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Sehgal-Thomas Type Fixed Point Theorems in Generalized Metric Spaces

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Abstract. A fixed point theorems for pointwise contractive semigroup of self-mappings in setting of generalized metric space are proved. Using the basic result some consequences are derived. This is a generalization of some well known fixed point results in metric spaces.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Fixed point theory is one of the most powerful and useful tools in nonlinear functional analysis. The application of this theory is remarkable in a wide scale of mathematical, engineering, economic, physical, computer science and other fields of science.The intrinsic subject of fixed point theorems is concerned with the conditions for the existence, uniqueness and exact methods of evaluation of fixed point of a mapping. The Banach contraction principle [4] is a simplest and limelight result in this direction. In many papers, following the Banach contraction principle, the existence of weaker contractive conditions combined with stronger additional assumptions on the mapping or on the space, is investigated. Moreover, since all these results are based on an iteration process, they can be implemented in almost all branches of quantitative sciences.

Consideration various generalizations of metric spaces (partial metric spaces, fuzzy metric spaces, probability metric spaces, quasi-metric, uniform spaces, ultra metric spaces, b-metric spaces, cone metric spaces) leads to opportunity to use distinct advantages by creating topological structure suitable for application in some cases when the classical metric does not give the answer.

Sehgal [22] initiated the study of fixed point for mappings with contractive iterate at a point. This result was extended and applied by many authors and we quote some of them [7], [11], [12], [17], [18], [21].

In metric space (X, d), the mapping $T : X \to X$ is said to be with contractive iterate at a point $x \in X$ if there is a positive integer such that for all $y \in X$

$$d(T^{n(x)}x, T^{n(x)}y) \leq q(d(x, y)),$$

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where $q \in (0, 1)$.

In [21], V.M.Sehgal and J.W.Thomas proved a common fixed point result for a family of pointwise contractive self-mappings in metric space (X, d).

Let (X, d) be a complete metric space and $M \subseteq X$. Let F be a commutative semigroup of self-mappings (not necessarily continuous) of M. The semigroup F is pointwise contractive in M if for each $x \in M$, there is an $f_x \in F$ such that

$$d(f_x(y), f_x(x) \leq \varphi(d(y, x)),$$

for all $y \in M$, where φ is some real valued function defined on the nonnegative reals.

Theorem 1.1. *[21] Let M be a closed subset of X and F a commutative semigroup of self-mappings of M, which is pointwise contractive in M for some* $\varphi : [0, \infty) \to [0, \infty)$*, where* φ *is nondecreasing, continuous on the right and satisfies* $\varphi(r) < r$ *for all* $r > 0$*. If for some* $x_0 \in M$*,*

$$\sup\{d(f(x_0), x_0) : f \in F\} < \infty,$$

then, there exists a unique $\xi \in M$ *such that* $f(\xi) = \xi$ *for each* $f \in F$*. Moreover, there is a sequence* $\{g_n\} \subset F$ *with* $g_n(x) \to \xi$ *for each* $x \in M$*.*

In our paper we consider the related result in setting of G-metric spaces. The aim is to show that this result is valid in a more general class of spaces and wide class of functions φ.

2. Preliminaries

On 1963. S. Gähler introduced 2-metric spaces, but other authors proved that there is no relation between two distance functions and there is no easy relationship between results obtained in these two settings. B. C. Dhage introduced a new concept of the measure of nearness between three or more objects. But topological structure of so called D-metric spaces was incorrect. Finally, Z. Mustafa and B. Sims [14] introduced correct definition of a generalized metric space as follows.

Definition 2.1. *[14] Let X be a nonempty set, and let* $G : X \times X \times X \to \mathbb{R}^+$ *be a function satisfying the following properties*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$*;*

(G2) $0 < G(x, x, y)$*, for all* $x, y \in X$*, with* $x \neq y$*;*

(G3) $G(x, x, y) \leq G(x, y, z)$*, for all* $x, y, z \in X$*, with* $z \neq y$*;*

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$*, (symmetry in all three variables);*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$*, for all* $x, y, z, a \in X$*.*

Then function G is called a generalized metric, abbreviated G-metric on X, and the pair (X, G) *is called a G-metric space.*

Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y and $z \in \mathbb{R}^2$, moreover taking a in the interior of the triangle shows that (G5) is the best possible.

Example 1.1[14] Let (X, d) be an ordinary metric space, then (X, d) defines G-metrics on X by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z),$$

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}.$$

Example 1.2[14] Let $X = \{a, b\}$. Define G on $X \times X \times X$ by

$$G(a, a, a) = G(b, b, b) = 0, \quad G(a, a, b) = 1, \quad G(a, b, b) = 2,$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear the (X, G) is a G-metric space.

The following useful properties of a G-metric are readily derived from the axioms.

Proposition 2.2. *[14] Let (X, G) be a G-metric space, then for any x, y, z and a from X it follows that:*

1. *if $G(x, y, z) = 0$, then $x = y = z$,*
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
3. $G(x, y, y) \leq 2G(y, x, x)$,
4. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
5. $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
6. $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 2.3. *[14] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the* limit *of the sequence $\{x_n\}$ if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to x.*

Definition 2.4. *[14] Subset $B \subseteq X$ is bounded if there exists $r > 0$ such that for some $x_0 \in B$, $G(x_0, x_0, x) \leq r$ for all $x \in B$.*

Proposition 2.5. *[14] Let (X, G) be a G-metric space, then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$ the following are equivalent:*

1. *$\{x_n\}$ is G-convergent to x,*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*
3. *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

Definition 2.6. *[14] Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Proposition 2.7. *[14] In a G-metric space (X, G), the following are equivalent:*

1. *the sequence $\{x_n\}$ is G-Cauchy,*
2. *for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq n_0$.*

A G-metric space (X, G) is G-complete (or complete G-metric), if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.8. *[14] Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.9. *(X, G) is symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Fixed point theorems in symmetric G-metric space are mostly consequences of the related fixed point results in metric spaces. In this paper we discuss non-symmetric case.

In [8] it was shown that if (X,G) is a G-metric space, putting $\delta(x, y) = G(x, y, y)$, (X, δ) is a quasi metric space (generally, δ is not symmetric). It is well known that any quasi metric induces different metrics and mostly used are

(μ) $\mu(x, y) = \delta(x, y) + \delta(y, x)$,

(ρ) $\rho(x, y) = \max\{\delta(x, y), \delta(y, x)\}$.

The following result is an immediate consequence of above definitions and relations.

Theorem 2.10. *let (X, G) be a G-metric space and let $D \in \{\delta, \rho\}$. Then*

1. *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, D);*
2. *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, D);*
3. *(X, G) is G-complete if and only if (X, D) is complete.*

Recently, Samet at all [19] and Jleli,Samet [8] observed that some fixed point theorems in context of G-metric space can be proved (by simple transformation) using related existing results in the setting of (quasi) metric space. Namely, if the contraction condition of the fixed point theorem on G-metric space can be reduced to two variables, then one can construct an equivalent fixed point theorem in setting of usual metric space. This idea is not completely new, but it was not successfully used before (see [15]). Karapinar and Agarval in [9] continued to develop Jleli-Samet technique in G-metric space, but, on the other side, they proved fixed point theorems on the context of G-metric space for which Jleli-Samet technique is not applicable. So, in some cases, as it is noticed even in Jleli-Samet paper [8], when the contraction condition is of nonlinear type, this strategy cannot be always successfully used. This is exactly the case in our paper where the use of Jleli-Simet technique does not give satisfactory results. Namely, if the assumption (contraction inequality) imposed on the function f is dependent of the variable $x \in X$, then f_x is not the same function for all $x \in X$, which is the case in Sehgal-Thomas type fixed point theorems. It implies that conditions which the contractor φ in related metric space must satisfy become significantly more restrictive if the Jleli-Simet technique is used. But, using directly G-metric G, the proofs of theorems in our paper are given. The conclusion is that results from our paper cannot be deduced from the usual one in metric or quasi metric space and cannot be derived from the results of Samet et al [19] and Jleli, Samet [8].

For more fixed point results in generalized metric spaces related to our paper, we refer the reader to [1], [2], [3], [6], [18], [20].

3. Main Results

On 1975 Matkowski introduced the following class of mappings:

Definition 3.1. *[10] Let T be a mapping on a metric space (X, d). Then T is called a weak contraction if there exists a function φ from $[0, \infty)$ to itself satisfying the following:*

i) *φ is nondecreasing,*

ii) $\lim_n \varphi^n(t) = 0$ *for all* $t > 0$,

iii) $d(Tx, Ty) \le \varphi(d(x, y))$ *for all* $x, y \in X$.

In the same paper he proved the existence and uniqueness of a fixed point for such type of mappings. This result is significant because the concept of weak contraction of Matkowski type is independent of Meir-Keeler contraction [13], and it was generalized in different directions [1], [12], [11], [16], [17], [18], [20], [23]. Matkowski generalized his own result proving a theorem of Sehgal-Guseman type [7].

Theorem 3.2. *[11] Let (X, d) be a complete metric space, $T : X \to X$, and $\varphi : [0, \infty) \to [0, \infty)$. If φ is nondecreasing, $\lim_{t\to\infty}(t - \varphi(t)) = \infty$, $\lim_{k\to\infty} \varphi^k(t)) = 0$ for $t > 0$, and for each $x \in X$ there is a positive integer $n = n(x)$ such that for all $y \in X$,*

$$d(T^{n(x)}x, T^{n(x)}y) \le \varphi(d(x, y)),$$

then T has a unique fixed point $a \in X$. Moreover, for each $x \in X$, $\lim_{k\to\infty} T^k(x) = a$.

The aim of this paper is to show that this result is valid in a more general class of spaces and wide class of functions φ.

Let (X, G) be a complete G-metric space and $B \subseteq X$. Let $\mathcal{F}$ be a commutative semigroup of self-mappings (not necessarily continuous) of B. The semigroup $\mathcal{F}$ is pointwise contractive in B if for each $x \in B$, there is an $f_x \in \mathcal{F}$ such that

$$G(f_x(y), f_x(x), f_x(x)) \leq \varphi(G(y, x, x)) \tag{3.1}$$

for all $y \in B$, and some mapping $\varphi : [0, +\infty) \to [0, +\infty)$.

Remark 3.3. *A generalization of the contraction principle can be obtained using different type of a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$. The most usual additional properties imposed on φ are:*

(A_1) $\varphi(0) = 0$,

(A_2) *φ is right continuous and $\varphi(t) < t$ for all $t > 0$,*

(A_3) $\lim_{i\to\infty} \varphi^i(t) = 0$, *for all $t > 0$,*

(A_4) $\{t_i\} \subset [0, \infty)$ *is a sequence such that $t_{i+1} \leq \varphi(t_i)$, then $\lim_{i\to\infty} t_i = 0$.*

It is well know that

$$(A_4) \Leftrightarrow (A_3) \Leftrightarrow (A_2) \Rightarrow (A_1).$$

Theorem 3.4. *Let B be a closed subset of complete G-metric space (X, G) and $\mathcal{F}$ a commutative semigroup of self-mapping of B, which is pointwise contractive in B for some $\varphi : [0, \infty) \to [0, \infty)$, where φ is nondecreasing continuous on the right and satisfies $\varphi(t) < t$ for all $t > 0$. If for some $x_0 \in B$*

$$\sup\{G(x_0, x_0, f(x_0)) | \ f \in \mathcal{F}\} < \infty, \tag{3.2}$$

then, there exists a unique $u \in B$ such that $f(u) = u$ for each $f \in \mathcal{F}$. Moreover, there is a sequence $\{g_n\} \subseteq \mathcal{F}$ with $g_n(x) \to u$, for each $x \in B$.

Proof. If $d_0 = \sup\{G(x_0, x_0, f(x_0)) | \ f \in \mathcal{F}\}$, then $\varphi^n(d_0) \to 0, \ n \to \infty$.

Let $f_0 = f_{x_0}$ and inductively $f_n = f_{x_n}$, where $x_{n+1} = f_n(x_n)$. Then, for a fixed integer $k \geq 0$,

$$\sup_{n\geq k} G(x_{n+1}, x_{k+1}, x_{k+1}) = \sup_{n\geq k} G(f_n \circ f_{n-1} \circ ... \circ f_k(x_k), f_k(x_k), f_k(x_k))$$

Let $h_n = f_n \circ f_{n-1} \circ f_{k+1}$. It follows that

$$\begin{aligned} \sup_{n\geq k} G(x_{n+1}, x_{k+1}, x_{k+1}) &= \sup_{n\geq k} G(f_k(h_n(x_k)), f_k(x_k), f_k(x_k)) \\ &\leq \sup_{n\geq k} \varphi(G(h_n(x_k), x_k, x_k)) \\ &\leq \sup_{n\geq k} \varphi^{k+1}(G(h_n(x_0), x_0, x_0)) \leq \varphi^{k+1}(d_0) \to 0, \end{aligned}$$

when $k \to \infty$. Thus, the sequence $\{x_n\}$ is Cauchy. Let $\lim_n x_n = u \in B$. By hypotheses, there is an $f_u \in \mathcal{F}$ such that

$$G(f_u(x_n), f_u(u), f_u(u)) \leq \varphi(G(x_n, u, u)) \to 0, \quad \text{as} \ \ n \to \infty$$

so

$$\lim_n f_u(x_n) = f_u(u)$$

and therefore $\lim_n G(f_u(x_n), x_n, x_n) = G(f_u(u), u, u)$. On the other side

$$\begin{aligned} G(f_u(x_n), x_n, x_n) &\leq \varphi(G(f_u(x_{n-1}), x_{n-1}, x_{n-1})) \\ &\leq \varphi^n(G(f_u(x_0), x_0, x_0)) \\ &\leq \varphi^n(d_0) \to 0,\ n \to \infty. \end{aligned}$$

Hence $f_u(u) = u$. Using (3.1) one can prove that u is a unique fixed point of f_u on B. Furthermore, since $\mathcal{F}$ is commutative, for any $f \in \mathcal{F}$ we have that

$$f(u) = f(f_u(u)) = f_u(f(u))$$

and therefore $f(u) = u$, for each $f \in \mathcal{F}$.

For each nonnegative integer n, set $g_n = f_n \circ f_{n-1} \circ ... \circ f_0$. Obviously $g_n \in \mathcal{F}$. We are going to prove that $g_n(x) \to u,\ n \to \infty$, for each $x \in B$. For any fixed $x \in B$

$$G(g_n(x), u, u) \leq G(g_n(x), x_{n+1}, x_{n+1}) + G(x_{n+1}, u, u).$$

Since $x_n \to u,\ n \to \infty$, it is suffices to prove that $G(g_n(x), x_{n+1}, x_{n+1}) \to 0,\ n \to \infty$. However

$$\begin{aligned} G(g_n(x), x_{n+1}, x_{n+1}) &= G(f_n(g_{n-1}(x)), f_n(x_n), f_n(x_n)) \\ &\leq \varphi(G(g_{n-1}(x), x_n, x_n)) \\ &\leq \varphi^{n+1}(G(x, x_0, x_0)) \to 0,\ n \to \infty. \end{aligned}$$

Thus $G(g_n(x), x_{n+1}, x_{n+1}) \to 0,\ n \to \infty$ and proof is completed. □

If B is a bounded subset of X, condition (3.2) holds for each $x_0 \in B$.

Corollary 3.5. *Let B be a closed bounded subset of complete G-metric space (X, G), and $\mathcal{F}$ a commutative semigroup of self-mappings of B which is pointwise contractive in B, for some $\varphi : [0, \infty) \to [0, \infty)$ where φ is nondecreasing right continuous function and $\varphi(t) < t,\ t > 0$. Then there exists a unique $u \in B$ and sequence $\{g_n\} \subseteq \mathcal{F}$ such that $f(u) = u$ for any $f \in \mathcal{F}$ and $g_n(x) \to u,\ n \to \infty$, for each $x \in B$.*

Corollary 3.6. *Let B be a closed bounded subset of complete G-metric space (X, G) and f a self-mapping of B. If f satisfies the condition: for each $x \in B$, there exists on integer $n(x) \geq 1$ such that for all $y \in B$*

$$G\left(f^{n(x)}(y), f^{n(x)}(x), f^{n(x)}(x)\right) \leq \varphi(G(y, x, x)) \tag{3.3}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is nondecreasing right continuous function and $\varphi(t) < t$ for $t > 0$, then there exists a unique $u \in B$ such that $f(u) = u$ and $\lim_k f^k(x) = u$, for any $x \in B$.

Proof. Family $\mathcal{F} = \{f^k |\ k \in \mathbb{N}\}$ is a commutative semigroup pointwise contractive in B, so by Corollary 3.5, there exists an unique fixed point u of f and there is $\{g_n\} \subseteq \{f^k |\ k \in \mathbb{N}\}$ such that $\lim_n g_n(x) = u$ for any $x \in B$. Let us prove that in fact $\lim_k f^k(x) = u$, for any $x \in B$. For k sufficiently large we have $k = r \cdot n(u) + s$, with $r > 0$ and $0 \leq s < n(u)$ and therefore

$$\begin{aligned} G\left(f^k(x), u, u\right) &= G\left(f^{r \cdot n(u)+s}(x), f^{n(u)}(u), f^{n(u)}(u)\right) \leq \varphi^r(G(f^s(x), u, u)) \\ &\leq \varphi^r(d_0), \end{aligned}$$

for $d_0 = diam\ B$. Since $\varphi^r(d_0) \to 0,\ r \to \infty$, it follows that $\lim_k f^k(x) = u$. □

Remark 3.7. *For a bounded G-metric spaces, Corollary 3.6. improves T.M.Sehgal [22] and L.F. Guseman [7] results.*

Following Matkowski [11], we provide boundness of the orbit with some additional condition on contractive function φ. In the next two theorems we shall use, among the other assumptions, properties (A_5) and (A_6) of φ:

(A_5) for any $y \geq 0$ there exists a $t(y) \geq 0,\ t(y) = \sup_{t\geq 0}\{t \leq y + \varphi(t)\}$,

(A_6) $\lim_{t\to\infty}(t - \varphi(t)) = \infty$.

It is easy to show that $(A_5) \Leftrightarrow (A_6)$.

Theorem 3.8. *Let (X, G) be a complete G-metric space, $f : X \to X$, where nondecreasing right continuous function φ satisfies (A_5) or (A_6) together with (A_2) or (A_3) and for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$G\left(f^{n(x)}(x), f^{n(x)}(x), f^{n(x)}(y)\right) \leq \varphi\left(G(x, x, y)\right), \tag{3.4}$$

for all $y \in X$. Then f has a unique fixed point $u \in X$. Moreover, for each $x \in X$, $\lim_k f^k(x) = u$.

Proof. Boundness of the orbit $\{f^k(x)\}_k$, for every $x \in X$, will be proved by mathematical induction.

Let conditions (A_3) and (A_6) be satisfied (weak contraction in the sense of Matkowski). Fix $x \in X$, fix integer s, $0 \leq s < n = n(x)$ and put

$$\begin{aligned} u_k &= G\left(x, x, f^{kn(x)+s}(x)\right), \quad k = 0, 1, 2, ..., \\ h &= \max\left\{G(x, x, f^{n(x)}(x)),\ G(x, x, f^s(x))\right\}. \end{aligned}$$

By (A_6) there exists $c,\ c > h$, such that

$$t - \varphi(t) > h,\ t > c.$$

The last inequalities imply that $u_0 < c$. Suppose that there exists a positive integer j such that $u_j \geq c$, but $u_i < c$ for $i < j$.

Using (3.4), we get

$$\begin{aligned} u_j &= G\left(x, x, f^{jn(x)+s}(x)\right) \\ &\leq G\left(x, x, f^{n(x)}(x)\right) + G\left(f^{n(x)}(x), f^{n(x)}(x), f^{jn(x)+s}(x)\right) \\ &\leq h + \varphi(u_{j-1}) \leq h + \varphi(u_j), \end{aligned}$$

i.e. $u_j - \varphi(u_j) \leq h$ which contradicts the choice of c. Therefore $u_j < c$ for $j = 0, 1, ...$, and consequently the orbit $\{f^k(x)\}_k$ is bounded, so $\sup_k G\left(x, x, f^k(x)\right) = M < \infty$.

Now, we can apply Corollary 2.2, which finishes the proof. □

Remark 3.9. *If (X, G) is symmetric G-metric space, (3.4) becomes*

$$\rho(f^{n(x)}(x), f^{n(x)}(y)) \leq \varphi(\rho(x, y))$$

and the proof follows immediately from Matkowski fixed point theorem [11].

Theorem 3.10. *Let f be a self-mapping of a complete G-metric space (X, G). If there exists a subset B of X such that $f(B) \subseteq B$, f satisfies (3.4) over B, where nondecreasing right continuous function φ satisfies (A_5) or (A_6) together with (A_2) or (A_3) and for some $x_0 \in X$, $\overline{\{f^n(x_0) : n \geq 1\}} \subseteq B$, then there exists a unique $u \in B$ such that $f(u) = u$ and $\lim_{k\to\infty} f^k(y) = u$ for each $y \in B$. Furthermore, if f satisfies (3.4) over X, then u is unique fixed point in X and $\lim_{k\to\infty} f^k(y) = u$ for each $y \in X$.*

Remark 3.11. *Taking $\varphi(t) = q \cdot t,\ 0 < q < 1$, by Theorem 3.10 we obtain the fixed point result from [5], so Theorem 3.10 is also a generalization of Guseman fixed point result from [7].*

The next theorem is also a Guseman type of fixed point theorem in G- metric space. The assumptions about contractor φ is different with respect to Theorem 3.10 Similarly as in previous analysis, the next theorem can be applied in metric space and also in cases where some special form of function φ is used.

Theorem 3.12. *Let $f : X \to X$, where (X, G) is G-metric space and let $\varphi : [0, \infty) \to [0, \infty)$ be a subadditive mapping satisfying $\sum_{i=1}^{\infty} \varphi^i(t) < \infty$ for all $t > 0$. If for some $x_0 \in X$ the closure of orbit $\overline{O(f; x_0)}$ is complete and for each $x \in O(f; x_0)$ there exists an $n(x) \in \mathbb{N}$ such that*

$$G\left(f^{n(x)}(y), f^{n(x)}(x), f^{n(x)}(x)\right) \le \varphi\left(G(y, x, x)\right), \tag{3.5}$$

for all $y \in O(f; x_0)$, then the sequence $x_{i+1} = f^{n(x_i)}(x_i)$, $i \in \mathbb{N}_0$, converges to some $x^* \in X$.

If inequality (3.5) holds for all $x \in \overline{O(f; x_0)}$, then $f^{n(x^*)}(x^*) = x^*$ and $\lim_i f^i(x) = x^*$ for every $x \in \overline{O(f; x_0)}$. If $f(\overline{O(f; x_0)}) \subseteq \overline{O(f; x_0)}$, then x^* is the fixed point of f.

Proof. First, we show that $\{x_i\}_{i\in\mathbb{N}_0} \subset X$ is a Cauchy sequence. For sufficiently large $m \in \mathbb{N}$, there exist $k, r \in \mathbb{N}$, $1 \le r < n(x_0)$ such that $m = k \cdot n(x_0) + r$. Using (3.5), we get

$$\begin{aligned}
G\left(f^m(x_0), x_0, x_0\right) &\le G\left(f^{kn(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\le \varphi\left(G\left(f^{(k-1)n(x_0)+r}(x_0), x_0, x_0\right)\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\le \varphi\left(G\left(f^{(k-1)n(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right)\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\le \varphi^2\left(G\left(f^{(k-2)n(x_0)+r}(x_0), x_0, x_0\right)\right) + \varphi\left(G\left(f^{n(x_0)}(x_0), x_0, x_0\right)\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\le \cdots \\
&\le \varphi^k\left(G\left(f^r(x_0), x_0, x_0\right)\right) + \sum_{i=1}^{k-1} \varphi^i\left(G\left(f^{n(x_0)}(x_0), x_0, x_0\right)\right).
\end{aligned}$$

Putting $A = \max\{G\left(f^p(x_0), x_0, x_0\right) : 1 \le p \le n(x_0)\}$, for all $m \in \mathbb{N}$ the next inequality holds

$$G\left(f^m(x_0), x_0, x_0\right) \le \sum_{s=1}^{k} \varphi^s(A) \le \sum_{s=1}^{\infty} \varphi^s(A) = B < \infty, \tag{3.6}$$

and consequently,

$$\begin{aligned}
G\left(x_m, x_m, x_{m+1}\right) &= G\left(f^{n(x_{m-1})}(x_{m-1}), f^{n(x_{m-1})}(x_{m-1}), f^{n(x_m)} f^{n(x_{m-1})}(x_{m-1})\right) \\
&\le \varphi\left(G\left(x_{m-1}, x_{m-1}, f^{n(x_m)}(x_{m-1})\right)\right) \\
&\le \cdots \le \varphi^m\left(G\left(x_0, x_0, f^{n(x_m)}(x_0)\right)\right) \le \varphi^m(B)
\end{aligned}$$

for all $m \in \mathbb{N}$. Using the last inequality, for every $i, j \in \mathbb{N}$, $i < j$, we have

$$G\left(x_i, x_i, x_j\right) \le G\left(x_i, x_i, x_{i+1}\right) + \cdots + G\left(x_{k-1}, x_{k-1}, x_k\right) \le \sum_{s=i}^{j} \varphi^s(B)$$

implying that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Since $\overline{O(f; x_0)}$ is a complete, there exists an $x^* \in \overline{O(f; x_0)}$ such that $\lim_{i\to\infty} x_i = x^*$.

If the inequality (3.5) holds for all $x \in \overline{O(f; x_0)}$, then the elements x_i of the sequence $\{x_i\}_{i\in\mathbb{N}}$ from the previous part of the proof, satisfy next two relations

$$G\left(f^{n(x^*)}(x^*), f^{n(x^*)}(x^*), f^{n(x^*)}(x_i)\right) \le \varphi\left(G\left(x^*, x^*, x_i\right)\right) < G\left(x^*, x^*, x_i\right), \tag{3.7}$$

and

$$G\left(f^{n(x^*)}(x_i), x_i, x_i\right) = G\left(f^{n(x^*)} f^{n(x_{i-1})}(x_{i-1}), f^{n(x_{i-1})}(x_{i-1}), f^{n(x_{i-1})}(x_{i-1})\right) \le$$

$$\varphi\left(G\left(f^{n(x^*)}(x_{i-1}), x_{i-1}, x_{i-1}\right)\right) \le \varphi^i\left(G\left(f^{n(x^*)}(x_0), x_0, x_0\right)\right). \tag{3.8}$$

By (3.7)

$$\lim_{i\to\infty} f^{n(x^*)}(x_i) = f^{n(x^*)}(x^*)$$

and by (3.8)

$$\lim_{i\to\infty} G\left(f^{n(x^*)}(x_i), x_i, x_i\right) = G\left(f^{n(x^*)}(x^*), x^*, x^*\right) = 0.$$

Hence, $f^{n(x^*)}(x^*) = x^*$.

Next, we claim that $\lim_i f^i(x) = x^*$, for each $x \in \overline{O(f;x_0)}$. Putting $i = kn(x^*) + s$, $s \in \mathbb{N}$, $0 \le s < n(x^*)$, we get

$$G\left(f^{kn(mz)+s}(x), x^*, x^*\right) \le \varphi(G\left(f^{(k-1)n(mz)+s}(x), x^*, x^*\right) \le \cdots$$

$$\le \varphi^k(G\left(f^s(x), x^*, x^*\right) = \varphi^k(M),$$

where $M = \max\{G\left(f^s(x), x^*, x^*\right) \ : \ 0 \le s < n(x^*)\}$. Since $\sum_i \varphi^i(t) < \infty \Rightarrow \lim_i \varphi^i(t) = 0$, $\lim_i f^i(x) = x^*$.

To show that x^* is a unique fixed point of $f^{n(x^*)}$ in $\overline{O(f;x_0)}$, we assume that there exists another point $x^{**} \in \overline{O(f;x_0)}$ with the same property. Then

$$G\left(x^{**}, x^*, x^*\right) = G\left(f^{n(x^*)}(x^{**}), f^{n(x^*)}(x^*), f^{n(x^*)}(x^*)\right) \le \varphi(G\left(x^{**}, x^*, x^*\right)),$$

that is $x^{**} = x^*$. Further, if $f(\overline{O(f;x_0)}) \subseteq \overline{O(f;x_0)}$, then

$$f(x^*) = f(f^{n(x^*)}(x^*)) = f^{n(x^*)}(f(x^*))$$

implying $f(x^*) = x^*$. □

4. Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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