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On Multivalued G-Monotone Ćirić and Reich Contraction Mappings

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Abstract. In this work, we investigate the existence of fixed points for multivalued G-monotone Ćirić quasi-contraction and Reich contraction mappings in a metric space endowed with a graph G.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

In recent years, mathematicians working in fixed point theory got interested into the class of monotone single-valued mappings defined in a metric space endowed with a partial ordering. Recall that the multivalued version of the Banach contraction principle was discovered by Nadler [7] who obtained the following result:

Theorem 1.1. *Let (X, d) be a complete metric space. Denote by $\mathcal{CB}(\mathcal{X})$ the set of all nonempty closed bounded subsets of X. Let $S : X \to \mathcal{CB}(\mathcal{X})$ be a multivalued mapping. If there exists $k \in [0, 1)$ such that*

$$H(S(u), S(v)) \le k\, d(u, v)$$

for all $u, v \in X$, where H is the Pompeiu-Hausdorff metric on $\mathcal{CB}(\mathcal{X})$, then S has a fixed point in X.

Following its publication, many extensions and generalizations were discovered; see for instance [4, 6] and references cited therein.

Investigation of the fixed point theory of monotone single-valued mappings in metric spaces endowed with a partial order was initiated by Ran and Reurings in [9] (see also [11, 12]) who proved the following result:

Theorem 1.2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that $(X, \preceq)$ is a lattice. Let $T : X \to X$ be a continuous monotone contraction. Assume there exists $x_0 \in X$ such that x_0 and $T(x_0)$ are comparable. Then $\{T^n(x_0)\}$ converges to the unique fixed point ω of T. Moreover, we have*

$$\lim_{n\to+\infty} T^n(x) = \omega,$$

for any $x \in X$.

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Nieto et al. in [8] extended Theorem 1.2 to prove the existence of solutions to some differential equations. Afterward, Jachymski [5] extended the results of [[8], [9]] to the case of single-valued mappings in metric spaces endowed with a graph. Most recently, the author [1] extended Nadler's mutlivalued fixed point theorem to the case of monotone multivalued mappings in metric spaces endowed with a graph.

In [3] Ćirić presented the concept of quasi-contraction mappings and quasi-contraction multivalued mappings as a generalization of the Banach contraction principle. The extension of Ran and Reurings and Jachymski fixed point theorems for monotone contraction mappings in metric spaces endowed with a partial order or a graph to the case of quasi-contraction mappings was carried in [2].

The aim of this paper is to extend the conclusion of Theorem 1.2 and the result of [5] to the case of multivalued monotone quasi-contraction and Reich (a, b, c)-contraction mappings in metric spaces endowed with a graph.

2. Preliminaries

Let (X, d) be a metric space. Consider a directed graph G such that the set of its vertices $V(G)$ coincides with X and the set of its edges $E(G)$ contains all loops i.e., $(x, x) \in E$ for each $x \in V(G)$. Such a digraph is called reflexive. We assume that G has no parallel edges.

Definition 2.1. *A digraph G is called transitive if*

$$(u, v) \in E(G) \quad and \quad (v, w) \in E(G) \Rightarrow (u, w) \in E(G) \quad for\ all \quad u, v, w \in V(G).$$

As Jachymski [5] did, we introduce the following property:
The triplet (X, G, d) has Property (P) if

(P) For any sequence $(x_n)_{n\geq 1}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, for $n \geq 1$, then there is a subsequence $(x_{k_n})_{n\geq 1}$ with $(x_{k_n}, x) \in E(G)$, for $n \geq 1$.

Note that if G is transitive, then the Property (P) implies the following property:

(PP) For any $(x_n)_{n\geq 1}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, for $n \geq 1$, then $(x_n, x) \in E(G)$, for every $n \geq 1$.

Throughout we assume that (X, d) is a metric space, $\mathcal{CB}(\mathcal{X})$ is the class of all nonempty closed and bounded subsets of X and G is a reflexive digraph defined on X. Moreover, we assume that the triplet (X, G, d) has property (P) and G-intervals are closed. Recall that a G-interval is any of the subsets $[x, \to) = \{u \in X; (x, u) \in E(G)\}$ and $(\leftarrow, y] = \{u \in X; (u, y) \in E(G)\}$, for any $x, y \in X$.

3. *G*-Monotone Quasi-Contraction Mappings in Metric Spaces

Let (X, d) be a metric space. The Pompeiu-Hausdorff distance on $\mathcal{CB}(\mathcal{X})$ is defined by

$$H(A, B) := \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

for $A, B \in \mathcal{CB}(\mathcal{X})$, where $d(a, B) := \inf_{b\in B} d(a, b)$.

The following technical result is useful to explain our definition later on.

Lemma 3.1. *Let (X, d) be a metric space. For any $A, B \in \mathcal{CB}(\mathcal{X})$ and $\varepsilon > 0$, we have:*

(i) for $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \varepsilon;$$

(ii) for $b \in B$, there exists $a \in A$ such that

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Note that from Lemma 3.1, whenever one uses multivalued mappings which involves the Pompeiu-Hausdorff distance, then one must assume that the multivalued mappings have bounded values. Otherwise, one has only to assume that the multivalued mappings have nonempty closed values.

Definition 3.2. *Let (X, d) be a metric space. We denote by $C(X)$ the collection of all nonempty closed subsets of X. A multivalued mapping $J : X \to C(\mathcal{X})$ is called quasi-contraction if there exists $k \in [0, 1)$ such that for any $a, b \in X$ and any $A \in J(a)$, there exists $B \in J(b)$ such that*

$$d(A, B) \leq k \ \max\Big(d(a, b); d(a, A); d(b, B); d(a, B); d(b, A)\Big). \tag{1}$$

Definition 3.3. *Let (X, G, d) be as above. A multivalued mapping $J : X \to C(\mathcal{X})$ is called G-monotone quasi-contraction if there exists $k \in [0, 1)$ such that for any $a, b \in X$ with $(a, b) \in E(G)$ and any $A \in J(a)$, there exists $B \in J(b)$ such that $(A, B) \in E(G)$ and*

$$d(A, B) \leq k \ \max\Big(d(a, b); d(a, A); d(b, B); d(a, B); d(b, A)\Big). \tag{2}$$

The point $a \in X$ is called a fixed point of J if $a \in J(a)$.

Example 3.4. *Let $X = \{0, 1, 2, 3\}$ and $d(x, y) = |x - y|$, $\forall x, y \in X$. Define the multivalued map $J : X \to C(\mathcal{X})$ by:*

$$J(x) = \{0, 2, 3\} \text{ for } x \in \{0, 1\} \text{ and } J(x) = \{3\} \text{ for } x \in \{2, 3\}.$$

Then J is a G-monotone quasi-contraction with $k = \frac{1}{3}$, where $G = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (2, 3)\}$, but is not a multivalued quasi-contraction since $d(0, 3) > \frac{1}{3} \ \max\Big(d(1, 2); d(1, 0); d(2, 3); d(1, 3); d(2, 0)\Big)$.

Next we discuss the existence of fixed points for such mappings. First, let J be as in Definition 3.3. For any $u_0 \in X$, the sequence $\{u_n\}$ defines an orbit of J at u_0 if $u_n \in J(u_{n-1})$, $n = 1, 2, \cdots$.

The following technical lemma is crucial to prove the main result of this section.

Lemma 3.5. *Let (X, G, d) be as above. Let $J : X \to C(\mathcal{X})$ be a G-monotone multivalued quasi-contraction mapping. Let $u_0 \in X$ be such that $(u_0, u_1) \in E(G)$ for some $u_1 \in J(u_0)$. Assume that $k < \frac{1}{2}$, where k is the constant associated with the G-monotone quasi-contraction definition of J. Then there exists an orbit $\{u_n\}$ of J at u_0 such that $(u_n, u_{n+1}) \in E(G)$, for any $n \in \mathbb{N}$ and*

$$d(u_n, u_{n+1}) \leq \left(\frac{k}{1-k}\right)^n \ d(u_0, u_1).$$

Proof. By induction we will construct the orbit $\{u_n\}$ of J at u_0. Assume $\{u_0, u_1, \cdots, u_n\}$ are constructed such that $u_{i+1} \in J(u_i)$ and $(u_i, u_{i+1}) \in E(G)$ and

$$d(u_i, u_{i+1}) \leq \left(\frac{k}{1-k}\right)^i \ d(u_0, u_1), \ i = 0, \cdots, n-1.$$

Since J is a G-monotone multivalued quasi-contraction mapping, there exists $u_{n+1} \in J(u_n)$ such that

$$d(u_n, u_{n+1}) \leq k \ \max\Big(d(u_{n-1}, u_n); d(u_n, u_{n+1}); d(u_{n-1}, u_{n+1}); d(u_n, u_n)\Big).$$

Obviously this will imply

$$\begin{aligned} d(u_n, u_{n+1}) &\leq k \ \max\Big(d(u_{n-1}, u_n); d(u_{n-1}, u_{n+1})\Big) \\ &\leq k \ \max\Big(d(u_{n-1}, u_n); d(u_{n-1}, u_n) + d(u_n, u_{n+1})\Big) \\ &\leq k \ \Big(d(u_{n-1}, u_n) + d(u_n, u_{n+1})\Big). \end{aligned}$$

Hence

$$d(u_n, u_{n+1}) \le \frac{k}{1-k}\, d(u_{n-1}, u_n) \le \left(\frac{k}{1-k}\right)^n d(u_0, u_1).$$

By the induction argument, the proof of Lemma 3.5 is complete. □

Next we state the main result of this section.

Theorem 3.6. *Let (X, d) be a complete metric space and G be a reflexive digraph defined on X such that (X, G, d) has Property (P). Let $J : X \to \mathcal{C}(X)$ be a G-monotone multivalued quasi-contraction mapping. Let $u_0 \in X$ be such that $(u_0, u_1) \in E(G)$ for some $u_1 \in J(u_0)$. Assume that $k < \frac{1}{2}$, where k is the constant associated with the G-monotone quasi-contraction definition of J. Then there exists an orbit $\{u_n\}$ of J at u_0 which converges to $\omega \in X$, a fixed point of J.*

Proof. The orbit sequence $\{u_n\}$ of J at u_0 obtained in Lemma 3.5 is Cauchy. Since X is complete, then there exists $\omega \in X$ such that $\{u_n\}$ converges to ω. Since $(u_n, u_{n+1}) \in E(G)$, for any $n \ge 1$, Property (P) implies that there is a subsequence $\{u_{k_n}\}$ such that $(u_{k_n}, \omega) \in E(G)$, for any $n \ge 0$. Next we prove that ω is a fixed point of J, i.e., $\omega \in J(\omega)$. Since $u_{k_n+1} \in J(u_{k_n})$ and $(u_{k_n}, \omega) \in E(G)$, then there exists $\omega_n \in J(\omega)$ such that

$$d(u_{k_n+1}, \omega_n) \le k\ \max\Big(d(u_{k_n}, \omega); d(u_{k_n}, u_{k_n+1}); d(\omega, \omega_n); d(u_{k_n}, \omega_n); d(u_{k_n+1}, \omega)\Big),$$

for any $n \ge 1$. In particular, we have

$$d(u_{k_n+1}, \omega_n) \le k\,\Big(d(u_{k_n}, \omega) + d(u_{k_n}, u_{k_n+1}) + d(\omega, \omega_n) + d(u_{k_n}, \omega_n) + d(u_{k_n+1}, \omega)\Big),$$

for any $n \ge 1$. Since $d(\omega, \omega_n) - d(\omega, u_{k_n+1}) \le d(u_{k_n+1}, \omega_n)$, we get

$$(1-2k) d(\omega, \omega_n) - d(\omega, u_{k_n+1}) \le k\,\Big(d(u_{k_n}, \omega) + d(u_{k_n}, u_{k_n+1}) + d(u_{k_n}, \omega) + d(u_{k_n+1}, \omega)\Big),$$

for any $n \ge 1$. Hence

$$(1-2k) \limsup_{n\to+\infty} d(\omega_n, \omega) \le 0,$$

which implies $\lim_{n\to+\infty} d(\omega_n, \omega) = 0$ since $k < 1/2$. Therefore $\{\omega_n\}$ converges to ω and since $J(\omega)$ is closed, we conclude that $\omega \in J(\omega)$, i.e., ω is a fixed point of J. □

If we assume that G is transitive in Theorem 3.6, then we have $(u_0, \omega) \in E(G)$.

Remark 3.7. *It is not clear to us whether the conclusion of Theorem 3.6 is still valid when $k < 1$.*

4. *G*-Monotone Reich Contraction Mappings in Metric Spaces

Reich in [10] proved that any multivalued Reich (a, b, c)-contraction on a complete space has a fixed point. In this section, we define the notation of multivalued G-monotone Reich contraction mappings and then obtain a fixed point theorem for such mappings. Let us first define the multivalued Reich contraction map.

Definition 4.1. *Let (X, d) be a metric space. A multivalued mapping $J : X \to \mathcal{C}(X)$ is called Reich (a, b, c)-contraction if there exists nonnegative numbers a, b, c with $a + b + c < 1$ such that for any $u, w \in X$ and any $U \in J(u)$ there exists $W \in J(w)$ such that*

$$d(U, W) \le a\, d(u, w) + b\, d(u, U) + c\, d(w, W). \tag{3}$$

Definition 4.2. *Let (X, G, d) be as above. A multivalued mapping $J : X \to \mathcal{C}(X)$ is called G-monotone Reich (a, b, c)-contraction if there exists nonnegative numbers a, b, c with $a + b + c < 1$ such that for any $u, w \in X$ with $(u, w) \in E(G)$ and any $U \in J(u)$, there exists $W \in J(w)$ such that $(U, W) \in E(G)$ and*

$$d(U, W) \le a\, d(u, w) + b\, d(u, U) + c\, d(w, W). \tag{4}$$

Example 4.3. *Let $X = \{0, 1, 2, 3\}$ and $d(x, y) = |x - y|$, $\forall x, y \in X$. Define the multivalued map $J : X \to \mathcal{C}(X)$ by:*

$$J(x) = \{0, 2, 3\} \text{ for } x \in \{0, 1\} \text{ and } J(x) = \{1, 3\} \text{ for } x \in \{2, 3\}.$$

Then J is a G-monotone Reich $(\frac{1}{3}, 0, \frac{1}{3})$-contraction, where $G = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (0, 2), (2, 3)\}$, but is not a multivalued Reich $(\frac{1}{3}, 0, \frac{1}{3})$-contraction since $d(0, 1) > \frac{1}{3}\ d(1, 2) + 0\ d(1, 0) + \frac{1}{3}\ d(2, 1)$ and $d(0, 3) > \frac{1}{3}\ d(1, 2) + 0\ d(1, 0) + \frac{1}{3}\ d(2, 3)$.

Such an example enforces the idea that the study of multivalued G-monotone Reich contraction is worthy of consideration. Next we state the main result of this section.

Theorem 4.4. *Let (X, d) be a complete metric space and G be a reflexive digraph defined on X such that (X, G, d) has Property (P). Let $J : X \to \mathcal{C}(X)$ be a multivalued G-monotone Reich (a, b, c)-contraction mapping. Let $u_0 \in X$ be such that $(u_0, u_1) \in E(G)$ for some $u_1 \in J(u_0)$. Then there exists an orbit $\{u_n\}$ of J at u_0 which converges to $\omega \in X$, a fixed point of J.*

Proof. Since $(u_0, u_1) \in E(G)$ and as J is a G-monotone Reich contraction mapping, there exists $u_2 \in J(u_1)$ such that $(u_1, u_2) \in E(G)$ and

$$d(u_1, u_2) \le a\ d(u_0, u_1) + b\ d(u_0, u_1) + c\ d(u_1, u_2).$$

Thus we have,

$$d(u_1, u_2) \le \frac{(a + b)}{1 - c}\ d(u_0, u_1).$$

Denote $\alpha = \frac{(a+b)}{1-c}$. By an inductive procedure, we obtain a sequence $\{u_n\}_{n \in \mathbb{N}}$ such that $(u_n, u_{n+1}) \in E(G)$ with

$$d(u_n, u_{n+1}) \le \alpha^n\ d(u_0, u_1).$$

Clearly, $\{u_n\}_{n \in \mathbb{N}}$ is Cauchy. Since (X, d) is complete then there exists $\omega \in X$ such that $u_n \to \omega$. Since (X, G, d) has Property (P), then there is a subsequence (u_{k_n}) such that $(u_{k_n}, \omega) \in E(G)$, for every $n \ge 0$. Next we prove that ω is a fixed point of J, i.e., $\omega \in J(\omega)$. Since $u_{k_n+1} \in J(u_{k_n})$ and $(u_{k_n}, \omega) \in E(G)$, then there exists $\omega_n \in J(\omega)$ such that

$$d(u_{k_n+1}, \omega_n) \le a\ d(u_{k_n}, \omega) + b\ d(u_{k_n}, u_{k_n+1}) + c\ d(\omega, \omega_n),$$

for any $n \ge 1$. Thus we have,

$$d(\omega, \omega_n) - d(u_{k_n+1}, \omega) \le a\ d(u_{k_n}, \omega) + b\ d(u_{k_n}, u_{k_n+1}) + c\ d(\omega, \omega_n),$$

for any $n \ge 1$. Therefore,

$$(1 - c)d(\omega, \omega_n) \le a\ d(u_{k_n}, \omega) + b\ d(u_{k_n}, u_{k_n+1}) + d(u_{k_n+1}, \omega),$$

for any $n \ge 1$. Hence

$$(1 - c) \limsup_{n \to +\infty} d(\omega, \omega_n) \le 0,$$

which implies $\lim\limits_{n \to +\infty} d(\omega, \omega_n) = 0$ since $c < 1$. Therefore $\{\omega_n\}$ converges to ω and since $J(\omega)$ is closed, we conclude that $\omega \in J(\omega)$, i.e., ω is a fixed point of J. □

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