a]^\perp \cap \mathbb{B}_n$,

$$\varphi_a^1(z) = \frac{a - Pz}{1 - (z, a)}, \quad \varphi_a^2(z) = \frac{-s_a Qz}{1 - (z, a)}$$

and $\delta(a, z) = |\varphi_a(z)|$.

Then one can check that

(I1) The restriction of φ_a onto U^a is automorphisam of U^a and the restriction onto B_z maps it bi-holomorphically mapping onto $B_{\tilde{z}}$.

A domain U is called complete circular if whenever $z \in U$ and $|\lambda| \leq 1$ then $\lambda z \in U$. Note in passing that a complete circular domain automatically contains 0.

We need a few results from Rudin [24].

For a we define $s = s_a = \sqrt{1 - |a|^2}$.

Theorem 2.1 (2.2.2 [24]). *For every $a \in \mathbb{B}$, φ_a has the following properties:*
(i) $\varphi_a(0) = a$ *and* $\varphi_a(a) = 0$
(ii) $\varphi_a'(0) = -s^2 P - sQ$, $\varphi_a'(a) = -P/s^2 - Q/s$
(iii) *the identity*

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - (z, a)|^2},$$

(iv) φ_a *is an involution:* $\varphi_a(\varphi_a(z)) = z$
(v) φ_a *is a homeomorphism of* $\overline{\mathbb{B}}$ *onto* $\overline{\mathbb{B}}$, *and* $\varphi_a \in \mathrm{Aut}(\mathbb{B})$.
(vi) $\mathrm{Aut}(\mathbb{B})$ *acts transitively on* $\mathbb{B}$.

We only outline a proof. Since $(1 - (z, a))^{-1} = 1 + < z, a > + O(|z|^2)$ and $|a|^2 Pz = a < z, a >$, $\varphi_a(z) = a - (P + sQ)z + a < z, a > + O(|z|^2)$. Hence

$$\varphi_a(z) - \varphi_a(0) = -s^2 Pz - sQz + O(|z|^2)$$

and therefore the first formula in (ii) follows; the second one follows from

$$\varphi_a(a + h) = \frac{-Ph - sQh}{s^2 - < h, a >}.$$

From (iv), it follows that φ_a is one-to-one of $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}}$, and that $\varphi_a^{-1} = \varphi_a$. If $a, b \in \mathbb{B}$, $\varphi_b \circ \varphi_a$ is an automorphism of B that takes a to b.

If $f \in Aut(\mathbb{B})$, $a = f^{-1}(0)$, $J_{\mathbb{R}} f$ denotes real Jacobian, then

$$J_{\mathbb{R}} f(z) = \left(\frac{(1 - |a|^2)}{|1 - (z, a)|^2}\right)^{n+1}. \tag{2}$$

Proposition 2.2 (Theorem 8.1.2). *Suppose that (i) G and G' are complete circular domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively,*
(ii) G' convex and bounded
(iii) $F : G \to G'$ holomorphic

Then
(a) $F'(0)$ *maps* G *into* G'
(b) $F(rG) \subset rG'$ $(0 < r \leq 1)$ *if* $F(0) = 0$.

The following is an immediate corollary of Proposition 2.2:

Corollary 2.3. *Suppose that* $f \in O(\mathbb{B}_n, \mathbb{B}_m)$. *If* $f(0) = 0$, *then*
(A1) $|f'(0)| \leq 1$.

We give another proof which is more in spirit of this paper.

Proof. For $z^* = z/|z|$ define $D_z = \{\zeta z^* : \zeta \in \mathbb{U}\}$ and $F(\zeta) = f(\zeta z^*)$, $\zeta \in \mathbb{U}$. Let p be projection of $\mathbb{B}_m$ on the slice $D_{f(z)}$. By one dim version of Schwarz lemma $|F(\zeta)| \leq |\zeta|$ and in particular for $\zeta = |z|$, $|f(z)| \leq |z|$. Hence (A1) $|f'(0)| \leq 1$. □

Proposition 2.4 (Theorem 8.1.4 [24]). *Suppose that* $f : \mathbb{B}_n \to \mathbb{B}_m$ *holomorphic,* $a \in \mathbb{B}_n$ *and* $b = f(a)$.
Then
$|\varphi_b(f(z))| \leq |\varphi_a(z)|$, $z \in \mathbb{B}_n$
or equivalently

$$\frac{|1-(fz,fa)|^2}{(1-|fa|^2)(1-|fz|^2)} \leq \frac{|1-(z,a)|^2}{(1-|a|^2)(1-|z|^2)}. \tag{3}$$

Set

$$\sigma_n(z,a) := \frac{|1-(z,a)|^2}{(1-|a|^2)(1-|z|^2)}.$$

For $z, w \in \mathbb{C}^n$, $|1- < z, w > |^2 = 1 + | < z, w > |^2 - (|z|^2 + |w|^2) + |z - w|^2$ and therefore
(A1) $|1- < z, w > |^2 \leq (s_z s_w)^2 + |z - w|^2$ and $|1- < z, w > |^2 = (s_z s_w)^2 + |z - w|^2$, $z, w \in \mathbb{C}$, that is

$$(B1) \qquad \sigma_n(z,w) \leq 1 + \frac{|z-w|^2}{(s_z s_w)^2}, \quad \sigma_1(z,w) = 1 + \frac{|z-w|^2}{(s_z s_w)^2}, z, w \in \mathbb{C}.$$

Theorem 2.5. *Suppose that* $f \in O(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ *and* $b = f(a)$.
(i) *Then* $s_a^2|f'(a)| \leq s_b$, *i.e.* $(1-|a|^2)|f'(a)| \leq \sqrt{1-|f(a)|^2}$.
(ii) *If* $m = 1$, *then* $s_a^2|f'(a)| \leq s_b^2$, *and*
(iii) *If* $m > 1$, *the inequality (a)* $\sigma_m(fz, fw) \leq \sigma_n(z,w)$, $z, w \in \mathbb{B}_n$, *does not hold in general, but if* $f \in O(\mathbb{B}_n, \mathbb{B}_1)$ *then* $\sigma_1(fz, fw) \leq \sigma_n(z,w)$, *that is the following inequality holds:*

$$\sigma_1(fz, fa) = \frac{|fz - fa|^2}{(1-|fa|^2)(1-|fz|^2)} \leq \frac{|z-a)|^2}{(1-|a|^2)(1-|z|^2)}, \quad z \in \mathbb{B}_n. \tag{4}$$

Proof. (i) Suppose first that $f(0) = 0$ and take $z \in \mathbb{B}_n$. Hence (A1) $|f'(0)| \leq 1$.
For $u \in T_a C^m$, by Theorem 2.1(ii), $v = \varphi_a'(a)u = -Pu/s^2 - Qu/s$ and, by Pitagora's theorem,

$$|u| = \sqrt{|Pu|^2 + |Qu|^2}, |v|^2 = |Pu|^2/s^4 + |Qu|^2/s^2$$

and therefore we find

$$(B1) \qquad \frac{|u|}{s} \leq |\varphi_a'(a)u| \leq \frac{|u|}{s^2}.$$

If $f(a) = b$, set $h = \varphi_b \circ f \circ \varphi_a$. By the chain rule $h'(0) = \varphi_b'(b) \circ f'(a) \circ \varphi_a'(0)$.
Set $u \in T_a C^n$, $v = f'(a)u \in T_a C^m$, $u' = \varphi_a'(a)u$ and $v' = \varphi_b'(b)v$. By (A1), $|v'| \leq |u'|$. Since, by (B1),

$$\frac{|v|}{s_b} \leq |v'| \quad \text{and} \quad |u'| \leq \frac{|u|}{s_a^2},$$

hence $s_a^2|f'(a)| \leq s_b$, i.e. $(1-|a|^2)|f'(a)| \leq \sqrt{1-|f(a)|^2}$ and therefore (i) is proved.

(ii) If $m=1$, then $s_b^2|v'| = |v|$ and (ii) follows.
(iii) By (B1) and (3),

$$\sigma_1(fz, fa) = 1 + \frac{|fz - fa|^2}{(s_{fz}s_{fa})^2} \leq \sigma_n(z,a) = \frac{|1-(z,w)|^2}{(s_z s_a)^2} \leq 1 + \frac{|z-a|^2}{(s_z s_a)^2}$$

and therefore (4). If z tends a, (ii) also follows from (4). If (a1) holds, then (b1) $s_a^2|f'(a)| \leq s_b^2$. For function $f_0 = \varphi_b \circ \varphi_a$ we have $(1-|a|^2)|f_0'(a)| = (1-|b|^2)$ which yields a contradiction with (b1). For more details see Section 5. □

3. Contraction Properties of Holomorphic Functions with Respect to Kobayashi Distances

The author also published a paper [17] about holomorphic fixed point theorem on Riemann surfaces.

Let G be bounded connected open subset of complex Banach space, $p \in G$ and $\mathbf{v} \in T_pG$. We define $k_G(p,\mathbf{v}) = \inf\{|\mathbf{h}|\}$, where infimum is taking over all $\mathbf{h} \in T_0\mathbb{C}$ for which there exists a holomorphic function such that $\phi : \mathbb{U} \to G$ such that $\phi(0) = p$ and $d\phi(\mathbf{h}) = \mathbf{v}$.

We define the distance function on G by integrating the pseudometric k_G: for $z, z_1 \in G$

$$\mathrm{Kob}_G(z,z_1) = \inf_\gamma \int_0^1 k_G(\gamma(t), \dot\gamma(t))\, dt \tag{5}$$

where the infimum is over all piecewise paths $\gamma\colon [0,1] \to G$ with $\gamma(0) = z$ and $\gamma(1) = z_1$.

One can prove

Theorem 3.1. *Suppose that G and G_1 are bounded connected open subset of complex Banach space and $f : G \to G_1$ is holomorphic. Then*

$$\mathrm{Kob}_{G_1}(fz, fz_1) \leq \mathrm{Kob}_G(z,z_1) \tag{6}$$

for all $z, z_1 \in G$.

Let $A = \{1 < |x| < 4\}$, $A^* = \{2 < |x| < 3\}$, $l(t) = 2 + \frac{1}{3}(t-1)$ and$f(x) = -l(|x|)x$. l maps the interval $(1,4)$ onto the interval $(2,3)$ and therefore f maps A onto $A^* \subset A$, but f has no fixed point (there is no point $x \in A$ such that $f(x) = x$. Hence this example shows that there is no a metric d on G such that f is a contraction wrt d. The situation is completely different for analytic functions.

Theorem 3.2. *Suppose that G is bounded connected open subset of complex Banach space and $G_* \subset G$, $s_0 = dist(G_*, G^c)$, $d_0 = diam(G)$ and $q_0 = \frac{d_0}{d_0+s_0}$. Then*
(i) $\mathrm{Kob}_G \leq q_0 \mathrm{Kob}_{G_*}$ *on G_*.*
(ii) *In addition if $f : G \to G_*$ is holomorphic, then*

$$\mathrm{Kob}_{G_*}(fz, fz_1) \leq q_0 \mathrm{Kob}_{G_*}(z,z_1) \tag{7}$$

for $z, z_1 \in G_*$.

$$\mathrm{Kob}_G(fz, fz_1) \leq q_0 \mathrm{Kob}_G(z,z_1) \tag{8}$$

for $z, z_1 \in G$.

Proof. Suppose that $p \in G_*$, $v \in T_pG_*$ and $\phi : \mathbb{U} \to G$ is a holomorphic function such that $\phi(0) = p$ and $d\phi(h) = v$. Set $R_s = \frac{d_0+s}{d_0}$ and $q_s = \frac{d_0}{d_0+s}$. For $h \in \mathbb{U}$ define $\phi_s(h) = p + R_s(\phi(h) - p)$. Then $\phi_s(h) - \phi(h) = (R_s - 1)(\phi(h) - p)$ and therefore $|\phi_s(h) - \phi(h)| \leq s$. For $s < s_0$, ϕ_s maps $\mathbb{U}$ into G and $d\phi_s(h) = R_s v$. Hence $k_G(p,v) \leq q_s k_{G_*}(p,v)$ and if s approaches s_0 we first get (i) $k_G(p,v) \leq q_0 k_{G_*}(p,v)$ and by a standard procedure $\mathrm{Kob}_G \leq q_0 \mathrm{Kob}_{G_*}$. Now, by (6), we have (ii) $\mathrm{Kob}_{G_*}(fz, fz_1) \leq \mathrm{Kob}_G(z,z_1)$. Combining (i) and (ii) we get (7) and (8). □

If $d_0 = diam(G)$ is not finite, elementary example: $H_a = \{z : Imz > a\}$ with $f(z) = z + ia$ which maps H onto H_a, shows that the theorem does not hold.

Theorem 3.3. *Let $D \subset C^n$ domain for which Kobayshi (Carthéodory) pseudo-distance is distance and $f : D \to D$ holomorphic mapping such that $\overline{f(D)}$ is a compact subset of D. Then f is contraction with respect to Kobayshi (Carthéodory) metric on D. In particular f has fixed points in D.*

It is a corollary of Theorem 3.2. A version of Theorems 3.1-3.2 was proved in 1968 by Clifford Earle and Richard Hamilton [5] (see Sections 5 for further comments).

3.1. Hyperbolic Riemann surfaces

We can relax the condition $s_0 > 0$ in Theorem 3.2 if we we consider Riemann surface, see Theorem 3.9.

Every Riemann surface admits as its universal covering surface the unit disc, the finite plane, or extended plane.

Theorem 3.4. *Let S be an arbitrary Riemann surface, let D be its universal covering surface, and G the covering group of D over S. Then S is conformally equivalent to the Riemann surface D/G.*

The limit set od a Fuschian group actin on a disk D is either the whole boundary ∂D or a nowhere dense subset of ∂D. If G is a Fuschian group of the first kind acting on H, then the fixed points of G are dense everywhere on the real axis.

Let S be an arbitrary Riemann surface, whose universal covering is $\mathbb{H}$ and $p : \mathbb{H} \to S$ projection. By c_S, k_S we denote Carthéodory, Kobayashy distance respectively.

Several years ago, the author communicated at Belgrade seminar, the following results (probably known to the experts in the subject, see also [17]):

Theorem 3.5. *Let S be an arbitrary Riemann surface, whose universal covering is $\mathbb{H}$. Then $c_S \le k_S$. If equality holds for some pair of points $p, q \in S$, then S is conformaly equivalent with $\mathbb{H}$.*

Theorem 3.6. *Let S be an arbitrary Riemann surface, whose universal covering is $\mathbb{H}$ and $f : S \to S$ holomorphic mapping. If f is not an isometry (or not onto), then f is contraction on any compact $K \subset S$ with respect to Carthéodory-Poincaré metric on S.*

Proof is based on

Lemma 3.7. *Let M be hyperbolic Riemann surface and $f : M \to M$ analytic function. If there $p, q \in M$, $p \neq q$, such that $d(f(p), f(q)) = d(p, q)$, then $\tilde{f}$ is Mobius. In particular, f is onto.*

Lemma 3.8. *Let $f : \mathbb{H} \to \mathbb{H}$ holomorphic mapping and $\omega = \xi + i\eta \in \mathbb{H}$ and $v = Imf$. Then*

$$\lim_{(z,w)\to(\omega,\omega)} \frac{d(f(z), f(w))}{d(z, w)} = |f'(\omega)| \frac{\eta}{v(f(\omega))} . \tag{9}$$

$f(z) = z + i$ has no fixed point.

If X is a metric space and if f has a fixed point p then there exist a compact set F such that $f(F) \subset F$; take for example $F = \{p\}$.

Let M and N be hyperbolic Riemann surfaces and $f : M \to N$ an analytic function. If p is fixed point, then $|f'(p)| \le 1$.

Theorem 3.9. (i) *Let M be hyperbolic Riemann surface and $f : M \to M$ analytic function and F compact subset of M.*

If f is not isometry, then f is contraction on F.

In addition, if $f(F) \subset F$, then there is a unique fixed point $p_0 = f(p_0) \in F$.

Proof. Define $h : M^2 \setminus \{(p,p) : p \in M\} \to \mathbb{R}^+$ by

$$h(p,q) = \frac{d(f(p), f(q))}{d(p,q)} . \tag{10}$$

Since M is hyperbolic, there is a covering $\pi : \mathbb{U} \to M$. If $(p,q) = (r,r)$, $r \in M$ there is $z \in \mathbb{H}$ such that $\pi(z) = r$. By $h(r,r) = |\tilde{f}'(z)|\frac{y}{v(z)}$, where $z = x + iy$, $\tilde{f}$ is a liffiting of f and $v = Im\tilde{f}$, we extend h to M^2.

Then h is continuous on M^2 and therefore there is (p_0, q_0) such that $k_0 = h(p_0, q_0) = \max\{h(p,q) : (p,q) \in M^2\}$. By Lemma 3.7, $k_0 < 1$. □

Under hypothesis (i), we can associate to f a continuous mapping $h = \underline{f} : M^2 \to (0,1]$. If M is compact, then f is a contraction or $f \in \mathrm{Aut}(M^2)$. In particular f has a fixed point.

It was Minda who noticed that all self-mappings of a hyperbolic Riemann surface with two fixed points must be automorphisms.

Theorem 3.10. (i) *Let M be hyperbolic Riemann surface and $f : M \to M$ analytic function and F compact subset of M.*
If f is not isometry, then f is contraction on F.
In addition, if $f(F) \subset F$, then there is a unique fixed point $p_0 = f(p_0) \in F$.

Let M be a hyperbolic Riemann surface and $f : M \to M$ an analytic function. If p is fixed point of f, then $|f'(p)| \leq 1$.

4. Further Research

In this section we discus a few results which may give orientation for further research. There are many results related to subject of this paper. We will mention only a few of them which are selected on the basis of personal taste and which can be the basis for further research in our opinion. In particular it seems that continuation of research related to the Gromov-Schwarz Lemma is good possibility.
We plan to discuss subject related rigidity of holomorphic mappings and a new Schwarz Lemma at the boundary, see D.M. Burns and S.G. Krantz [4], S.G. Krantz [15], Ornek [21] and the literature cited there, in a forthcoming paper.

First we need some notations. For a function h, we use notation $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient. We use the notation $\lambda_f(z) = |\partial f(z)| - |\bar{\partial} f(z)|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar{\partial} f(z)|$, if $\partial f(z)$ and $\bar{\partial} f(z)$ exist. For a hyperbolic plane domain D, we denote respectively by $\lambda = \lambda_D$ (or if we wish to be more specific by Hyp_D) and δ_D (in some papers we use also notation σ_D) the hyperbolic and pseudo-hyperbolic metric on D respectively. If D is the unit disk we simply write λ for hyperbolic metric.

By $\mathrm{Hyp}_D(z)$ we also denote the hyperbolic density at $z \in D$.

4.1. *Pseudo-distances defined by pluriharmonic functions*

In [6] A. Khalfallah constructs $\alpha_{M,P}$, a new holomorphically invariant pseudo-distance on a complex Banach manifold M using the set of real pluriharmonic functions on M with values in P, a proper open interval of $\mathbb{R}$. It is well known that the Kobayashi pseudo-distance is the largest and the Carathéodory pseudo-distance is the smallest one which can be assigned to complex Banach manifolds by a Schwarz-Pick system. Therefore $C_M \leq \alpha_{M,P} \leq \mathrm{Kob}_M$.

4.2. Schwarz, Yau, Royden lemma

Yau [25] proved the following generalization of Schwarz lemma.

Theorem 4.1 (Yau). *Let M be a complete Kähler manifold with Ricci curvature bounded from below by a constant, and N be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant. Then any holomorphic mapping f from M into N decrease distances up to a constant depending only on the curvature of M and N.*

Royden [23] improved the estimate in Yau theorem.

Theorem 4.2 (Royden). *Let M be a complete Hermitian manifold with holomorphic sectional curvature bounded from below by a constant $k \leq 0$, and N be another Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant $K < 0$. Assume either that M has Riemann sectional curvature bounded from below or that M is Kähler with holomorphic bisectional curvature bounded from below. Then any holomorphic mapping f from M into N satisfies*

$$||df||^2 \leq \frac{k}{K}\,.$$

In [25], Yau mentioned that in order to draw a useful conclusion in the case of harmonic mappings between Riemannian manifolds, it seems that one has to assume the mapping is quasi-conformal.
Since we can consider Theorem 5.1 [9] as a version of Schwarz lemma for harmonic-quasiregular maps between surfaces it seems natural to ask whether there exists a version of Yau-Royden theorem for harmonic-quasiregular maps.

4.3. Schwarz lemma for harmonic-quasiregular maps

The author with M. Knežević proved:

Proposition 4.3 (the unit disk euclidean-qch version,[9]). *Let f be a k-quasiconformal euclidean harmonic mapping from the unit disc $\mathbb{U}$ into itself. Then for all $z \in \mathbb{U}$ we have*

$$|f_z(z)| \leq \frac{1}{1-k}\,\frac{1-|f(z)|^2}{1-|z|^2}.$$

Using $L_f(z) \leq (1+k)|f_z(z)|$, we get
(A) $\lambda(fz', fz) \leq K\lambda(z', z)$, $z', z \in \mathbb{U}$.

In proof we use the metric density $\sigma_f(z) = (1-k)^2\lambda(f(z))|f_z(z)|^2$ and check that the curvature $\mathbf{K}(\sigma)(z) \leq -1$.

In communication with Pavlović appears a question:

Question 2. Whether (A) holds if f is k-qr? We announce a positive answer to this question:

Theorem 4.4. *Let f be a k-quasiregular euclidean harmonic mapping from the unit disc $\mathbb{U}$ into itself. Then for any two points z_1 and z_2 in $\mathbb{U}$ we have*

$$\lambda(f(z_1), f(z_2)) \leq \frac{1+k}{1-k}\,\lambda(z_1, z_2)\,.$$

4.4. Schwarz Lemma for harmonic functions

There is tightly connection between harmonic and holomorphic functions. A few year ago I had in mind the following result:

Theorem 4.5. *Suppose that D is a hyperbolic plane domain and $G = S(a,b) = (a,b)\times\mathbb{R}$, $-\infty < a < b \leq \infty$, plane domain and $f : D \to G$ is a complex harmonic on hyperbolic domain D. Let $z \in D$, $h \in T_z\mathbb{C}$, $|h| = 1$, and $df_z(h) = \lambda v$, $\lambda > 0$, $p = f(z)$ and $v \in T_p\mathbb{C}$. If the measure of the angle between v and $e_1 = e_1(p) \in T_pC$,$p = f(z)$,is α, then*
(I) $\lambda \cos\alpha\, \mathrm{Hyp}_G(f(z)) \leq \mathrm{Hyp}_D(z)$.
(II) *If f is real valued, then*
$\lambda\, \mathrm{Hyp}_G(fz) \leq \mathrm{Hyp}_D(z)$.
Hence $\mathrm{Hyp}_G(f(z_1), f(z_2) \leq \mathrm{Hyp}_D(z_1, z_2)$.

In the case $D = \mathbb{D}$, (II) is proved for $G = S(-1, 1)$ in [7], and for $G = S(0, \infty)$ in [16]. We use Ahlfors-Schwarz lemma to give a simple approach to Kalaj-Vuorinen results [7] (shortly KV-results) and to put it into a broader perspective. But, it turns out that our methods (results) unify very recent approaches by D. Kalaj-M. Vuorinen, H. Chen, K. Dyakanov, D. Kalaj, M. Marković, A. Khalfallah and P.Melentijević.

Note also that the Schwarz theory for pluriharmonic functions is studied in [26] by Z. Xu, including the Schwarz lemma, the Julia lemma, and the behavior of invariant metric for pluriharmonic functions in the unit ball [1)].

4.5. Gromov-Schwarz Lemma

On any almost Hermitian manifold, we can introduce a fundamental 2-form (or cosymplectic structure) that depends only on the chosen metric and the almost complex structure. This form is always non-degenerate. With the extra integrability condition that it is closed (i.e., it is a symplectic form), we get an almost Kähler structure. If both the almost complex structure and the fundamental form are integrable, then we have a Kähler structure. A Hermitian manifold is a complex manifold G with a Hermitian metric H on its holomorphic tangent space: $H_a(\lambda u, v) = \lambda H_a(u, v)$, $\lambda \in \mathbb{C}$, $H_a(u, v) = \overline{H_a(v, u)}$, for $u, v \in T_aG$, $a \in G$.

Likewise, an almost Hermitian manifold is an almost complex manifold with a Hermitian metric on its holomorphic tangent space. Set $g = ReH$ and $\omega = -ImH$. Note that $g = ReH$ is Riemannian metric $H_a(u, v) = \omega(u, iv) - \omega(u, v)$. Let M be a smooth manifold. An almost complex structure J on M is a linear complex structure (that is, a linear map which squares to -I) on each tangent space of the manifold, which varies smoothly on the manifold. In other words, we have a smooth tensor field J of degree (1, 1) such that $J^2 = -I$ when regarded as a vector bundle isomorphism $J : TM \to TM$ on the tangent bundle. A manifold equipped with an almost complex structure is called an almost complex manifold.

A symplectic form on a manifold M is a closed non-degenerate differential 2-form ω. Let (M, ω) be a compact symplectic manifold. A symplectic manifold consists of a pair (M, ω), of a manifold M and a symplectic form ω. Assigning a symplectic form ω to a manifold M is referred to as giving M a symplectic structure.

We say that an almost-complex structure J on M is compatible with ω if $g(u, v) = \omega(Ju, v)$ defines a metric on M.

Pseudoholomorphic version of the Schwarz Lemma (known as Gromov-Schwarz Lemma) is important tool in symplectic geometry.

Theorem 4.6 (Gromov-Schwarz). *Let (M, J, ω) be a compact Hermitian manifold. There exist positive constants $\epsilon_0 > 0$ and $c > 0$ with the following properties.*

If $g : \mathbb{U} \to M$ is a J-holomorphic such that $g(\mathbb{U})$ is contained in some ϵ_0-ball $B_{\epsilon_0} \subseteq M$, then the norm of the differential of g at the origin is bounded by c: $|dg_0| < c$.

After writing a version of manuscript [18], Petar Melentijević [20] sent me his preprint (at 18 Jan 2017) and turns my attention on D. Kalaj manuscript [8] in which the parts (i) and (ii) of Theorem 2.5 are also proved.[2)]

5. Appendix

In this section we will add additional consideration related to Theorem 2.5.

[1)] We received this information after the paper has been accepted for publication and a galley proof is made.

[2)] see also Pavlović https://www.researchgate.net/publication/262967183_Kalaj-Vuorinen's_Schwarz_lemma

5.1. Another proof of Theorem 2.5(i)

For $\epsilon > 0$ set $E(a,\epsilon) = \varphi_a(\epsilon B)$. Then $z \in E(a,\epsilon)$ iff $|\varphi_a(z)| < \epsilon$

$$\frac{|Pz - c|^2}{\epsilon^2\rho^2} + \frac{|Qz|^2}{\epsilon^2\rho} < 1,$$

where $P = P_a$, $Q = Q_a$, $c = c(a,\epsilon) = \frac{(1-\epsilon^2)a}{1-\epsilon^2|a|^2}$ and $\rho = \rho(a,\epsilon) = \frac{1-|a|^2}{1-\epsilon^2|a|^2}$ is an ellipsoid with centre c.

The intersection of $E(a,\epsilon)$ with $[a]$ is a disk of radius $\epsilon\rho$, which is roughly ϵs^2 when ϵ is small; its intersection with the real$(2n-2)$- dimensional space perpendicular to $[a]$ at c is a ball of much larger radius $\epsilon\sqrt{\rho} \sim \epsilon s$.

It is clear that $B(c,\epsilon\rho) \subset E(a,\epsilon) \subset B(c,\epsilon\sqrt{\rho})$.

Suppose that $f : \mathbb{B}_n \to \mathbb{B}_m$ is holomorphic, $a \in \mathbb{B}_n$ and $b = f(a)$. Then

$|f'(c)| \leq \frac{\sqrt{\rho_*}}{\rho}$, where $\rho_* = \rho(b,\epsilon)$. Since c, $|f'(c)|$, ρ and ρ_* tend to a, $|f'(a)|$, $1-|a|^2$ and $1-|b|^2$ respectively, we get $s_a|f'(a)| \leq \sqrt{s_b}$. □

5.2. Addition to the proof of Theorem 2.5(iii) *and and Theorems 3.1-3.2*

The Schwarz-Pick lemma states that every holomorphic function from the unit disk $\mathbb{U}$ to itself, or from the upper half-plane $\mathbb{H}$ to itself, will not increase the Poincaré distance between points.

It is convenient to introduce a pseudo-distance

$$\delta(z,\omega) = |\varphi_z(\omega)| = \left|\frac{z-\omega}{1-\overline{\omega}z}\right|, \quad z,\omega \in \mathbb{U} \tag{11}$$

which is a *conformal invariant*.

Shwarz-Pick lemma: If f holomorphic function from the unit disk $\mathbb{U}$ to itself, then

$$\delta(f(z), f(\omega)) \leq \delta(z,\omega), \quad z,\omega \in \mathbb{U} \tag{12}$$

with equality only if f is a Möbius transformation of $\mathbb{D}$ onto itself.

For $z, w \in \mathbb{C}$, set $a = (1-|z|^2)(1-|w|^2)$, $b = |z-w|^2$, $A = (1-|fz|^2)(1-|fw|^2)$, and $B = |fz - fw|^2$. By this notation,

(A2) $|1- <z,w>|^2 = 1 + |<z,w>|^2 - (|z|^2+|w|^2) + |z-w|^2 = a+b$,

(B2) $|1- <fz,fw>|^2 = A + B$.

If $f \in \mathcal{O}(\mathbb{B}_1,\mathbb{B}_1)$, using (A2) and (B2) Shwarz-Pick lemma can be rewritten in the form $\frac{B}{b} \leq \frac{A+B}{a+b}$ and therefore $Ba \leq Ab$, that is

(I) $$|fz - fw|\sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)} \leq \sqrt{(1-|fz|^2)}\sqrt{(1-|fw|^2)}|z-w|.^{3)}$$

Question 1 (D. Jocić). If $f \in \mathcal{O}(\mathbb{B}_n,\mathbb{B}_m)$ whether (I) holds?

For $z,w \in \mathbb{C}^n$, $|z-w|^2 = |z|^2 + |w|^2 - 2Re <z,w>$, and $|1- <z,w>|^2 = 1 - 2Re <z,w> + |<z,w>|^2$. Hence

$|1- <z,w>|^2 = 1 + |<z,w>|^2 - (|z|^2+|w|^2) + |z-w|^2$ and

$|1- <fz,fw>|^2 = 1 + |<fz,fw>|^2 - (|fz|^2+|fw|^2) + |fz-fw|^2$ and

By Cauchy-Shwarz inequality $|<z,w>|^2 \leq |z||w|$ and therefore

(C2) $|1- <z,w>|^2 \leq a_n + b_n$, where $a_n = (1-|z|^2)(1-|w|^2)$, and $b_n = |z-w|^2$.

Set $A_m = (1-|fz|^2)(1-|fw|^2)$ and $B_m = |fz-fw|^2$. By (C2) and (3) (Section 2),

$$\sigma_1(fz,fw) = \frac{A_1+B_1}{A_1} \leq \sigma_n(z,w) = \frac{|1-(z,w)|^2}{a_n} \leq \frac{a_n+b_n}{a_n}$$

3) D. Jocić turns my attantion on this form and after communication with him we have added the proof of (4))

and therefore (4) (Section 4).

We show that (I) does not hold in general. Contrary suppose that (I) holds and that $f \in O(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ and $b = f(a)$.

Recall if $m > 1$ we proved,

(II) $(1 - |a|^2)|f'(a)| \leq \sqrt{1 - |f(a)|^2}$.

Note that for function $f_0 = \varphi_b \circ \varphi_a$ we have equality in (II).

If (I) holds and z tends a then we have,

(III) $(1 - |a|^2)|f'(a)| \leq (1 - |b|^2)$.

An application of (II) and (III) tof_0 shows that $s_b \leq s_b^2$ and consequently $s_b \geq 1$. Since $s_b < 1$ for $b \neq 0$, we have a contradiction.

5.3. Further comments related to Theorems 3.1-3.2

We have worked on the subject from time to time between 1980 -1990 and in that time we proved Theorems 3.1-3.2([4])). But we realized these days that it is a version of the Earle-Hamilton (1968) fixed point theorem, which may be viewed as a holomorphic formulation of Banach's contraction mapping theorem. A version of this result was proved in 1968 (when I enroled Math Faculty) by Clifford Earle and Richard Hamilton [5] by showing that, with respect to the Carathéodory metric on the domain, the holomorphic mapping becomes a contraction mapping to which the Banach fixed-point theorem can be applied. Perhaps there are applications of this result in the Teichmüller theory.

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[4])we found a my hand written manuscript 1990 and did not pay much attention to it at that time