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Fixed Points of Multivalued Non-Linear $\mathcal{F}$-Contractions with Application to Solution of Matrix Equations

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Abstract. In the present paper, we introduce the notion of α-type $\mathcal{F}$-τ-contraction and establish related fixed point results in metric spaces. An example is also given to illustrate our main results and to show that our results are proper generalization of Altun et al. (2015), Miank et al. (2015), Altun et al. (2016) and Olgun et al. (2016). We also obtain fixed point results in the setting of partially ordered metric spaces. Finally, an application is given to set up the existence of positive definite solution of non-linear matrix equation.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Let $(\mathcal{X}, d)$ be a metric space. $2^{\mathcal{X}}$ denotes the family of all nonempty subsets of $\mathcal{X}$, $C(\mathcal{X})$ denotes the family of all nonempty, closed subsets of $\mathcal{X}$, $CB(\mathcal{X})$ denotes the family of all nonempty, closed, and bounded subsets of $\mathcal{X}$ and $K(\mathcal{X})$ denotes the family of all nonempty compact subsets of $\mathcal{X}$. It is clear that, $K(\mathcal{X}) \subseteq CB(\mathcal{X}) \subseteq C(\mathcal{X}) \subseteq P(\mathcal{X})$. For $\mathcal{A}, \mathcal{B} \in C(\mathcal{X})$, let

$$H(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{x\in\mathcal{A}} D(x, \mathcal{B}), \sup_{y\in\mathcal{B}} D(y, \mathcal{A})\right\},$$

where $D(x, \mathcal{B}) = \inf\{d(x, y) : y \in \mathcal{B}\}$. Then H is called generalized Pompeiu Hausdorff distance on $C(\mathcal{X})$. It is well known that H is a metric on $CB(\mathcal{X})$, which is called Pompeiu Hausdorff metric induced by d. For detail see ([5], [8], [19]).
An interesting generalization of the Banach contraction principle is Nadler's fixed point theorem [23], he proved that every multivalued contraction on complete metric space has a fixed point. After this many authors extended Nadler's fixed point theorem in many directions (see [1, 9, 10, 14, 15, 22, 26] and references there in). The following generalization of it is given by Klim et al. [20].

Theorem 1.1 ([20]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$. Assume that the following conditions hold:*

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1. *the map $x \to D(x, \mathcal{T}x)$ is lower semi-continuous;*

2. *there exists $b \in (0,1)$ and a function $\varphi : [0,\infty) \to [0,b)$ satisfying*

$$\lim_{t\to s^+}\sup \varphi(t) < b \qquad \text{for} \quad s \geq 0$$

and for any $x \in \mathcal{X}$, there is $y \in I_b^x$ satisfying

$$D(y, \mathcal{T}y) \leq \varphi(d(x,y))d(x,y),$$

where $I_b^x = \{y \in \mathcal{T}x : bd(x,y) \leq d(x, \mathcal{T}x)\}$.

Then $\mathcal{T}$ has a fixed point.

Above mentioned results were extended by Ćirić in [11], see also [14].

In 2012, Samet et al. [28] defined α-admissible mappings and established fixed point theorems and Asl et al. [6] extended these concepts to multivalued mappings.

Definition 1.2 ([18]). *Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a function, then $\mathcal{T}$ is an α_*-admissible mapping if*

$$\alpha(y,z) \geq 1 \text{ implies that } \alpha_*(\mathcal{T}y, \mathcal{T}z) \geq 1, \quad y, z \in \mathcal{X},$$

where

$$\alpha_*(\mathcal{A}, \mathcal{B}) = \inf_{y\in\mathcal{A}, z\in\mathcal{B}} \alpha(y,z).$$

Further, Ali et al [2] generalized the Definition 1.2 in the following way:

Definition 1.3 ([2]). *Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha : X \times X \to \mathbb{R}_+$ be two functions. We say that $\mathcal{T}$ is generalized α_*-admissible mapping if*

$$\alpha(y,z) \geq 1 \text{ implies that } \alpha(u,v) \geq 1, \quad \text{for all } u \in \mathrm{T}y,\ v \in \mathrm{T}z.$$

Recently, Wardowski defined $\mathcal{F}$-contraction [29] and then $\mathcal{F}$-weak-contraction [30] and proved fixed point results as a generalization of the Banach contraction principle for these contractions. Further, Hussain et al. [17] broadened this idea to α-$\mathcal{GF}$-contraction with respect to a general family of functions $\mathcal{G}$. Many authors did work in this direction (see [13], [16], [27] and references there in). Following Wordowski, we denote by $\mathfrak{F}$, the set of all functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying following conditions:

($\mathcal{F}_1$) $\mathcal{F}$ is strictly increasing;

($\mathcal{F}_2$) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$;

($\mathcal{F}_3$) there exist $0 < k < 1$ such that $\lim_{n\to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$,

$\mathfrak{F}_*$, if $\mathcal{F}$ also satisfies the following:

($\mathcal{F}_4$) $\mathcal{F}(\inf A) = \inf \mathcal{F}(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$,

Following this direction of research, Gopal et al. [12] introduced the concepts of α-type-$\mathcal{F}$-contractive mappings and proved some fixed point theorems concerning such contractions as:

Definition 1.4 ([12]). *Let $(\mathcal{X}, d)$ be a complete metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be an α-type $\mathcal{F}$-contraction on $\mathcal{X}$ if there exists $\tau > 0$ and two functions $\mathcal{F} \in \mathfrak{F}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in \mathcal{X}$ satisfying $d(\mathcal{T}x, \mathcal{T}y) > 0$, the following inequality holds*

$$\tau + \alpha(x,y)\mathcal{F}(d(\mathcal{T}x, \mathcal{T}y)) \leq \mathcal{F}(d(x,y)).$$

Definition 1.5 ([12]). *Let $(\mathcal{X}, d)$ be a complete metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be an α-type $\mathcal{F}$-weak-contraction on $\mathcal{X}$ if there exists $\tau > 0$ and two functions $\mathcal{F} \in \mathfrak{F}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to \{-\infty\} \cup (0, +\infty)$ such that for all $x, y \in \mathcal{X}$ satisfying $d(\mathcal{T}x, \mathcal{T}y) > 0$, the following inequality holds*

$$\tau + \alpha(x, y)\mathcal{F}(d(\mathcal{T}x, \mathcal{T}y)) \le \mathcal{F}\left(\max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2}\right\}\right).$$

Theorem 1.6 ([12]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an α-type $\mathcal{F}$-weak-contraction satisfying the following conditions:*

1. *$\mathcal{T}$ is α-admissible;*
2. *there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, \mathcal{T}x_0) \ge 1$;*
3. *$\mathcal{T}$ is continuous.*

Then T has a fixed point.

On unifying the concepts of Wardowski's and Nadler's, Altun et al. [3] gave the concept of multivalued $\mathcal{F}$-contractions and found some fixed point results.

Definition 1.7 ([3]). *Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{T} : \mathcal{X} \to CB(\mathcal{X})$ be a mapping. Then $\mathcal{T}$ is a multivalued $\mathcal{F}$-contraction, if $\mathcal{F} \in \mathfrak{F}$ and there exists $\tau > 0$ such that for all $x, y \in \mathcal{X}$,*

$$H(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \tau + \mathcal{F}(H(\mathcal{T}x, \mathcal{T}y)) \le \mathcal{F}(d(x, y)).$$

Theorem 1.8 ([3]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be a multivalued $\mathcal{F}$-contraction, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Theorem 1.9 ([3]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to CB(\mathcal{X})$ be a multivalued $\mathcal{F}$-contraction. Suppose $\mathcal{F} \in \mathfrak{F}_*$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Olgun et al. [24] proved the non-linear cases of Theorems 1.8 and 1.9 as:

Theorem 1.10 ([24]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$. If there exists $\mathcal{F} \in \mathfrak{F}$ and $\tau : (0, \infty) \to (0, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > 0 \quad \textit{for all} \quad s \ge 0$$

and for all $x, y \in \mathcal{X}$,

$$H(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \tau(d(x, y)) + \mathcal{F}(H(\mathcal{T}x, \mathcal{T}y)) \le \mathcal{F}(d(x, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Theorem 1.11 ([24]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to CB(\mathcal{X})$. If there exists $\mathcal{F} \in \mathfrak{F}_*$ and $\tau : (0, \infty) \to (0, \infty)$ satisfying all the condition of Theorem 1.10, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

On the other side, Minak et al. [21], extended the results of Wardowski's as:

Theorem 1.12 ([21]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exists $\tau > 0$ such that for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying*

$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \le \mathcal{F}(d(z, y)),$$

where

$$\mathcal{F}_\sigma^z = \{y \in \mathcal{T}z : \mathcal{F}(d(z, y)) \le \mathcal{F}(D(z, \mathcal{T}z)) + \sigma\},$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to d(z, \mathcal{T}z)$ is lower semi-continuous.

Theorem 1.13 ([21]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfying all the assumption of Theorem 1.12. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Minak et al. [21] also showed that $\mathcal{F}_\sigma^z \neq \emptyset$ in both cases when $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{F} \in \mathfrak{F}_*$. Very recently, Altun et al. [4] generalized Theorem 1.1 by adopting the concept of [21] as follows:

Theorem 1.14 ([4]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. Assume that the following conditions hold:*

1. *the map $z \to D(z, \mathcal{T}z)$ is lower semi-continuous;*
2. *there exists $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \qquad \text{for} \quad s \geq 0$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)).$$

Then $\mathcal{T}$ has a fixed point.

Theorem 1.15 ([4]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ satisfying all the conditions of Theorem 1.14. Then $\mathcal{T}$ has a fixed point.*

By considering the above facts, we define α-type $\mathcal{F}$-τ-contraction for multivalued mappings and prove non-linear form of Mizouguchi-Takahashi's type fixed point theorems. Our results generalize and extend many existing results in literature including the works in [3], [4], [12], [21] and [24].

2. Main Results

We begin this section with the following definition.

Definition 2.1. *Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued mapping on a metric space $(\mathcal{X}, d)$, then $\mathcal{T}$ is said to be an α-type $\mathcal{F}$-τ-contraction on $\mathcal{X}$, if there exists $\sigma > 0$, $\tau : (0, \infty) \to (\sigma, \infty)$, $\mathcal{F} \in \mathfrak{F}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to \{-\infty\} \cup (0, +\infty)$ such that for all $z \in \mathcal{X}$, $y \in \mathcal{F}_\sigma^z$ with $D(z, \mathcal{T}z) > 0$ satisfying*

$$\tau(d(z, y)) + \alpha(z, y)\mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(M(z, y)), \tag{2.1}$$

where,

$$M(z, y) = \max\left\{d(z, y), D(z, \mathcal{T}z), D(y, \mathcal{T}y), \frac{D(y, \mathcal{T}z) + D(z, \mathcal{T}y)}{2}, \frac{D(y, \mathcal{T}y)[1 + D(z, \mathcal{T}z)]}{1 + d(z, y)}, \frac{D(y, \mathcal{T}z)[1 + D(z, \mathcal{T}y)]}{1 + d(z, y)}\right\}. \tag{2.2}$$

Theorem 2.2. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be an α-type $\mathcal{F}$-τ-contraction satisfying the following assertions:*

1. *$\mathcal{T}$ is generalized α_*-admissible mapping;*
2. *the map $z \to D(z, \mathcal{T}z)$ is lower semi-continuous;*
3. *there exists $z_0 \in \mathcal{X}$ and $z_1 \in \mathcal{T}z_0$ such that $\alpha(z_0, z_1) \geq 1$;*

4. *τ satisfies*

$$\lim_{t\to s^+}\inf \tau(t) > \sigma \qquad \text{for all} \quad s \geq 0$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Let $z_0 \in \mathcal{X}$, since $\mathcal{T}z \in K(\mathcal{X})$ for every $z \in \mathcal{X}$, the set $\mathcal{F}_\sigma^z$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in \mathcal{F}_\sigma^{z_0}$ and by hypothesis $\alpha(z_0, z_1) \geq 1$. Assume that $z_1 \notin \mathcal{T}z_1$, otherwise z_1 is the fixed point of $\mathcal{T}$. Then, since $\mathcal{T}z_1$ is closed, $D(z_1, \mathcal{T}z_1) > 0$, so, from (2.1), we have

$$\tau(d(z_0, z_1)) + \alpha(z_0, z_1)\mathcal{F}(D(z_1, \mathcal{T}z_1)) \leq \mathcal{F}(M(z_0, z_1)), \tag{2.3}$$

where

$$\begin{aligned} M(z_0, z_1) = \max\Bigg\{ & d(z_0, z_1), D(z_0, \mathcal{T}z_0), D(z_1, \mathcal{T}z_1), \frac{D(z_1, \mathcal{T}z_0) + D(z_0, \mathcal{T}z_1)}{2}, \\ & \frac{D(z_1, \mathcal{T}z_1)[1 + D(z_0, \mathcal{T}z_0)]}{1 + d(z_0, z_1)}, \frac{D(z_1, \mathcal{T}z_0)[1 + D(z_0, \mathcal{T}z_1)]}{1 + d(z_0, z_1)} \Bigg\}. \end{aligned} \tag{2.4}$$

Since $\mathcal{T}z_0$ and $\mathcal{T}z_1$ are compact, so we have

$$\begin{aligned} M(z_0, z_1) &= \max\Bigg\{ d(z_0, z_1), d(z_0, z_1), d(z_1, z_2), \frac{d(z_1, z_1) + d(z_0, z_2)}{2}, \\ &\qquad \frac{d(z_1, z_2)[1 + d(z_0, z_1)]}{1 + d(z_0, z_1)}, \frac{d(z_1, z_1)[1 + d(z_0, z_2)]}{1 + d(z_0, z_1)} \Bigg\} \\ &= \max\left\{ d(z_0, z_1), d(z_1, z_2), \frac{d(z_0, z_2)}{2} \right\}. \end{aligned} \tag{2.5}$$

Since $\frac{d(z_0,z_2)}{2} \leq \frac{d(z_0,z_1)+d(z_1,z_2)}{2} \leq \max\{d(z_0, z_1), d(z_1, z_2)\}$, it follows that

$$M(z_0, z_1) \leq \max\{d(z_0, z_1), d(z_1, z_2)\}. \tag{2.6}$$

Suppose that $d(z_0, z_1) < d(z_1, z_2)$, then (2.3) implies that

$$\begin{aligned} \tau(d(z_0, z_1)) + \mathcal{F}(D(z_1, \mathcal{T}z_1)) &\leq \tau(d(z_0, z_1)) + \alpha(z_0, z_1)\mathcal{F}(D(z_1, \mathcal{T}z_1)) \\ &\leq \mathcal{F}(d(z_1, z_2)), \end{aligned} \tag{2.7}$$

consequently,

$$\tau(d(z_0, z_1)) + \mathcal{F}(d(z_1, z_2)) \leq \mathcal{F}(d(z_1, z_2)), \tag{2.8}$$

or, $\mathcal{F}(d(z_1, z_2)) \leq \mathcal{F}(d(z_1, z_2)) - \tau(d(z_0, z_1))$, which is a contradiction. Hence $M(d(z_0, z_1)) \leq d(z_0, z_1)$, therefore by using $\mathcal{F}_1$, (2.3) implies that

$$\tau(d(z_0, z_1)) + \alpha(z_0, z_1)\mathcal{F}(d(z_1, z_2)) \leq \mathcal{F}(d(z_0, z_1)). \tag{2.9}$$

Now for $z_1 \in \mathcal{X}$ there exists $z_2 \in \mathcal{F}_\sigma^{z_1}$ with $z_2 \notin \mathcal{T}z_2$, otherwise z_2 is the fixed point of $\mathcal{T}$, since $\mathcal{T}z_2$ is closed, so, $D(z_2, \mathcal{T}z_2) > 0$. Since $\mathcal{T}$ is generalized α_*-admissible, then $\alpha(z_1, z_2) \geq 1$. Again by using (2.1), we get

$$\tau(d(z_1, z_2)) + \alpha(z_1, z_2)\mathcal{F}(D(z_2, \mathcal{T}z_2)) \leq \mathcal{F}(M(z_1, z_2)), \tag{2.10}$$

where

$$\begin{aligned} M(z_1, z_2) = \max\Bigg\{ & d(z_1, z_2), D(z_1, \mathcal{T}z_1), D(z_2, \mathcal{T}z_2), \frac{D(z_2, \mathcal{T}z_1) + D(z_1, \mathcal{T}z_2)}{2}, \\ & \frac{D(z_2, \mathcal{T}z_2)[1 + D(z_1, \mathcal{T}z_1)]}{1 + d(z_1, z_2)}, \frac{D(z_2, \mathcal{T}z_1)[1 + D(z_1, \mathcal{T}z_2)]}{1 + d(z_1, z_2)} \Bigg\}. \end{aligned} \tag{2.11}$$

Since $\mathcal{T}z_1$ and $\mathcal{T}z_2$ are compact, so we have

$$M(z_1,z_2) = \max\left\{d(z_1,z_2), d(z_2,z_3), \frac{d(z_1,z_3)}{2}\right\}. \tag{2.12}$$

Since $\frac{d(z_1,z_3)}{2} \le \frac{d(z_1,z_2)+d(z_2,z_3)}{2} \le \max\{d(z_1,z_2), d(z_2,z_3)\}$, it follows that

$$M(z_1,z_2) \le \max\{d(z_1,z_2), d(z_2,z_3)\}. \tag{2.13}$$

Suppose that $d(z_1,z_2) < d(z_2,z_3)$, then (2.10) implies that $\mathcal{F}(d(z_2,z_3)) \le \mathcal{F}(d(z_2,z_3)) - \tau(d(z_1,z_2))$, which is a contradiction. Hence $M(d(z_1,z_2)) \le d(z_1,z_2)$, therefore by using $\mathcal{F}_1$, (2.10) implies that

$$\tau(d(z_1,z_2)) + \alpha(z_1,z_2)\mathcal{F}(d(z_2,z_3)) \le \mathcal{F}(d(z_1,z_2)). \tag{2.14}$$

On continuing recursively, we get a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$, where $z_{n+1} \in \mathcal{F}_\sigma^{z_n}$, $z_{n+1} \notin \mathcal{T}z_{n+1}$, $\alpha(z_n,z_{n+1}) \ge 1$, $M(z_n,z_{n+1}) \le d(z_n,z_{n+1})$ and

$$\tau(d(z_n,z_{n+1})) + \mathcal{F}(D(z_{n+1},\mathcal{T}z_{n+1})) \le \mathcal{F}(d(z_n,z_{n+1})). \tag{2.15}$$

Since $z_{n+1} \in \mathcal{F}_\sigma^{z_n}$ and $\mathcal{T}z_n$ and $\mathcal{T}z_{n+1}$ are compact, we have

$$\tau(d(z_n,z_{n+1})) + \mathcal{F}(d(z_{n+1},z_{n+2})) \le \mathcal{F}(d(z_n,z_{n+1})) \tag{2.16}$$

and

$$\mathcal{F}(d(z_n,z_{n+1})) \le \mathcal{F}(d(z_n,z_{n+1})) + \sigma. \tag{2.17}$$

Combining (2.16) and (2.17) gives

$$\mathcal{F}(d(z_{n+1},z_{n+2})) \le \mathcal{F}(d(z_n,z_{n+1})) + \sigma - \tau(d(z_n,z_{n+1})) \tag{2.18}$$

Let $d_n = d(z_n,z_{n+1})$ for $n \in \mathbb{N}$, then $d_n > 0$ and from (2.18) $\{d_n\}$ is decreasing. Therefore, there exists $\delta \ge 0$ such that $\lim_{n\to\infty} d_n = \delta$. Now let $\delta > 0$. From (2.18), we get

$$\begin{aligned}\mathcal{F}(d_{n+1}) &\le \mathcal{F}(d_n) + \sigma - \tau(d_n)\\ &\le \mathcal{F}(d_{n-1}) + 2\sigma - \tau(d_n) - \tau(d_{n-1})\\ &\;\vdots\\ &\le \mathcal{F}(d_0) + n\sigma - \tau(d_n) - \tau(d_{n-1}) - \cdots - \tau(d_0).\end{aligned} \tag{2.19}$$

Let $\tau(d_{p_n}) = \min\{\tau(d_0), \tau(d_1), \cdots, \tau(d_n)\}$ for all $n \in \mathbb{N}$. From (2.19), we get

$$\mathcal{F}(d_{n+1}) \le \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_n})). \tag{2.20}$$

From (2.15), we also get

$$\mathcal{F}(D(z_{n+1},\mathcal{T}z_{n+1})) \le \mathcal{F}(D(z_0,\mathcal{T}z_0)) + n(\sigma - \tau(d_{p_n})). \tag{2.21}$$

Now consider the sequence $\{\tau(d_{p_n})\}$. We distinguish two cases.
Case 1. For each $n \in \mathbb{N}$, there is $m > n$ such that $\tau(d_{p_n}) > \tau(d_{p_m})$. Then we obtain a subsequence $\{d_{p_{n_k}}\}$ of $\{d_{p_n}\}$ with $\tau(d_{p_{n_k}}) > \tau(d_{p_{n_{k+1}}})$ for all k. Since $d_{p_{n_k}} \to \delta^+$, we deduce that

$$\liminf_{k\to\infty} \tau(d_{p_{n_k}}) > \sigma.$$

Hence $\mathcal{F}(d_{n_k}) \le \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_{n_k}}))$ for all k. Consequently, $\lim_{k\to\infty} \mathcal{F}(d_{n_k}) = -\infty$ and by ($\mathcal{F}2$), we obtain $\lim_{k\to\infty} d_{p_{n_k}}) = 0$, which contradicts that $\lim_{n\to\infty} d_n > 0$.
Case 2. There is $n_0 \in \mathbb{N}$ such that $\tau(d_{p_{n_0}}) > \tau(d_{p_m})$ for all $m > n_0$. Then $\mathcal{F}(d_m) \le \mathcal{F}(d_0) + m(\sigma - \tau(d_{p_{n_0}}))$ for all $m > n_0$. Hence $\lim_{m\to\infty} \mathcal{F}(d_m) = -\infty$, so $\lim_{m\to\infty} d_m = 0$, which contradicts that $\lim_{m\to\infty} d_m > 0$. Thus, $\lim_{n\to\infty} d_n = 0$. From ($\mathcal{F}_3$), there exists $0 < r < 1$ such that

$$\lim_{n\to\infty} (d_n)^r \mathcal{F}(d_n) = 0. \tag{2.22}$$

By (2.20), we get for all $n \in \mathbb{N}$

$$(d_n)^r \mathcal{F}(d_n) - (d_n)^r \mathcal{F}(d_0) \leq (d_n)^r n(\sigma - \tau(d - p_n)) \leq 0. \tag{2.23}$$

Letting $n \to \infty$ in (2.23), we obtain

$$\lim_{n\to\infty} n(d_n)^r = 0 \tag{2.24}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n(d_n)^r \leq 1$, or, $d_n \leq \frac{1}{n^{1/r}}$, for all $n > n_1$. Next, for $m > n \geq n_1$ we have

$$d(z_n, z_m) \leq \sum_{i=n}^{m-1} d(z_i, z_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}},$$

since $0 < k < 1$, $\sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$ converges. Therefore, $d(z_n, z_m) \to 0$ as $m, n \to \infty$. Thus, $\{z_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, there exists $z^* \in \mathcal{X}$ such that $z_n \to z^*$ as $n \to \infty$. From equations (2.21) and $\mathcal{F}_2$, we have

$$\lim_{n\to\infty} D(z_n, \mathcal{T} z_n) = 0.$$

Since $z \to D(z, \mathcal{T} z)$ is lower semi-continuous, then

$$0 \leq D(z, Tz) \leq \lim_{n\to\infty} \inf D(z_n, \mathcal{T} z_n) = 0.$$

Thus, $\mathcal{T}$ has a fixed point. □

In the following theorem we take $C(\mathcal{X})$ instead of $K(\mathcal{X})$, then we need to take $\mathcal{F} \in \mathfrak{F}_*$ in Definition 2.1.

Theorem 2.3. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ be an α-type $\mathcal{F}$-τ-contraction with $\mathcal{F} \in \mathfrak{F}_*$ satisfying all the assertions of Theorem 2.2. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Proof. Let $z_0 \in \mathcal{X}$, since $\mathcal{T} z \in C(\mathcal{X})$ for every $z \in \mathcal{X}$ and $\mathcal{F} \in \mathfrak{F}_*$, the set $\mathcal{F}_\sigma^z$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in \mathcal{F}_\sigma^{z_0}$ and by hypothesis $\alpha(z_0, z_1) \geq 1$. Assume that $z_1 \notin \mathcal{T} z_1$, otherwise z_1 is the fixed point of $\mathcal{T}$. Then, since $\mathcal{T} z_1$ is closed, $D(z_1, \mathcal{T} z_1) > 0$, so, from (2.1), we have

$$\tau(d(z_0, z_1)) + \alpha(z_0, z_1)\mathcal{F}(D(z_1, \mathcal{T} z_1)) \leq \mathcal{F}(M(z_0, z_1)), \tag{2.25}$$

where

$$\begin{aligned} M(z_0, z_1) = \max \Bigg\{ & d(z_0, z_1), D(z_0, \mathcal{T} z_0), D(z_1, \mathcal{T} z_1), \frac{D(z_1, \mathcal{T} z_0) + D(z_0, \mathcal{T} z_1)}{2}, \\ & \frac{D(z_1, \mathcal{T} z_1)[1 + D(z_0, \mathcal{T} z_0)]}{1 + d(z_0, z_1)}, \frac{D(z_1, \mathcal{T} z_0)[1 + D(z_0, \mathcal{T} z_1)]}{1 + d(z_0, z_1)} \Bigg\}. \end{aligned} \tag{2.26}$$

The rest of the proof can be completed as in the proof of Theorem 2.2 by considering the closedness of $\mathcal{T} z$, for all $z \in \mathcal{X}$. □

Example 2.4. *Let $\mathcal{X} = \left\{ \frac{1}{2^{n-1}} : n \in \mathbb{N} \right\} \cup \{0\}$ with usual metric d. Then $(\mathcal{X}, d)$ is a complete metric space. Define $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, and $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ by*

$$\mathcal{T} z = \begin{cases} \left\{ \frac{1}{2^n}, 1 \right\} & \text{if } z = \frac{1}{2^{n-1}} \\ \left\{ 0, \frac{1}{2} \right\} & \text{if } z = 0, \end{cases}$$

$$\alpha(z, y) = \begin{cases} 2^{n+1} & \text{if } z = \frac{1}{2^{n-1}}, y = \frac{1}{2^n} \\ 2 & \text{if } z, y \in \left\{ \frac{1}{2^{n-1}}, 1 \right\} \\ 0 & \text{if } z = 0 \end{cases}$$

and $\mathcal{F}(r) = \ln(r)$.
Then

$$D(z,\mathcal{T}z) = \begin{cases} \frac{1}{2^n} & \text{if } z = \frac{1}{2^{n-1}}, n > 1 \\ 0 & \text{if } z = 0, 1, \end{cases}$$

hence $z \to D(z,\mathcal{T}z)$ *is lower semi-continuous. Let* $D(z,\mathcal{T}z) > 0$ *and* $\tau(t) = \frac{1}{t} + \frac{1}{2}$ *for* $\sigma = \frac{1}{2}$, *then* $z = \frac{1}{2^{n-1}}, n > 1$, *so,* $\mathcal{T}z = \left\{\frac{1}{2^n}, 1\right\}$. *Thus for* $y = \frac{1}{2^n} \in \mathcal{T}z$, *we have*

$$\mathcal{F}(d(z,y)) - \mathcal{F}D(z,\mathcal{T}z) = \mathcal{F}\left(\frac{1}{2^n}\right) - \mathcal{F}\left(\frac{1}{2^n}\right) = 0.$$

Therefore, $y \in \mathcal{F}_\sigma^z$ *for* $\sigma = \frac{1}{2}$ *and*

$$M(z,y) = \max\left\{\frac{1}{2^n}, \frac{1}{2^n}, \frac{1}{2^{n+1}}, 0, \frac{1}{2^{n+1}}, 0\right\} = \frac{1}{2^n}.$$

This implies that

$$\begin{aligned} \tau(d(z,y)) + \alpha(z,y)\mathcal{F}(D(y,\mathcal{T}y)) &= \tau\left(\frac{1}{2^n}\right) + 2^{n+1}\mathcal{F}\left(\frac{1}{2^{n+1}}\right) \\ &= 2^n + \frac{1}{2} - 2^{n+1}\ln\left(2^{n+1}\right) \\ &< -\ln(2^n) = \ln\left(\frac{1}{2^n}\right) \\ &= \mathcal{F}(M(z,y)). \end{aligned}$$

Hence $\mathcal{T}$ *is* α*-type* $\mathcal{F}$*-*τ*-contraction.*
Since $\alpha(z,y) \geq 1$ *when* $z, y \in \left\{\frac{1}{2^{n-1}}, 1\right\}$ *and* $z = \frac{1}{2^{n-1}}, y = \frac{1}{2^{n-1}}$, *this implies that* $\alpha(u,v) > 1$ *for all* $u \in \mathcal{T}z$ *and* $v \in \mathcal{T}y$.
Hence $\mathcal{T}$ *is generalized* α_* *admissible mapping. Thus, all conditions of Theorem 2.3 hold and* 0 *is a fixed point of* $\mathcal{T}$.
On the other hand, for $z = \frac{1}{2}$ *there exists* $y = \frac{1}{4} \in \mathcal{F}_\sigma^z$ *for* $\sigma = \frac{1}{2}$ *such that*

$$\begin{aligned} \tau(d(z,y)) + \mathcal{F}(D(y,\mathcal{T}y)) &= \tau\left(\frac{1}{4}\right) + \mathcal{F}\left(\frac{1}{8}\right) \\ &= 4 + \frac{1}{2} - \ln(8) = 2.421 \\ &> -1.386 = \ln\left(\frac{1}{4}\right) \\ &= \mathcal{F}(d(z,y)). \end{aligned}$$

That is, Theorem 1.14 can not be applied for this example.

By taking $\alpha(z_0, z_1) = 1$ in Theorems 2.2 and 2.3, we get the following:

Corollary 2.5. *Let* $(\mathcal{X}, d)$ *be a complete metric space,* $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ *and* $\mathcal{F} \in \mathfrak{F}$. *If there exists* $\sigma > 0$, *and a function* $\tau : (0,\infty) \to (\sigma,\infty)$ *such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \qquad \text{for all} \quad s \geq 0$$

and for all $z \in \mathcal{X}$ *with* $D(z,\mathcal{T}z) > 0$, *there exists* $y \in \mathcal{F}_\sigma^z$ *satisfying*

$$\tau(d(z,y)) + \mathcal{F}(D(y,\mathcal{T}y)) \leq \mathcal{F}(M(z,y)), \tag{2.27}$$

where $M(z,y)$ *is given in* (2.2). *Then* $\mathcal{T}$ *has a fixed point in* $\mathcal{X}$ *provided that* $z \to D(z,\mathcal{T}z)$ *is lower semi-continuous.*

Corollary 2.6. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. If there exists $\sigma > 0$, and a function $\tau : (0,\infty) \to (\sigma,\infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \qquad \text{for all} \quad s \geq 0$$

and for all $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(D(y,\mathcal{T}y)) \leq \mathcal{F}(M(z,y)), \tag{2.28}$$

where $M(z,y)$ is given in (2.2)*. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided that $z \to D(z,\mathcal{T}z)$ is lower semi-continuous.*

Remark 2.7. *Corollary 2.5 and Corollary 2.6 generalize the Theorem 1.15 and Theorem 1.14, respectively. If we take τ as a constant function, then Corollary 2.5 and Corollary 2.6 is a generalization of Theorem 1.12 and 1.13, respectively.*

As an application of Theorems 2.2 and 2.3, we obtain the following:

Theorem 2.8. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}$. Assume that the following assertions hold:*

1. *$\mathcal{T}$ is generalized α_*-admissible mapping;*
2. *there exists $z_0 \in \mathcal{X}$ and $z_1 \in \mathcal{T}z_0$ such that $\alpha(z_0, z_1) \geq 1$;*
3. *there exists $\tau : (0,\infty) \to (0,\infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > 0 \qquad \text{for all} \quad s \geq 0$$

and for all $z \in \mathcal{X}$ with $H(\mathcal{T}z, \mathcal{T}y) > 0$, there exist a function $\alpha : \mathcal{X} \times \mathcal{X} \to \{-\infty\} \cup (0, +\infty)$ satisfying

$$\tau(d(z,y)) + \alpha(z,y)\mathcal{F}(H(\mathcal{T}z,\mathcal{T}y)) \leq \mathcal{F}(M(z,y)), \tag{2.29}$$

where $M(z,y)$ is defined in (2.2)*.*

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Since $\mathcal{T}$ is continuous if and only if it is both upper and lower semi-continuous, then $\mathcal{T}$ is upper semi-continuous. Therefore, the function $z \to D(z,\mathcal{T}z)$ is lower semi-continuous (see the proposition 4.2.6 of [5]). On the other hand, for any $z \in \mathcal{X}$ with $D(z,\mathcal{T}z) > 0$ and $y \in \mathcal{F}_\sigma^z$, we have

$$\begin{aligned} \tau(d(z,y)) + \alpha(z,y)\mathcal{F}(D(y,\mathcal{T}y)) &\leq \tau(d(z,y)) + \alpha(z,y)\mathcal{F}(H(\mathcal{T}z,\mathcal{T}y)) \\ &\leq \mathcal{F}(M(z,y)). \end{aligned}$$

Thus, all conditions of Theorem 2.2 are satisfied. Hence $\mathcal{T}$ has a fixed point. □

By similar arguments of Theorem 2.8 and using Theorem 2.3, we state the following:

Theorem 2.9. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}_*$ satisfying all assertions of Theorem 2.8. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

By considering $\alpha(z,y) = 1$ reduces Theorems 2.8 and 2.9 to the following:

Corollary 2.10. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}$. If there exists $\tau : (0,\infty) \to (0,\infty)$ such that*

$$\liminf_{t\to s^+} \tau(t) > 0 \qquad \text{for all} \quad s \geq 0$$

and for all $z \in \mathcal{X}$ with $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(H(\mathcal{T}z,\mathcal{T}y)) \leq \mathcal{F}(M(z,y)),$$

where $M(z,y)$ is defined in (2.2)*. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Corollary 2.11. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}_*$ satisfying all assertions of Corollary 2.10. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Remark 2.12. *Corollary 2.10 and Corollary 2.11 generalize the Theorems 1.10 and Theorem 1.11, respectively. If we take τ as a constant function, then Corollary 2.5 and Corollary 2.6 is a generalization of Theorems 1.8 and 1.9, respectively.*

From Theorems 2.8 and 2.9, we get the following fixed point result for single valued mappings:

Theorem 2.13. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}$. Assume that the following assertions hold:*

1. *$\mathcal{T}$ is α-admissible mapping;*
2. *there exists $z_0, z_1 \in \mathcal{X}$ such that $\alpha(z_0, z_1) \geq 1$;*
3. *there exists $\tau : (0,\infty) \to (0,\infty)$ such that*

$$\liminf_{t\to s^+} \tau(t) > 0 \qquad \text{for all} \quad s \geq 0$$

and for all $z \in \mathcal{X}$ with $d(\mathcal{T}z, \mathcal{T}y) > 0$, there exist a function $\alpha : \mathcal{X} \times \mathcal{X} \to \{-\infty\} \cup (0, +\infty)$ satisfying

$$\tau(d(z,y)) + \alpha(z,y)\mathcal{F}(d(\mathcal{T}z,\mathcal{T}y)) \leq \mathcal{F}(m(z,y)), \tag{2.30}$$

where

$$\begin{aligned} m(z,y) = \max\Bigg\{ & d(z,y), d(z,\mathcal{T}z), d(y,\mathcal{T}y), \frac{d(y,\mathcal{T}z) + d(z,\mathcal{T}y)}{2}, \\ & \frac{d(y,\mathcal{T}y)[1 + d(z,\mathcal{T}z)]}{1 + d(z,y)}, \frac{d(y,\mathcal{T}z)[1 + d(z,\mathcal{T}y)]}{1 + d(z,y)} \Bigg\}. \end{aligned} \tag{2.31}$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Remark 2.14. *If we take τ, a constant function in* (2.30)*, Theorem 2.13 generalizes the Theorem 1.8.*

3. Fixed point results in partially ordered metric space

Let $(\mathcal{X}, d, \preceq)$ be a partially ordered metric space and $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued mapping. For $A, B \in 2^{\mathcal{X}}$, $A \preceq B$ implies that $a \preceq b$ for all $a \in A$ and $b \in B$. We say that $\mathcal{T}$ is monotone increasing if $\mathcal{T}y \preceq \mathcal{T}z$ for all $y, z \in \mathcal{X}$, for which $y \preceq z$. There are many applications in differential and integral equations of monotone mappings in ordered metric spaces (see [7, 16, 25] and references therein). In this section, we derive following new results in partially ordered metric spaces from our main results.

Definition 3.1. *Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued mapping on a partially ordered metric space $(\mathcal{X}, d, \preceq)$, then $\mathcal{T}$ is said to be an ordered $\mathcal{F}$-τ-contraction on $\mathcal{X}$, if there exists $\sigma > 0$ and $\tau : (0, \infty) \to (\sigma, \infty)$, $\mathcal{F} \in \mathfrak{F}$ such that for all $z \in \mathcal{X}$, $y \in \mathcal{F}_\sigma^z$ with $z \preceq y$ and $D(z, \mathcal{T}z) > 0$ satisfying*

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(M(z, y)), \tag{3.1}$$

where,

$$M(z, y) = \max\left\{d(z, y), D(z, \mathcal{T}z), D(y, \mathcal{T}y), \frac{D(y, \mathcal{T}z) + D(z, \mathcal{T}y)}{2}, \frac{D(y, \mathcal{T}y)[1 + D(z, \mathcal{T}z)]}{1 + d(z, y)}, \frac{D(y, \mathcal{T}z)[1 + D(z, \mathcal{T}y)]}{1 + d(z, y)}\right\}. \tag{3.2}$$

Theorem 3.2. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space and $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be an ordered $\mathcal{F}$-τ-contraction satisfying the following assertions:*

1. *$\mathcal{T}$ is monotone increasing;*
2. *the map $z \to D(z, \mathcal{T}z)$ is lower semi-continuous;*
3. *there exists $z_0 \in \mathcal{X}$ and $z_1 \in \mathcal{T}z_0$ such that $z_0 \preceq z_1$;*
4. *τ satisfies*

$$\liminf_{t \to s^+} \tau(t) > \sigma \qquad \text{for all} \quad s \geq 0$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Define $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(z, y) = \begin{cases} 1 & z \preceq y \\ 0 & \text{otherwise,} \end{cases}$$

then for $z, y \in \mathcal{X}$ with $z \preceq y$, $\alpha(z, y) \geq 1$ implies $\alpha_*(\mathcal{T}z, \mathcal{T}y) = 1$. This shows that $\mathcal{T}$ is generalized α_*-admissible mapping. Also, from (3.1), we get

$$\begin{aligned} \tau(d(z, y)) + \alpha(z, y)\mathcal{F}(D(y, \mathcal{T}y)) &\leq \tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \\ &\leq \mathcal{F}(M(z, y)). \end{aligned}$$

So, $\mathcal{T}$ is α-type $\mathcal{F}$-τ-contraction. Thus, all the conditions of Theorem 2.2 hold true. Hence, $\mathcal{T}$ has a fixed point in $\mathcal{X}$. □

By similar arguments as in Theorem 3.2 and by using Theorem 2.3, we get the following:

Theorem 3.3. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space and $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ be an ordered $\mathcal{F}$-τ-contraction with $\mathcal{F} \in \mathfrak{F}_*$ satisfying all the assertions of Theorem 3.2. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Theorem 3.4. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}$. Assume that the following assertions hold:*

1. *$\mathcal{T}$ is monotone increasing;*

2. *there exists $z_0 \in \mathcal{X}$ and $z_1 \in \mathcal{T}z_0$ such that $z_0 \preceq z_1$;*

3. *there exists $\tau : (0,\infty) \to (0,\infty)$ such that*

$$\liminf_{t\to s^+} \tau(t) > 0 \qquad \textit{for all} \quad s \geq 0$$

and for all $z, y \in \mathcal{X}$ with $z \preceq y$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(H(\mathcal{T}z,\mathcal{T}y)) \leq \mathcal{F}(M(z,y)), \tag{3.3}$$

where $M(z,y)$ is defined in (3.2).

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. By defining $\alpha : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ as in Theorem (3.2) and by using Theorem (2.8), we get the required result. □

By similar arguments as in Theorem 3.4 and by using Theorem 2.9, we get the following:

Theorem 3.5. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}_*$ satisfying all assertions of Theorem 3.4. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

From Theorems 3.4 and 3.5, we get the following fixed point result for single valued mapping.

Theorem 3.6. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}$. Assume that the following assertions hold:*

1. *$\mathcal{T}$ is monotone increasing;*

2. *there exists $z_0, z_1 \in \mathcal{X}$ such that $z_0 \preceq z_1$;*

3. *there exists $\tau : (0,\infty) \to (0,\infty)$ such that*

$$\liminf_{t\to s^+} \tau(t) > 0 \qquad \textit{for all} \quad s \geq 0$$

and for all $z, y \in \mathcal{X}$ with $z \preceq y$ and $d(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(d(\mathcal{T}z,\mathcal{T}y)) \leq \mathcal{F}(m(z,y)), \tag{3.4}$$

where

$$\begin{aligned} m(z,y) = \max\Bigg\{ & d(z,y), d(z,\mathcal{T}z), d(y,\mathcal{T}y), \frac{d(y,\mathcal{T}z)+d(z,\mathcal{T}y)}{2}, \\ & \frac{d(y,\mathcal{T}y)[1+d(z,\mathcal{T}z)]}{1+d(z,y)}, \frac{d(y,\mathcal{T}z)[1+d(z,\mathcal{T}y)]}{1+d(z,y)} \Bigg\}. \end{aligned} \tag{3.5}$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

4. Application to Non-Linear Matrix Equation

Let $H(n)$ denote the set of all $n \times n$ Hermitian matrices, $P(n)$ the set of all $n \times n$ Hermitian positive definite matrices, $S(n)$ the set of all $n \times n$ positive semidefinite matrices. Instead of $X \in P(n)$ we will write $X > 0$. Furthermore, $X \geq 0$ means $X \in S(n)$. Also we will use $X \geq Y$ $(X \leq Y)$ instead of $X - Y \geq 0$ $(Y - X \geq 0)$. The symbol $\|.\|$ denotes the spectral norm, that is,

$$\|A\| = \sqrt{\lambda^+(A^*A)},$$

where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. We denote by $\|.\|_1$ the Ky Fan norm defined by

$$\|A\|_1 = \sum_{i=1}^{n} s_i(A),$$

where $s_i(A)$, $i = 1, ..., n$, are the singular values of A. Also,

$$\|A\|_1 = tr((A^*A)^{1/2}),$$

which is $tr(A)$ for (Hermitian) nonnegative matrices. Then the set $H(n)$ endowed with this norm is a complete metric space. Moreover, $H(n)$ is a partially ordered set with partial order $\preceq$, where $X \preceq Y \Leftrightarrow Y \leq X$. In this section, denote $d(X, Y) = \|Y - X\|_1 = tr(Y - X)$.
Now, consider the non-linear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i^* \gamma(X) A_i, \tag{4.1}$$

where Q is a positive definite matrix, A_i, $i = 1, ..., m$, are arbitrary $n \times n$ matrices and γ is a mapping from $H(n)$ to $H(n)$ which maps $P(n)$ into $P(n)$. Assume that γ is an order-preserving mapping (γ is order preserving if $A, B \in H(n)$ with $A \preceq B$ implies that $\gamma(A) \preceq \gamma(B)$). Now we prove the following result.

Theorem 4.1. *Let $\gamma : H(n) \to H(n)$ be an order-preserving mapping which maps $P(n)$ into $P(n)$ and $Q \in P(n)$. Assume that there exists a positive number N for which $\sum_{i=1}^{m} A_i A_i^* \prec N I_n$ and $\sum_{i=1}^{m} A_i^* \gamma(Q) A_i \succ 0$ such that for all $X \preceq Y$ we have*

$$d(\gamma(X), \gamma(Y)) \leq \frac{1}{N} m(Y, X) e^{-\left(\frac{2+d(X,Y)}{2d(X,Y)}\right)}, \tag{4.2}$$

where

$$m(X, Y) = \max\left\{d(X, Y), d(X, \mathcal{T}X), d(Y, \mathcal{T}Y), \frac{d(Y, \mathcal{T}Y) + d(X, \mathcal{T}X)}{2}, \frac{d(Y, \mathcal{T}Y)[1 + d(X, \mathcal{T}X)]}{1 + d(X, Y)}, \frac{d(Y, \mathcal{T}X)[1 + d(X, \mathcal{T}Y)]}{1 + d(X, Y)}\right\}.$$

Then 4.1 has a solution in $P(n)$.

Proof. Define $\mathcal{T} : H(n) \to H(n)$ and $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{T}(X) = Q + \sum_{i=1}^{m} A_i^* \gamma(X) A_i \tag{4.3}$$

and $F(r) = \ln r$ respectively. Then a fixed point of $\mathcal{T}$ is a solution of (4.1). Let $X, Y \in H(n)$ with $X \preceq Y$, then $\gamma(X) \preceq \gamma(y)$. So, for $d(X, Y) > 0$ and $\tau(t) = \frac{1}{t} + \frac{1}{2}$, we have

$$\begin{aligned}
d(TX, TY) &= \|\mathcal{T}Y - \mathcal{T}X\|_1 \\
&= tr(\mathcal{T}Y - \mathcal{T}X) \\
&= \sum_{i=1}^{m} tr(A_i A_i^*(\gamma(Y) - \gamma(X))) \\
&= tr\left(\left(\sum_{i=1}^{m} A_i A_i^*\right)(\gamma(Y) - \gamma(X))\right) \\
&\leq \left\|\sum_{i=1}^{m} A_i A_i^*\right\| \|\gamma(Y) - \gamma(X)\|_1 \\
&\leq \frac{\left\|\sum_{i=1}^{m} A_i A_i^*\right\|}{N} m(Y, X) e^{-\left(\frac{2+\|Y-X\|_1}{2\|Y-X\|_1}\right)} \\
&< m(Y, X) e^{-\left(\frac{2+\|Y-X\|_1}{2\|Y-X\|_1}\right)},
\end{aligned}$$

and so,

$$\begin{aligned}
\ln(\|\mathcal{T}Y - \mathcal{T}X\|_1) &< \ln\left(m(Y, X) e^{-\left(\frac{2+\|Y-X\|_1}{2\|Y-X\|_1}\right)}\right) \\
&= \ln(m(X, Y)) - \frac{2 + \|Y - X\|_1}{2\|Y - X\|_1}.
\end{aligned}$$

This implies that

$$\frac{1}{\|Y - X\|_1} + \frac{1}{2} + \ln(\|\mathcal{T}Y - \mathcal{T}X\|_1) < \ln(m(X, Y)).$$

Consequently,

$$\tau(d(X, Y)) + \mathcal{F}(d(TX, TY)) < \mathcal{F}(m(X, Y)).$$

Also, from $\sum_{i=1}^{m} A_i^* \gamma(Q) A_i \succ 0$, we have $Q \preceq \mathcal{T}(Q)$. Thus, by using Theorem 3.6, we conclude that $\mathcal{T}$ has a fixed point and hence 4.1 has a solution in $P(n)$. □

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