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Monotone Asymptotic Pointwise Contractions

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Abstract. In this work, we extend the fixed point result of Kirk and Xu for asymptotic pointwise nonexpansive mappings in a uniformly convex Banach space to monotone mappings defined in a hyperbolic uniformly convex metric space endowed with a partial order.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Following the publication of the three main fixed point theorems for nonexpansive mappings in 1965 [3, 7, 12], many authors tried to extend these results to a larger class of mappings. The concept of asymptotic pointwise Lipschitzian mappings was introduced in [14, 15]. A simple proof of the existence of a fixed point for such mappings may be found in [16]. The authors in [8] extended this result to the nonlinear metric spaces. In this work, we introduce the concept of monotone asymptotic pointwise Lipschitzian mappings. Then we prove some fixed point results for these mappings. It was inspired from the recent work [1]. Metric fixed point theory for monotone mappings has been initiated in the works [5, 18] (see also [19]). A good reference to the metric fixed point theory is the book [11].

2. Basic Definitions and Results

The basic definitions and properties of hyperbolic metric spaces are easily found in the literature. We suggest the reference [10].

Let (M, d) be a metric space. Suppose that any two points $x, y \in M$ are endpoints of a unique metric segment denoted $[x, y]$, i.e. $[x, y]$ is an isometric image of the interval $[0, d(x, y)] \subset \mathbb{R}$. The unique point $z \in [x, y]$ defined by

$$d(x, z) = \beta d(x, y) \;\text{ and }\; d(z, y) = (1 - \beta) d(x, y)$$

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will be denoted by $(1-\beta)x \oplus \beta y$, where $\beta \in [0,1]$. This kind of metric spaces are known as convex metric spaces [17]. We will say that M is a hyperbolic metric space whenever

$$d\left(\frac{1}{2}a \oplus \frac{1}{2}x, \frac{1}{2}a \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d(x,y),$$

for all a, x, y in M. Note that if M is hyperbolic then we have

$$d((1-\beta)x \oplus \beta y, (1-\beta)z \oplus \beta w) \leq (1-\beta)d(x,z) + \beta d(y,w),$$

for all x, y, z, w in M and $\beta \in [0,1]$. A subset C of a hyperbolic metric space M is said to be convex if $[x,y] \subset C$, whenever $x, y \in C$.

Normed linear spaces are an example of linear hyperbolic metric spaces while CAT(0) spaces [2, 13], Hadamard manifolds [4], and the Hilbert open unit ball equipped with the hyperbolic metric [6] are examples of nonlinear hyperbolic metric spaces.

Definition 2.1. *Let (M,d) be a hyperbolic metric space. We define the modulus of uniform convexity of M by*

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\, d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\right\},$$

for any $a \in M$, for every $r > 0$ and $\varepsilon > 0$. We will say that (M,d) is uniformly convex if $\delta(r,\varepsilon) > 0$, for every $r > 0$ and $\varepsilon > 0$.

A metric property similar to the reflexivity in Banach spaces is the property (R) introduced in [9]. A hyperbolic metric space (M,d) is said to have the property (R) whenever any decreasing sequence of nonempty, closed, convex and bounded subsets of M has a nonempty intersection. We have the following result:

Theorem 2.2. *[10] Complete uniformly convex hyperbolic metric spaces have the property (R).*

In the next result, we discuss the properties of function types. Recall that $\tau : M \to [0,+\infty)$ is a type function if there exists a bounded sequence $\{x_n\} \in M$ such that

$$\tau(x) = \limsup_{n\to\infty}\, d(x_n, x).$$

It is easy to check that τ is continuous and convex provided (M,d) is hyperbolic. Recall that τ is convex if $\{x \in M;\ \tau(x) \leq r\}$ is a convex subset of M for any $r \geq 0$. The existence of a minimum point of such mappings is discussed in the following result.

Theorem 2.3. *([10], Theorem 2.4) Let (M,d) be a complete hyperbolic metric space and $C \subset M$ a nonempty closed, convex and bounded subset. Let τ be a type defined on C. If M satisfies the property (R), then τ has a minimum point $z \in C$, i.e.*

$$\tau(z) = \inf\{\tau(x);\ x \in C\} = \tau_0.$$

Moreover, if M is uniformly convex, then the minimum point is unique and any minimizing sequence $\{z_n\}$ in C, i.e. $\lim_{n\to+\infty} \tau(z_n) = \tau_0$, converges to z.

The concept of monotone mappings assumes the existence of a partial order in the underlying set. Assume that (M,d) is partially ordered by $\preceq$. We will say that $x, y \in M$ are comparable whenever $x \preceq y$ or $y \preceq x$. An order interval is one of the subsets $[a,\to) = \{x \in M;\ a \preceq x\}$, $(\leftarrow,a] = \{x \in M;\ x \preceq a\}$, or $[a,b] = \{x \in M;\ a \preceq x \preceq b\}$, for any $a, b \in M$.

Definition 2.4. *Let (M, d) be a metric space partially ordered by $\preceq$. A mapping $T : M \to M$ is said to be monotone if for any $x, y \in M$ such that $x \preceq y$, we have $T(x) \preceq T(y)$. We will say that T is monotone asymptotic pointwise Lipschitzian if T is monotone and there exists a sequence of mappings $k_n : M \to [0, \infty)$ such that for any $x \in M$, we have*

$$d(T^n(x), T^n(y)) \leq k_n(x)\, d(x, y), \quad \text{for any } y \text{ comparable to } x.$$

(i) If $\{k_n\}$ converges pointwise to $k : M \to [0, 1)$, then T is called a monotone asymptotic pointwise contraction.

(ii) If $\limsup_{n\to\infty} k_n(x) \leq 1$, then T is called a monotone asymptotic pointwise nonexpansive mapping.

A point $x \in M$ is said to be a fixed point of T if $T(x) = x$.

3. Main Results

The first result is the analogue to Theorem 3.1 of [16] for monotone monotone asymptotic pointwise contractions.

Theorem 3.1. *Let $(M, d, \preceq)$ be a complete hyperbolic metric space endowed with partially order $\preceq$ for which order intervals are convex and closed. Assume M satisfies the property (R). Let C be a nonempty convex closed bounded subset of M not reduced to one point. Let $T : C \to C$ be a monotone asymptotic pointwise contraction. Then T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable.*

Proof. Obviously if x is a fixed point of T, then x and $T(x) = x$ are comparable. Assume there exists $x_0 \in C$ such that $x_0 \preceq T(x_0)$. Since T is monotone, then we have $T^n(x_0) \preceq T^{n+1}(x_0)$, for every $n \in \mathbb{N}$. In other words, the orbit $\{T^n(x_0)\}$ is monotone increasing. For every $n \in \mathbb{N}$, the set $C_n = C \cap [T^n(x_0), \to)$ is a nonempty bounded closed convex subset of C. The family $\{C_n\}$ is decreasing and since M satisfies the property (R), we conclude that

$$C_\infty = \bigcap_{n\geq 0} [T^n(x_0), \to) \cap C = \bigcap_{n\geq 0} \{x \in C;\ T^n(x_0) \preceq x\} \neq \emptyset.$$

Define the type function $\tau : C_\infty \to [0, +\infty)$ generated by the orbit $\{T^n(x_0)\}$, i.e. $\tau(x) = \limsup_{n\to+\infty} d(T^n(x_0), x)$. First, note that $T(C_\infty) \subset C_\infty$. Indeed, let $x \in C_\infty$, then $T^n(x_0) \preceq x$, for any $n \in \mathbb{N}$. Since T is monotone and $x_0 \preceq T(x_0)$, we get

$$T^n(x_0) \preceq T^n(T(x_0)) = T^{n+1}(x_0) \preceq T(x),$$

for every $n \geq 0$, i.e. $T(x) \in C_\infty$. Using Theorem 2.3, we know that τ has a minimum point $z \in C_\infty$, i.e. $\tau(z) = \inf\{\tau(x); x \in C_\infty\} = \tau_0$. Since $z \in C_\infty$, we get $T^p(z) \in C_\infty$, for every $p \in \mathbb{N}$, which implies

$$\tau(T^p(z)) = \limsup_{n\to+\infty} d(T^n(x_0), T^p(z)) \leq k_p(x_0)\ \limsup_{n\to+\infty} d(T^{n-p}(x_0), z),$$

where $\{k_p(x_0)\}$ is given by Definition 2.4 such that $\lim_{p\to+\infty} k_p(x_0) = k < 1$. Hence $\tau_0 \leq \tau(T^p(z)) \leq k_p(x_0)\, \tau_0$, for every $p \in \mathbb{N}$, which implies $\tau_0 \leq k\, \tau_0$. Since $k < 1$, we conclude that $\tau_0 = 0$, which implies $\tau(T^p(z)) = 0$, for any $p \in \mathbb{N}$. In other words $\{T^n(x_0)\}$ converges to $T^p(z)$, for any $p \in \mathbb{N}$. The uniqueness of the limit will force $z = T(z)$, i.e. z is a fixed point of T. □

Remark 3.2. *It is not clear whether the property (R) may be omitted. It is similar to the weak-compactness of the compactness of the convexity structure in [8, 16]. This question is still unknown.*

The next result is the analogue to Theorem 3.5 of [16] for monotone monotone asymptotic pointwise nonexpansive.

Theorem 3.3. *Let $(M, d, \preceq)$ be a complete uniformly convex hyperbolic metric space endowed with partially order $\preceq$ for which order intervals are convex and closed. Let C be a nonempty convex closed bounded subset of M not reduced to one point. Let $T : C \to C$ be a continuous monotone asymptotic pointwise nonexpansive mapping. Then T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable.*

Proof. Obviously if x is a fixed point of T, then x and $T(x) = x$ are comparable. As we did in the proof of Theorem 3.1, assume there exists $x_0 \in C$ such that $x_0 \preceq T(x_0)$. Hence the orbit $\{T^n(x_0)\}$ is monotone increasing. For every $n \in \mathbb{N}$, the set $C_n = C \cap [T^n(x_0), \rightarrow)$ is a nonempty bounded closed convex subset of C. The family $\{C_n\}$ is decreasing and since M satisfies the property (R), we conclude that

$$C_\infty = \bigcap_{n\geq 0} [T^n(x_0), \rightarrow) \cap C = \bigcap_{n\geq 0} \{x \in C;\ T^n(x_0) \preceq x\} \neq \emptyset.$$

Define the type function $\tau : C_\infty \to [0, +\infty)$ generated by the orbit $\{T^n(x_0)\}$, i.e. $\tau(x) = \limsup_{n\to+\infty} d(T^n(x_0), x)$. Using Theorem 2.3, τ has a unique minimum point $z \in C_\infty$, i.e. $\tau(z) = \inf\{\tau(x); x \in C_\infty\} = \tau_0$. We claim that $\{T^n(z)\}$ is a minimizing sequence. First, note that $T(C_\infty) \subset C_\infty$. Indeed, let $x \in C_\infty$, then $T^n(x_0) \preceq x$, for any $n \in \mathbb{N}$. Since T is monotone and $x_0 \preceq T(x_0)$, we get

$$T^n(x_0) \preceq T^n(T(x_0)) = T^{n+1}(x_0) \preceq T(x),$$

for every $n \geq 0$, i.e. $T(x) \in C_\infty$. Since $z \in C_\infty$, we have $T^p(z) \in C_\infty$, for every $p \in \mathbb{N}$, which implies

$$\tau(T^p(z)) = \limsup_{n\to+\infty} d(T^n(x_0), T^p(z)) \leq k_p(x_0)\ \limsup_{n\to+\infty} d(T^{n-p}(x_0), z),$$

where $\{k_p(x_0)\}$ is given by Definition 2.4 such that $\lim_{p\to+\infty} k_p(x_0) = 1$. Hence $\tau_0 \leq \tau(T^p(z)) \leq k_p(x_0)\ \tau_0$, for every $p \in \mathbb{N}$, which implies

$$\lim_{p\to+\infty} \tau(T^p(z)) = \tau_0.$$

In other words $\{T^p(x)\}$ is a minimizing sequence of τ. Using Theorem 2.3 again, we conclude that $\{T^p(z)\}$ converges to z. Since T is continuous, then

$$T(z) = \lim_{n\to+\infty} T(T^n(z)) = \lim_{n\to+\infty} T^{n+1}(z) = z.$$

In other words, z is a fixed point of T. $\square$

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