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Fixed Point Results On θ-metric Spaces via Simulation Functions

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Abstract. In a recent article, Khojasteh et al. introduced a new class of simulation functions, $\mathcal{Z}$-contractions, with blending over known contractive conditions in the literature. Subsequently, in this paper, we extend and generalize the results in θ-metric context and we discuss some fixed point results in connection with existing ones. Also, we originate the notion of modified $\mathcal{Z}$-contractions and explore the existence and uniqueness of fixed points of such functions on the said spaces. Finally we include examples to instantiate our main results.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

With extensive and manifold applications, fixed point theory has been one of the most influential research topics in various fields of engineering and science. The most incredible result in this direction was stated by Banach, known as the Banach contraction principle [2]. This remarkable result has been generalized and extended in various abstract spaces using different conditions. However, the prospect of fixed point theory charmed many researchers and so there is a vast literature available for readers [3–7, 10, 11].

One of the most impressive generalizations of the notion of a metric is the concept of a fuzzy metric. Motivated from the definition of fuzzy metric spaces, recently Khojasteh et al. [8] introduced θ-metric by replacing the triangle inequality with a more generalized inequality.

Of late, Khojasteh et al. [9] introduced the concept of $\mathcal{Z}$-contractions by using simulation functions. This class of functions has received much recognition as these are convenient to exhibit a huge family of contractivity conditions that are renowned in fixed point theory. Later on, Olgun et al. [13] provided a new class of Picard operators on complete metric spaces using the concept of generalized $\mathcal{Z}$-contractions. In this exciting context, a lot of developments have been done in recent times [1, 12, 14].

In this manuscript, we use $\mathcal{Z}$-contractions to obtain the results on existence and uniqueness of fixed point in θ-metric spaces. Also, we introduce the concept of modified $\mathcal{Z}$-contractions there and go on to derive a fixed point result using them in the said spaces. Our main results are equipped with competent examples.

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This document unfolds with preliminaries section, where we review some definitions, examples and notable results that are involved in the sequel. The main results section comprises of some lemmas and fixed point results. These results extend, unify and generalize several results in the existing literature. Further we furnish some non-trivial examples to elicit the usability of the obtained theorems.

2. Preliminaries

At the outset, we dash some basic definitions and fundamental results off here. In the rest of this paper, $\mathbb{N}$ will stand for the set of all positive integers and $\mathbb{R}$ will denote the set of all real numbers.

Let $T : X \to X$ be a self-mapping. We say $x \in X$ is a fixed point of T if $Tx = x$.

The following notion of simulation functions was first introduced by Khojasteh et al. in [9].

Definition 2.1. *[9] The function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a simulation function, if the following properties hold:*

(ζ1) $\zeta(0,0) = 0$,

(ζ2) $\zeta(t,s) < s - t$ *for all* $s,t > 0$,

(ζ3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

The authors provided a wide range of examples of simulation functions to emphasize the promising applicability to the literature of fixed point theory. We list a few here.

Example 2.2. *[9] Suppose $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}, i = 1,2$ defined as:*

1. $\zeta_1(t,s) = \frac{s}{s+1} - t$ *for all* $t,s \in [0,\infty)$.
2. $\zeta_2(t,s) = \eta(s) - t$ *for all* $t,s \in [0,\infty)$, *where* $\eta : [0,\infty) \to [0,\infty)$ *be an upper semi continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$.
3. $\zeta_3(t,s) = s - \phi(s) - t$ *for all* $s,t \in [0,\infty)$ *where* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\phi(t) = 0 \Leftrightarrow t = 0$.

The collection of all the simulation functions is denoted by $\mathcal{Z}$.

Remark 2.3. *In recent times, Roldán et al. [14] slightly modified the previous definition and enlarged the family of simulation functions by changing (ζ3). It is redefined as:*

A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is said to be a simulation function, if it meets (ζ1), (ζ2) and

(ζ'3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

and $t_n < s_n$ *for all* $n \in \mathbb{N}$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

It is worthy to mention that every simulation function in the Khojasteh et al.'s sense (Definition 2.1) is also a simulation function in Roldán et al.'s (Remark 2.3) sense, but the converse is not true, see for example [14].

Remark 2.4. *Argoubi et al. [1] revised the previous Definition 2.1 a little by removing the condition (ζ1). For the sake of simplicity, we consider the following definition: A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the conditions (ζ2) and (ζ3) only.*

It has not escaped our notice that every simulation function in Khojasteh et al.'s sense (Definition 2.1) is a simulation function in Argoubi et al.'s sense (Remark 2.4), but the converse is not true, see [1].

Definition 2.5. *[9] Suppose $T : X \to X$ be any self-mapping and $\zeta \in \mathcal{Z}$ be a simulation function. Then T is said to be a $\mathcal{Z}$-contraction with respect to ζ, if for all $x, y \in X$,*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0$$

holds.

The Banach contraction is a perfect example of $\mathcal{Z}$-contraction. It satisfies the previous non-negativity restriction by taking $\zeta(t, s) = \lambda s - t$, where $\lambda \in [0, 1)$, as the corresponding simulation function.

Despite the examples in Example 2.2, there are several other examples of simulation functions and $\mathcal{Z}$-contractions, which can be found in [9].

Remark 2.6 (cf. [9]). *It can be easily said from the definition of the simulation function that for all $t \geq s > 0$, $\zeta(t, s) < 0$. So, if T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then for all $x, y \in X$,*

$$d(Tx, Ty) < d(x, y)$$

whenever $x \neq y$. This leads us to the conclusion that every $\mathcal{Z}$-contraction is contractive and hence continuous.

For our purposes, we need to enunciate the ideas of B-actions and θ-metrics here. In 2013, Khojasteh et al. [8] proposed the notion of θ-metric as a proper generalization of a metric.

Definition 2.7. *[8] Let $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a continuous mapping with respect to both the variables. Let $Im(\theta) = \{\theta(s, t) : s \geq 0, t \geq 0\}$. The mapping θ is called an B-action if and only if the following conditions hold:*

(B1) $\theta(0, 0) = 0$ *and* $\theta(s, t) = \theta(t, s)$ *for all* $s, t \geq 0$,

(B2)

$$\theta(s, t) < \theta(u, v) \quad \Rightarrow \quad \begin{cases} \text{either } s < u, t \leq v \\ \text{or } s \leq u, t < v, \end{cases}$$

(B3) for each $r \in Im(\theta)$ and for each $s \in [0, r]$, there exists $t \in [0, r]$ such that $\theta(t, s) = r$,

(B4) $\theta(s, 0) \leq s$, *for all* $s > 0$.

Example 2.8. *[8] The subsequent examples illustrate the definition.*

1. $\theta_1(s, t) = \frac{ts}{1+ts}$.
2. $\theta_2(s, t) = t + s + \sqrt{ts}$.

The set of all B-actions is denoted by Y.

The idea of B-action has been very much functional to formulate the notion of θ-metric spaces [8]. We here recall the definition of the said spaces.

Definition 2.9. *[8] Let X be a non-empty set. A mapping $d_\theta : X \times X \to [0, \infty)$ is called a θ-metric on X with respect to B-action $\theta \in Y$ if d_θ satisfies the following:*

(θ1) $d_\theta(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$,

(θ2) $d_\theta(x, y) = d_\theta(y, x)$ *for all* $x, y \in X$,

(θ3) $d_\theta(x, y) \leq \theta(d_\theta(x, z), d_\theta(z, y))$ *for all* $x, y, z \in X$.

Then the pair (X, d_θ) is called a θ-metric space.

Example 2.10. *[8] Here we provide a non-trivial example of θ-metric space.*
Let $X = \{x, y, z\}$ and $d_\theta : X \times X \to [0, \infty)$ is defined as:

$$d_\theta(x, y) = 5, d_\theta(y, z) = 12, d_\theta(z, x) = 13, d_\theta(x, y) = d_\theta(y, x),$$

$$d_\theta(y, z) = d_\theta(z, y), d_\theta(z, x) = d_\theta(x, z), d_\theta(x, x) = d_\theta(y, y) = d_\theta(z, z) = 0.$$

Taking $\theta(s, t) = \sqrt{s^2 + t^2}$, the mapping d_θ forms a θ-metric. And hence the pair (X, d_θ) is a θ-metric space.

Remark 2.11 (cf. [8]). *If (X, d_θ) is a θ-metric space and $\theta(s, t) = s + t$, for all $s, t \in [0, \infty)$, then (X, d_θ) is a metric space. Also we mention that a metric space is included in the class of θ-metric spaces if we consider the θ-metric as $\theta(s, t) = s + t$, for all $s, t \in [0, \infty)$.*

For further terminologies and derived results, see [8].

3. Main Results

In this section, we prove some fixed point theorems for self-mappings via simulation functions owing to the concept of θ-metric spaces and also we give illustrative examples. Before all else, we start with noting down following lemmas which will be crucial to our main results.

Lemma 3.1. *Let (X, d_θ) be any complete θ-metric space. Suppose $T : X \to X$ be any given $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T is an asymptotically regular mapping at any arbitrary $x \in X$.*

Proof. Let $x \in X$. With no loss of generality, we take $T^n x \neq T^{n+1} x$ for all $n \in \mathbb{N}$. Taking into account Remark 2.6, we have,

$$d_\theta(T^n x, T^{n+1} x) < d_\theta(T^{n-1} x, T^n x)$$

for all $n \in \mathbb{N}$. So $\{d_\theta(T^n x, T^{n+1} x)\}$ is a decreasing sequence of non-negative reals. Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d_\theta(T^n x, T^{n+1} x) = r.$$

Our claim is that $r = 0$. As T is a $\mathcal{Z}$-contraction with respect to ζ, we get,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d_\theta(T^{n+1} x, T^n x), d_\theta(T^n x, T^{n-1} x)) \\ &< 0. \end{aligned}$$

This contradiction proves that $r = 0$ and hence

$$\lim_{n\to\infty} d_\theta(T^n x, T^{n+1} x) = 0.$$

So T is an asymptotically regular mapping at every $x \in X$. □

Lemma 3.2. *Let (X, d_θ) be any complete θ-metric space. Suppose $T : X \to X$ be any $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then whenever T has a fixed point in X, it is unique.*

Proof. Let $u \in X$ be any fixed point of T. We take $v \in X$ as another fixed point of T. Therefore $Tu = u$ and $Tv = v$. Now by using $(B4)$ and $(\theta 3)$, we obtain

$$\begin{aligned} d_\theta(u, v) &= d_\theta(Tu, Tv) \\ &\leq \theta(d_\theta(Tu, u), d_\theta(u, Tv)) \\ &= \theta(d_\theta(u, u), d_\theta(u, Tv)) \\ &\leq d_\theta(u, Tv) \\ &\leq \theta(d_\theta(u, v), d_\theta(v, Tv)) \\ &\leq \theta(d_\theta(u, v), d_\theta(v, v)) \\ &\leq d_\theta(u, v). \end{aligned}$$

Since T is $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, owing to the Remark 2.6, above inequality turns out to be a contradiction and hence the theorem. □

The first main result of this article is the following one.

Theorem 3.3. *Let (X, d_θ) be a complete θ-metric space. Assume that $T : X \to X$ is any $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point u in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ converges to u.*

Proof. Let $x_0 \in X$ and $\{x_n\}$ be the corresponding Picard sequence, i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We claim that the sequence $\{x_n\}$ is bounded.

Reasoning by contradiction, we assume that $\{x_n\}$ is unbounded. So, we can construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for every $k \in \mathbb{N}$, n_{k+1} is the least integer such that

$$d_\theta(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d_\theta(x_m, x_{n_k}) \leq 1 \tag{1}$$

for $n_k \leq m \leq n_{k+1} - 1$. Now, using the triangle inequality ($\theta 3$) and (1), we have

$$\begin{aligned} 1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\ &\leq \theta(d_\theta(x_{n_{k+1}}, x_{n_{k+1}-1}), d_\theta(x_{n_{k+1}-1}, x_{n_k})) \\ &\leq \theta(d_\theta(x_{n_{k+1}}, x_{n_{k+1}-1}), 1). \end{aligned} \tag{2}$$

Letting $k \to \infty$ on both sides of (2) and then using Lemma 3.1 and ($B4$), we deduce that,

$$d_\theta(x_{n_{k+1}}, x_{n_k}) \to 1.$$

On the other hand, using ($\theta 3$) and (1), we derive that

$$\begin{aligned} 1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\ &\leq d_\theta(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq \theta(d_\theta(x_{n_{k+1}-1}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})) \\ &\leq \theta(1, d_\theta(x_{n_k}, x_{n_k-1})). \end{aligned}$$

So, as $k \to \infty$, we get,

$$d_\theta(x_{n_{k+1}-1}, x_{n_k-1}) \to 1.$$

For T is a $\mathcal{Z}$-contraction and $\zeta \in \mathcal{Z}$ is the respective simulation function, we derive that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(d_\theta(Tx_{n_{k+1}-1}, Tx_{n_k-1}), d_\theta(x_{n_{k+1}-1}, x_{n_k-1})) \\ &= \limsup_{k\to\infty} \zeta(d_\theta(x_{n_{k+1}}, x_{n_k}), d_\theta(x_{n_{k+1}-1}, x_{n_k-1})) \\ &< 0, \end{aligned}$$

and we arrive at a contradiction. So, the Picard sequence $\{x_n\}$ is bounded.

Now we will show that $\{x_n\}$ is Cauchy. For this, let

$$C_n = sup\{d_\theta(x_i, x_j) : i, j \geq n\}.$$

Note that $\{C_n\}$ is a decreasing sequence of non-negative reals. Thus there exists a $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C.$$

Our claim is that $C = 0$. Let us suppose that $C > 0$. Considering the formation of C_n, for any $k \in \mathbb{N}$, there exists n_k, m_k such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d_\theta(x_{m_k}, x_{n_k}) \leq C_k.$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} d_\theta(x_{m_k}, x_{n_k}) = C.$$

Now,

$$\begin{aligned} d_\theta(x_{m_k}, x_{n_k}) &\leq d_\theta(x_{m_k-1}, x_{n_k-1}) \\ &\leq \theta(d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{m_k}, x_{n_k-1})) \\ &\leq \theta(d_\theta(x_{m_k-1}, x_{m_k}), \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1}))). \end{aligned}$$

Letting $k \to \infty$ in the previous inequality and applying $(B4)$, we derive

$$\begin{aligned} C &\leq \lim_{k\to\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) \\ &\leq \theta(0, \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1}))) \\ &\leq \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})). \end{aligned} \tag{3}$$

Again taking limit as $k \to \infty$ in (3) and using $(B4)$, we get

$$\begin{aligned} C &\leq \lim_{k\to\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) \\ &\leq \theta(0, C) \\ &\leq C. \end{aligned}$$

As a consequence,

$$\lim_{k\to\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) = C.$$

As T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we derive that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(d_\theta(x_{m_k-1}, x_{n_k-1}), d_\theta(x_{m_k}, x_{n_k})) \\ &< 0, \end{aligned}$$

which is a contradiction. Consequently, $\{x_n\}$ is Cauchy.

Since (X, d_θ) is complete, there exists some $z \in X$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now we show that z is a fixed point of T. Conversely suppose, $Tz \neq z$. Then $d_\theta(z, Tz) > 0$. Again,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d_\theta(Tx_n, Tz), d_\theta(x_n, z)) \\ &\leq \limsup_{n\to\infty} [d_\theta(x_n, z) - d_\theta(x_{n+1}, Tz)] \\ &= -d_\theta(z, Tz). \end{aligned}$$

This contradiction proves that $d_\theta(z, Tz) = 0$, and hence, $Tz = z$. So we can conclude that z is a fixed point of T. Uniqueness is guaranteed from Lemma 3.2. □

Now we validate our fixed point result by the following examples.

Example 3.4. *Let* $X = [0,1]$ *be endowed with the Euclidean metric* $d_\theta(x,y) = |x-y|$. *Also we take* $\theta(s,t) = s+t+st$.
We define a mapping $T : X \to X$ *by* $Tx = \frac{x}{a} + b$, *where* $a > 1, x \in X$ *and* $b + \frac{1}{a} < 1$.
So we have,

$$\begin{aligned} d_\theta(Tx,Ty) &= |Tx - Ty| \\ &= |\frac{x}{a} + b - \frac{y}{a} - b| \\ &= \frac{1}{a}|x-y|. \end{aligned}$$

We claim that T is a $\mathcal{Z}$*-contraction with respect to the simulation function* $\zeta(t,s) = \lambda s - t$, *where* $\lambda > \frac{1}{a}$ *for all* $t, s \in [0,\infty)$.
So we have,

$$\begin{aligned} \zeta(d_\theta(Tx,Ty), d_\theta(x,y)) &= \lambda d_\theta(x,y) - d_\theta(Tx,Ty) \\ &= \lambda|x-y| - \frac{1}{a}|x-y| \\ &= (\lambda - \frac{1}{a})|x-y| \\ &\geq 0. \end{aligned}$$

Taking into account Theorem 3.3 we get, T has a unique fixed point and it is $u = \frac{ab}{a-1}$.
Since $b + \frac{1}{a} < 1$, *it is ensured that* $u \in X$.

Example 3.5. *Let* $X = [0,1]$ *be endowed with the Euclidean metric and* $\theta(s,t) = s + t + st$.
We define a mapping $T : X \to X$ *by* $Tx = \frac{1}{1+x}$, $x \in X$.
Our claim is that T is a $\mathcal{Z}$*-contraction with respect to the simulation function* $\zeta(t,s) = \frac{s}{s+1} - t$, *for all* $t, s \in [0,\infty)$.
So we have,

$$\begin{aligned} \zeta(d_\theta(Tx,Ty), d_\theta(x,y)) &= \frac{d_\theta(x,y)}{d_\theta(x,y)+1} - d_\theta(Tx,Ty) \\ &= \frac{|x-y|}{|x-y|+1} - |\frac{1}{x+1} - \frac{1}{y+1}| \\ &= \frac{|x-y|}{|x-y|+1} - \frac{|x-y|}{|x+1||y+1|} \\ &= |x-y|(\frac{1}{|x-y|+1} - \frac{1}{|x+1||y+1|}) \\ &\geq 0. \end{aligned}$$

Hence applying Theorem 3.3, T has a unique fixed point and it is $u = \frac{\sqrt{5}-1}{2} \in X$.

Here we introduce the new class of modified $\mathcal{Z}$-contractions.

Definition 3.6. *Let* (X, d_θ) *be a* θ*-metric space. We assume that a mapping* $T : X \to X$ *satisfies the following condition:*

$$\zeta(d_\theta(Tx,Ty), M(x,y)) \geq 0$$

for all $x, y \in X$, *where,*

$$M(x,y) = max\{d_\theta(x,y), d_\theta(x,Tx), d_\theta(y,Ty)\}.$$

Then T is said to be a modified $\mathcal{Z}$*-contraction with respect to* ζ.

Example 3.7. *Let $X = [0,1]$ be endowed with the Euclidean metric and $\theta(s,t) = s + t + st$. We define a mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{1}{7}, & x \in [0, \frac{1}{2}), \\ \frac{2}{7}, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then T is a modified $\mathcal{Z}$-contraction with respect to the simulation function $\zeta(t,s) = \frac{7}{8}s - t$.

Remark 3.8. *A modified $\mathcal{Z}$-contraction T is not necessarily continuous. The Example 3.7 completely validates our claim. Also, this fact indicates that T is not a $\mathcal{Z}$-contraction. So Theorem 3.3 is not applicable here.*

In this context, we deliver one of our main results related to modified $\mathcal{Z}$-contraction on the context of θ-metric spaces. This theorem assures us about the existence and uniqueness of the fixed point of a modified $\mathcal{Z}$-contraction. The subsequent lemma forms the basis for our result.

Lemma 3.9. *Given (X, d_θ) be any complete θ-metric space along with $T : X \to X$, a modified $\mathcal{Z}$-contraction with respect to some simulation function $\zeta \in \mathcal{Z}$. Then whenever T possesses any fixed point in X, it is unique.*

Proof. Let $u \in X$ be any fixed point of T. Now let $v \in X$ is another fixed point of T. This means that $Tu = u$ and $Tv = v$.

From Definition 3.6 and using the previous fact, we observe that

$$\begin{aligned} M(u,v) &= max\{d_\theta(u,v), d_\theta(u,Tu), d_\theta(v,Tv)\} \\ &= max\{d_\theta(u,v), d_\theta(u,u), d_\theta(v,v)\} \\ &= d_\theta(u,v). \end{aligned}$$

Using the definition of modified $\mathcal{Z}$-contraction, we attain that

$$\begin{aligned} 0 &\leq \zeta(d_\theta(Tu,Tv), M(u,v)) \\ &= \zeta(d_\theta(Tu,Tv), d_\theta(u,v)) \\ &= \zeta(d_\theta(u,v), d_\theta(u,v)). \end{aligned}$$

Considering Lemma 2.6, above inequality reaches a contradiction and hence the proof follows. □

Now, we are ready to state another main result here.

Theorem 3.10. *Let (X, d_θ) be a complete θ-metric space. Assume that $T : X \to X$ is any modified $\mathcal{Z}$-contraction with respect to some simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point u in X. Furthermore, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ converges to u.*

Proof. Let (X, d_θ) be a θ-metric space and $T : X \to X$ be a modified $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$.

Let x_0 be any arbitrary point and $\{x_n\}$ be the respective Picard sequence, i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now we suppose that $d_\theta(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Otherwise if there exists $n_p \in \mathbb{N} \cup \{0\}$ such that $x_{n_p} = x_{n_p+1}$, then x_{n_p} is a fixed point of T and we are done.

Next we define $d_\theta^n = d_\theta(x_n, x_{n+1})$. Then,

$$\begin{aligned} M(x_n, x_{n-1}) &= max\{d_\theta(x_n, x_{n-1}), d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} \\ &= max\{d_\theta^n, d_\theta^{n-1}\}. \end{aligned}$$

Using Remark 2.6, $\{d_\theta^n\}$ is a decreasing sequence of reals and hence $d_\theta^n < d_\theta^{n-1}$ for all $n \in \mathbb{N} \cup \{0\}$. So we get,

$$\begin{aligned} 0 &\leq \zeta(d_\theta(Tx_n, Tx_{n-1}), M(x_n, x_{n-1})) \\ &= \zeta(d_\theta^n, max\{d_\theta^n, d_\theta^{n-1}\}) \\ &= \zeta(d_\theta^n, d_\theta^{n-1}). \end{aligned}$$

Now $\{d_\theta^n\}$ is a decreasing sequence of non-negative real numbers and hence is convergent. Let

$$\lim_{n\to\infty} d_\theta^n = r.$$

If $r > 0$, we have,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d_\theta^n, d_\theta^{n-1}) \\ &< 0. \end{aligned}$$

We arrive at a contradiction and so $r = 0$.

We claim that the sequence $\{x_n\}$ is bounded. Reasoning by contradiction, we assume that $\{x_n\}$ is unbounded. So, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for every $k \in \mathbb{N}, n_{k+1}$ is the least integer such that

$$d_\theta(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d_\theta(x_m, x_{n_k}) \leq 1 \;\; for \;\; n_k \leq m \leq n_{k+1} - 1.$$

Now, using the triangle inequality, we have

$$\begin{aligned} 1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\ &\leq \theta(d_\theta(x_{n_{k+1}}, x_{n_{k+1}-1}), d_\theta(x_{n_{k+1}-1}, x_{n_k})) \\ &\leq \theta(d_\theta(x_{n_{k+1}}, x_{n_{k+1}-1}), 1). \end{aligned} \tag{4}$$

By taking the limit as $k \to \infty$ on both sides of (4) and using $(B4)$, we infer that,

$$d_\theta(x_{n_{k+1}}, x_{n_k}) \to 1.$$

Also, we have

$$\begin{aligned} 1 &< d_\theta(x_{n_{k+1}}, x_{n_k}) \\ &\leq M(x_{n_{k+1}-1}, x_{n_k-1}) \\ &= max\{d_\theta(x_{n_{k+1}-1}, x_{n_k-1}), d_\theta(x_{n_{k+1}-1}, x_{n_{k+1}}), d_\theta(x_{n_k-1}, x_{n_k})\} \\ &\leq max\{\theta(d_\theta(x_{n_{k+1}-1}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k-1})), d_\theta(x_{n_{k+1}-1}, x_{n_{k+1}}), d_\theta(x_{n_k-1}, x_{n_k})\} \\ &\leq max\{\theta(1, d_\theta(x_{n_k}, x_{n_k-1})), d_\theta(x_{n_{k+1}-1}, x_{n_{k+1}}), d_\theta(x_{n_k-1}, x_{n_k})\}. \end{aligned}$$

As $k \to \infty$, we derive, $1 \leq M(x_{n_{k+1}-1}, x_{n_k-1}) \leq 1$.

So,

$$\lim_{k\to\infty} M(x_{n_{k+1}-1}, x_{n_k-1}) = 1.$$

As T is a modified $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we obtain

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(d_\theta(x_{n_{k+1}}, x_{n_k}), M(x_{n_{k+1}-1}, x_{n_k-1})) \\ &< 0. \end{aligned}$$

This leads to a contradiction and hence $\{x_n\}$ is bounded.

Now we will show that $\{x_n\}$ is Cauchy. For this, we consider the real sequence

$$C_n = sup\{d_\theta(x_i, x_j) : i, j \geq n\}.$$

Note that $\{C_n\}$ is a decreasing sequence of non-negative reals. Thus there exists $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C.$$

Our claim is that $C = 0$. Let us suppose that $C > 0$. By the construction of C_n, for each $k \in \mathbb{N}$, there exists n_k, m_k such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d_\theta(x_{m_k}, x_{n_k}) \leq C_k.$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} d_\theta(x_{m_k}, x_{n_k}) = C,$$

and following the steps as in Theorem 3.3, we have

$$\lim_{k\to\infty} d_\theta(x_{m_k-1}, x_{n_k-1}) = C.$$

Now,

$$\begin{aligned} d_\theta(x_{m_k}, x_{n_k}) &\leq M(x_{m_k-1}, x_{n_k-1}) \\ &= max\{d_\theta(x_{m_k-1}, x_{n_k-1}), d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{n_k-1}, x_{n_k})\}, \\ &= max\{d_\theta(x_{m_k-1}, x_{n_k-1}), d_\theta(x_{m_k-1}, x_{m_k}), d_\theta(x_{n_k-1}, x_{n_k})\}. \end{aligned}$$

Consequently, taking $k \to \infty$, we get,

$$\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = C.$$

Now since T is a modified $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we derive that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(d_\theta(x_{m_k}, x_{n_k}), M(x_{m_k-1}, x_{n_k-1})) \\ &< 0, \end{aligned}$$

which is a contradiction. As a result, $C = 0$ and $\{x_n\}$ is Cauchy.

Since (X, d_θ) is complete, there exists some $z \in X$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now we show that z is a fixed point of T. Suppose, on the contrary, $Tz \neq z$. Then $d_\theta(z, Tz) > 0$. Now we employ Definition 3.6 and use Remark 2.6 to get,

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d_\theta(Tx_n, Tz), M(x_n, z)) \\ &\leq \limsup_{n\to\infty} [M(x_n, z) - d_\theta(x_{n+1}, Tz)] \\ &= -d_\theta(z, Tz). \end{aligned}$$

This contradiction attests that $d_\theta(z, Tz) = 0$, and so, $Tz = z$. Thus z is a fixed point of T. Uniqueness is guaranteed from Lemma 3.9. □

As an application of our earlier result, we furnish the next example which illustrates Theorem 3.10.

Example 3.11. *Let $X = [0, 1]$ be equipped with the usual Euclidean metric and $\theta(s, t) = s + t + st$. We define a mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{2}{9}, & x \in S_1 = [0, \frac{1}{2}), \\ \frac{1}{9}, & x \in S_2 = [\frac{1}{2}, 1]. \end{cases}$$

We argue that T is a modified $\mathcal{Z}$-contraction with respect to the simulation function $\zeta(t, s) = \frac{1}{2}s - t$.

Here we have, $0 \leq d_\theta(Tx, Ty) \leq \frac{1}{9}$ for all $x, y \in X$.

Now, if both $x, y \in S_1$ or S_2, then $d_\theta(Tx, Ty) = 0$ and we are done.

Otherwise, let $x \in S_1$ and $y \in S_2$.

We get $0 < d_\theta(x, y) \leq 1$. Also, $0 \leq d_\theta(x, Tx) \leq \frac{5}{9}$ and $\frac{7}{18} \leq d_\theta(y, Ty) \leq \frac{8}{9}$. Therefore $M(x, y) \geq \frac{7}{18}$. From the calculation, it is clear that

$$d_\theta(Tx, Ty) \leq \frac{1}{2} M(x, y).$$

So we have,

$$\zeta(d_\theta(Tx, Ty), M(x, y)) = \frac{1}{2} M(x, y) - d_\theta(Tx, Ty) \geq 0$$

for all $x, y \in X$.

As a consequence, T is a modified $\mathcal{Z}$-contraction. Taking into account Theorem 3.10, we can say that T has a unique fixed point. Here $u = \frac{2}{9}$ is that required fixed point.

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