Filomat 31:11 (2017), 3391–3406
https://doi.org/10.2298/FIL1711391B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Saturated Contraction Principles for Non Self Operators, Generalizations and Applications

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Abstract. Let (X, d) be a metric space, $Y \subset X$ a nonempty closed subset of X and let $f : Y \to X$ be a non self operator. In this paper we study the following problem: under which conditions on f we have all of the following assertions:

1. The operator f has a unique fixed point;
2. The operator f satisfies a retraction-displacement condition;
3. The fixed point problem for f is well posed;
4. The operator f has the Ostrowski property.

Some applications and open problems related to these questions are also presented.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and preliminaries

Let (X, d) be a complete metric space. Denote by $Y \in P_{cl}(X)$ the family of closed subsets of X, let $Y \in P_{cl}(X)$ and $f : Y \to X$ be a nonself operator. Let us consider the fixed point equation corresponding to f:

$$x = f(x). \tag{1.1}$$

Denote as usually the set of solutions of (1.1) by F_f.

Starting problems on the existence theory for equation (1.1) read as follows:

Problem 1.1. *Find $x_0 \in Y$ and conditions on Y, X and f such that*

$$f^n(x_0) \in Y \text{ for all } n \in \mathbb{N} \text{ and } f^n(x_0) \to x^* \in F_f \text{ as } n \to \infty.$$

2010 *Mathematics Subject Classification*. Primary 47H10; 54H25; Secondary 65J15; 65P40; 34KXX; 45GXX; 45N05

Keywords. metric space; generalized metric space; non self operator; fixed point; retraction; retraction-displacement condition; well posedness; Ostrowski property; data dependence; Ulam stability; demicontractive mapping; basin of attraction

Received: 10 November 2016; Accepted: 14 January 2017

Communicated by Vladimir Rakočević

The first author would like to acknowledge the support provided by the Deanship of Scientific Research at King Fahd University of Petroleum and Minerals for partially funding this work through the projects IN151014 and IN141047.

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If first part is solved, then find an invariant set U under f (i.e., a set satisfying $f(U) \subset U$) such that

$$x^* \in U \text{ and } f^n(x_0) \to x^*, \text{ for all } x_0 \in U.$$

Problem 1.2. *Find $H : Y \times [0,1] \to X$ with $H(\cdot,1) = f$ and suitable conditions on H such that*

$$F_{H(\cdot,0)} \neq \emptyset \Longrightarrow F_f \neq \emptyset.$$

Problem 1.3. *Find $h : Y \to Y$ with $F_f = F_h$ and appropriate conditions on f, Y and h such that*

$$F_h \neq \emptyset.$$

Problem 1.4. *If $Y \subset f(Y)$, under which conditions we have $F_f \neq \emptyset$?*

As references for the above problems we mention the following:

- Problem 1.1: [11], [52], [25], [5], [23], [29], [42], [47], [58], [70], [71], [16], [17], [21], [26], [31], [37], [40], [49], [60], [4], [6], [7], [8], [9], . . .
- Problem 1.2: [20], [34], [50], [18], [30], [51], [57], [24],[27], . . .
- Problem 1.3 and the retraction theory: [19], [65], [64], [62], [14], [32], [33], [36], [35], [41], [42], [43], [10], [61], [74], [28], . . .
- Problem 1.4: [1], [73], [39], [61], [44], . . .

In connection with these problems the following well known results are significant for the present study.

Theorem 1.5. (Reich and Zaslavski [58]) *Let (X,d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be an operator. We suppose that*

1. *f is an l-contraction;*
2. *There exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$.*

Then

1. $F_f = \{x^*\}$;
2. $f^n(x_n) \to x^*$ *as* $n \to \infty$.

Theorem 1.6. (Granas, [34]; see also [30])
Let (X,d) be a complete metric space, $U \subset X$ an open subset and let $h : \overline{U} \times [0,1] \to X$ be such that

1. *$h(\cdot,t) : \overline{U} \to X$, $t \in [0,1]$, are l-contractions;*
2. *$h(x,\cdot) : [0,1] \to X$, $x \in \overline{U}$, are L-Lipschitzian;*
3. *For all $x \in \partial U$ and $t \in [0,1]$ we have that $x \neq h(x,t)$.*

Then

1. $F_{h(\cdot,0)} \neq \emptyset \Longrightarrow F_{h(\cdot,t)} \neq \emptyset$, $\forall t \in [0,1]$;
2. *Let us denote by $x^*(t)$ the unique fixed point of $h(\cdot,t)$. Then the abstract function $x^* : [0,1] \to X$ is Lipschitzian.*

Theorem 1.7. (Brown, see [19] and [64]) *Let (X,d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be an operator. We suppose that*

1. *f is an l-contraction;*
2. *there exists a set retraction $r : X \to Y$ such that:*
   (a) *f is retractible with respect to r, i.e., $F_f = F_{r\circ f}$;*
   (b) *r is L-Lipschitzian;*
   (c) $l\,L < 1$.

Then $F_f = \{x^*\}$.

Theorem 1.8. (Caristi [22], see also [65]) *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$ *be an operator. We suppose that*

1. f *is an l-contraction;*
2. $x \in Y$, $f(x) \in X \setminus Y \Longrightarrow]x, f(x)]_d \cap Y \neq \emptyset$.

Then $F_f = \{x^*\}$.

Note. In Theorem 1.8 above, $]x, f(x)]_d \cap Y$ denotes the set of metrically inward points of f with respect to d, i.e., see [22],

$$]x, f(x)]_d := \{y \in X : d(x, f(x)) = d(x, y) + d(y, f(x)\}.$$

Theorem 1.9. (Reich and Zaslavski [59]) *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$ *be an operator. We suppose that*

1. f *is a contraction;*
2. *There exists a sequence* $\{y_n\}_{n\in\mathbb{N}^*}$ *in* Y *such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

Then $y_n \to x^*$ *as* $n \to \infty$ *and* $F_f = \{x^*\}$.

Theorem 1.10. (Folklore) *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$ *be an operator. We suppose that*

1. *there exists* $l > 1$ *such that*

$$d(f(x), f(y)) \geq l d(x, y), \forall x, y \in Y.$$

2. $Y \subset f(Y)$;
3. f *is an injective operator.*

Then $F_f = \{x^*\}$.

On the other hand, in [67], the following result has been proven for the case of self contractions.

Theorem 1.11. (Saturated principle of contraction) *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an l-contraction. Then:*

(i) There exists $x^* \in X$ *such that* $F_{f^n} = \{x^*\}$, $\forall n \in \mathbb{N}^*$.

(ii) For all $x \in X$, $f^n(x) \to x^*$ *as* $n \to \infty$.

(iii) $d(x, x^*) \leq \psi(d(x, f(x))$, $\forall x \in X$, *where* $\psi(t) = \frac{t}{1-l}$, $t \geq 0$, *i.e.,* f *is a* Ψ*-Picard operator.*

(iv) If $\{y_n\}_{n\in\mathbb{N}}$ *is a sequence in* X *such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty$$

then, $y_n \to x^*$ *as* $n \to \infty$, *i.e., the fixed point problem for* f *is well posed.*

(v) If $\{y_n\}_{n\in\mathbb{N}}$ *is a sequence in* X *such that*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty$$

then, $y_n \to x^*$ *as* $n \to \infty$, *, i.e., the operator* f *has the Ostrowski property.*

(vi) If $Y \subset X$ *is a closed subset such that* $f(Y) \subset Y$, *then* $x^* \in Y$. *Moreover, if, in addition,* Y *is bounded, then*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Starting from this background, the aim of this paper is to extend Theorem 1.11 to the case of nonself operators. Some generalizations, applications and open problems are also presented. The structure of the remaining part of our paper is the following

- Saturated fixed point principles for non self operators in metric spaces
- Saturated fixed point principles for non self operators in Banach spaces
- Applications
- Demicontractive nonself mappings
- The estimation of local convergence radius
- Other research directions

2. Saturated principle of nonself contraction in metric spaces

We start this section with the following result on the existence of fixed points of non self contractions.

Theorem 2.1. (Theorem of equivalent statements for nonself contraction)
Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ a contraction. Then the following statements are equivalent.

(a) *There exists $x^* \in Y$ such that $F_f = \{x^*\}$;*

(b) *There exists a sequence $\{y_n\}_{n\in\mathbb{N}^*}$ in Y such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

(c) *There exists a bounded subset $Z \subset X$ such that, if we denote*

$$Z_1 := f(Z), Z_2 = f(Z_1 \cap Y), \ldots, Z_{n+1} = f(Z_n \cap Y),$$

then $Z_n \neq \emptyset$, for all $n \in \mathbb{N}^*$.

(d) *There exists a bounded sequence $\{y_n\}_{n\in\mathbb{N}^*}$ in Y such that*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty.$$

(e) *There exists $U \in P_{cl}(Y)$ such that $f(U) \subset U$.*

(f) *There exists $U \in P_{cl}(Y)$ and a nonexpansive retraction $r : X \to U$ such that $f : U \to X$ is retractible with respect to r.*

Proof. First, we remark that (a) implies all of the statements (b)-(f).
$(b) \Longrightarrow (a)$. It follows by Theorem 1.9.
$(c) \Longrightarrow (a)$. It follows by Theorem 1.5.
It is also clear that $(c) \Longleftrightarrow (d)$.
$(d) \Longrightarrow (a)$. We have

$$\begin{aligned} d(y_{n+p+1}, y_{n+1}) &\leq d(y_{n+p+1}, f(y_{n+p})) + d(f(y_{n+p}), f(y_n)) + d(y_{n+1}, f(y_n)) \\ &\leq d(y_{n+p+1}, f(y_{n+p})) + ld(y_{n+p}), f(y_n) + d(y_{n+1}, f(y_n)) \\ &\leq d(y_{n+p+1}, f(y_{n+p})) + ld(y_{n+p}), f(y_{n+p-1})) + \cdots + l^n d(y_p), f(y_{p-1})) \\ &+ l^{n+1} d(y_p), f(y_0)) + l^n d(y_1), f(y_0)) + \cdots + l^1 d(y_n), f(y_{n-1})) + d(y_{n+1}), f(y_n)). \end{aligned}$$

From Cauchy lemma (see, for example, [62]), it follows that

$$d(y_{n+p+1}, y_{n+1}) \to 0 \text{ as } n \to \infty, \forall p \in \mathbb{N} \text{ or } p \to \infty.$$

Since (X, d) is complete, it follows that $\{y_n\}_{n\in\mathbb{N}^*}$ is convergent.

Let $y_n \to y^*$ as $n \to \infty$. By statement (d) and the continuity of the metric d and of the mapping f, we now conclude that $d(y^*, f(y^*)) = 0$, i.e., $y^* \in F_f$.

$(e) \Longrightarrow (a)$. It follows from the contraction principle.

$(f) \Longrightarrow (a)$. Since $f|_U$ is retractible with respect to r, it follows that $F_{f|_U} = F_{r\circ f|_U}$. But $r \circ f|_U : U \to U$ is a contraction.

□

Now, using Theorem 1.11, we can give a first answer to our problems.

Theorem 2.2. (Saturated principle of nonself contraction)
Let (X, d) be a metric space, $Y \subset X$ and $f : Y \to X$ an operator. We suppose that:

(a) f is an l-contraction.

(b) $F_f \neq \emptyset$.

Then:

(i) $F_f = \{x^*\}$. Moreover, if for some $y \in Y$ and $n \in \mathbb{N}^*$, $f^n(y)$ is defined and $f^n(y) = y$, then $y = x^*$.

(ii) $d(x, x^*) \leq \psi(d(x, f(x))$, $\forall x \in Y$, where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$.

(iii) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that, $y_n \to x^*$ as $n \to \infty$.

(iv) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

(v) Let $Z \subset Y$ be a bounded subset with $x^* \in Z$. Let us denote

$$Z_1 := f(Z), Z_2 = f(Z_1 \cap Y), \ldots, Z_{n+1} = f(Z_n \cap Y), \ldots.$$

Then

$$\bigcap_{n=1}^{\infty} Z_n = \{x^*\}.$$

Proof. (i) The uniqueness of the fixed point follows by the contraction condition. Let $y \in Y$ and $n \in \mathbb{N}^*$ be such that $f^n(y)$ is defined. Since $f^n(y) = y$, we have $f^{n+1}(y) = f(f^n(y)) = f(y)$ and $f^{n+1}(y) = f^n(f(y))$ which imply $y = f(y)$, i.e., $y \in F_f = \{x^*\}$.

(ii) It follows by the fact that

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + ld(x, x^*),$$

which yields the retraction-displacement condition.

(iii) It follows by (ii).

(iv) By (a), we obtain, in particular, that f is an l-quasicontraction, i.e.,

$$d(f(x), f(x^*)) \leq ld(x, x^*), \forall x \in Y.$$

We have

$$d(y_{n+1}, x^*) \leq d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \leq ld(y_n, x^*) + d(y_{n+1}, f(y_n)),$$

which, by Lemma 1.6, part (i) in [5], implies the conclusion. (An alternative proof can be given by using the Cauchy lemma, see [67].)

(v) This is a Reich-Zaslavski theorem (see [58], pp. 308). We present the proof given by Rus-Şerban ([70]). Let $Z \subset Y$ be a bounded closed subset such that $y_n \in Z$, $n \in \mathbb{N}$. We consider the following standard construction:

$$Z_1 := \overline{f(Z)},\ Z_2 := \overline{f(Z_1) \cap Z}, ..., Z_{n+1} := \overline{f(Z_n) \cap Z},\ n \in \mathbb{N}^*.$$

We remark that

a) $Z_{n+1} \subset Z_n,\ n \in \mathbb{N}^*$;

b) $f^n(y_n) \subset Z_n,\ n \in \mathbb{N}^*,\ so,\ Z_n \neq \emptyset,\ \forall n \in \mathbb{N}^*$.

On the other hand (δ denotes the diameter)

$$\begin{aligned}\delta(Z_{n+1}) &= \delta(\overline{f(Z_n \cap Z)}) = \delta(f(Z_n \cap Z)) \leq \delta(f(Z_n))\\ &\leq l\delta(Z_n) \leq ... \leq l^{n+1}(Z) \to 0,\ as\ n \to \infty.\end{aligned}$$

From Cantor intersection lemma we have

$$Z_\infty = \cap_{n\in\mathbb{N}^*} Z_n \neq \emptyset,\ \delta(Z_\infty) = 0 \text{ and } f(Z_\infty \cap Z) \subset Z_\infty.$$

These imply that $Z_\infty = \{x^*\}$. Since $f^n(y_n) \in Z_n$ we have that

$$f^n(y_n) \to 0 \text{ as } n \to \infty.$$

A simpler proof of (v) reads as follows. Since f is an l-quasicontraction, we have

$$\delta_d(Z_{n+1}, \{x^*\}) \leq l\delta_d(Z_n \cap Y, \{x^*\}) \leq l\delta_d(Z_n, \{x^*\}) \ldots$$

$$\leq l^{n+1}\delta_d(Z, \{x^*\}) \to 0 \text{ as } n \to \infty,$$

where δ_d denotes the diameter functional with two arguments. □

Remark 2.3.

1. *If (X, d) is a complete metric space and $Y \in P_{cl}(X)$, then we can replace condition (b) in Theorem 2.2 by any of the statements (b)-(f) in Theorem 1.11 (see also condition (b) in Theorem 1.5, Theorem 1.7, Theorem 1.8 and Theorem 1.9). Of course, we can also use Theorem 1.6 and Theorem 1.10 in order to have condition (b) in Theorem 2.2 satisfied.*
2. *Apart of those cases, condition (b) in Theorem 2.2 can also be replaced by the so called metrically inward condition, due to Caristi [22]:*
   (b)′ If $x \in Y$ is such that $f(x) \in X \setminus Y$, then there exists $y \in Y,\ y \neq x$, such that $d(x, y) + d(y, f(x)) = d(x, f(x))$.

The following variants of Theorem 2.2 are very useful in the case of generalized contractions and will be used in Section 3.

Theorem 2.4. *Let (X, d) be a complete metric space, $Y \subset X$ and $f : Y \to X$ an operator. We suppose that there exists a fixed point x^* of f and a constant l, $0 < l < 1$, such that*

$$d(f(x), x^*) \leq ld(x, x^*), \forall x \in Y. \tag{2.1}$$

Then all the conclusions in Theorem 2.2 hold.

Theorem 2.5. *Let (X, d) be a complete metric space, $Y \subset X$ and $f : Y \to X$ an operator with $F_f = \{x^*\}$. We suppose that for a metric ρ on X we have:*
(a) there exists $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y), \forall x, y \in X.$$

(b) there exists a constant $0 < l < 1$ *such that*

$$\rho(f(x), x^*) \leq l d(x, x^*), \forall x \in Y.$$

Then, in (X, d)*, we have all the conclusions of Theorem 2.2 satisfied with*

$(ii)'$ $d(x, x^*) \leq \dfrac{c_2}{c_1(1-l)} d(x, f(x)),\ \forall x \in Y.$

Proof. The conclusion follows by Theorem 2.4 in (X, ρ) and then by assumption (a) on the strongly equivalence of the metrics d and ρ. □

Remark 2.6. *For more considerations related to Theorem 2.5, see also [56] and [67].*

In literature, we can find some papers devoted to the relevance of a metric condition in the fixed point theory, see for example [69], [10], [55], [67], [68]. Now, in view of the previous results, we can consider the relevance of a metric condition in the following sense. Let (X, d) *be a complete metric space,* $Y \subset X$ *and* $f : Y \to X$ *an operator.*

We shall say that a metrical condition which implies the uniqueness of the fixed point of f *is relevant if this condition also ensures all the conclusions in Theorem 2.2.*

3. Saturated principles for nonself generalized contractions in Banach spaces

In this section we present three relevant metric conditions and the corresponding nonself saturated fixed point principles. For the sake of simplicity, the presentation should be given in the setting of a Banach space, but all results could be also established in convex metric spaces, too, see for example [72].

Let X be a Banach space, $Y \in P_{cl}(X)$, and $f : Y \to X$ a non-self mapping. If $x \in Y$ is such that $f(x) \notin Y$, then we suppose that we can choose an $y \in \partial Y$ (the boundary of Y) such that $y = (1-\lambda)x + \lambda f(x)$ $(0 < \lambda < 1)$, which actually expresses the fact that

$$d(x, f(x)) = d(x, y) + d(y, f(x)),\ y \in \partial Y, \tag{3.1}$$

where we denoted $d(x, y) = \|x - y\|$.

In general, the set K of all points y satisfying condition (3.1) above may contain more than one element. Note that K is included in the set of metrically inward points of f with respect to d, see [22],

In order to state our main results in this section we need the following concept from [11].

Definition 3.1. *Let X be a Banach space, Y a nonempty closed subset of X and* $f : Y \to X$ *a non-self mapping. Let* $x \in Y$ *with* $f(x) \notin Y$ *and let* $y \in \partial Y$ *be the corresponding elements defined by* (3.1)*. If, for any such elements x, we have*

$$d(y, f(y)) \leq d(x, f(x)), \tag{3.2}$$

for at least one corresponding element $y \in K$*, then we say that* f *has property* (M)*.*

Theorem 3.2. (Saturated principle for nonself Kannan type contractions)
Let (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$ *a Kannan non self mapping, i.e., a mapping for which there exists a constant* $a \in (0, 1/2)$ *such that*

$$d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \forall x, y \in Y. \tag{3.3}$$

Suppose also that:

(a) f *has property* (M)*;*

(b) f *satisfies Rothe's boundary condition*

$$T(\partial Y) \subset Y. \tag{3.4}$$

Then:

(i) $F_f = \{x^*\}$. *Moreover, if for some* $y \in Y$ *and* $n \in \mathbb{N}^*$, $f^n(y)$ *is defined and* $f^n(y) = y$, *then* $y = x^*$.

(ii) $d(x, x^*) \leq \dfrac{1}{1-l} d(x, f(x))$, $\forall x \in Y$, *where* $l = \dfrac{a}{1-a}$;

(iii) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ *in* Y *with*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that, $y_n \to x^*$ *as* $n \to \infty$.

(iv) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ *in* Y *with*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

(v) Let $Z \subset Y$ *be a bounded subset with* $x^* \in Z$. *Let us denote*

$$Z_1 := f(Z), Z_2 = f(Z_1 \cap Y), \ldots, Z_{n+1} = f(Z_n \cap Y), \ldots.$$

Then $Z_n \neq \emptyset$, *for all* $n \in \mathbb{N}^*$.

Proof. By (3.3) we obtain similarly to [3],

$$d(f(x), f(y)) \leq l d(x, y) + L d(y, f(x)), \forall x, y \in Y \tag{3.5}$$

and

$$d(f(x), f(y)) \leq l d(x, y) + L d(x, f(x)), \forall x, y \in Y. \tag{3.6}$$

with $l = \dfrac{a}{1-a} < 1$ and $L = \dfrac{2a}{1-a} > 0$.

Now, due to (3.5), by Theorem 3.3 in [11] we obtain that $F_f \neq \emptyset$, while, by (3.6) we obtain that the fixed point set of f is a singleton. Denote $F_f = \{x^*\}$.

Next, by (3.6) with $x := x^*$ and $y := x$, we get the quasi-contraction condition (2.1). The rest of the proof follows by Theorem 2.4. □

For a result related to the one given by Theorem 3.2, established in the setting of a Banach space endowed with a graph, we refer to [2].

Theorem 3.3. (Saturated principle for nonself Chatterjea type contractions)
Let (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$ *a Chatterjea non self mapping, i.e., a mapping for which there exists a constant* $b \in (0, 1/2)$ *such that*

$$d(f(x), f(y)) \leq b[d(x, f(y)) + d(x, f(y))], \forall x, y \in Y. \tag{3.7}$$

Suppose also that:

(a) f *has property* (M);

(b) f *satisfies Rothe's boundary condition* (3.4)

Then:

(i) $F_f = \{x^*\}$. *Moreover, if for some* $y \in Y$ *and* $n \in \mathbb{N}^*$, $f^n(y)$ *is defined and* $f^n(y) = y$, *then* $y = x^*$.

(ii) $d(x, x^*) \leq \dfrac{1}{1-l} d(x, f(x))$, $\forall x \in Y$, *where* $l = \dfrac{b}{1-b}$;

(iii) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ *in* Y *with*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that, $y_n \to x^*$ *as* $n \to \infty$.

(iv) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

(v) Let $Z \subset Y$ be a bounded subset with $x^* \in Z$. Let us denote

$$Z_1 := f(Z), Z_2 = f(Z_1 \cap Y), \ldots, Z_{n+1} = f(Z_n \cap Y), \ldots.$$

Then $Z_n \neq \emptyset$, for all $n \in \mathbb{N}^*$.

Proof. By (3.7) we obtain similarly to [3],

$$d(f(x), f(y)) \leq ld(x, y) + Ld(y, f(x)), \forall x, y \in Y \tag{3.8}$$

and

$$d(f(x), f(y)) \leq ld(x, y) + Ld(x, f(x)), \forall x, y \in Y. \tag{3.9}$$

with $l = \dfrac{b}{1-b} < 1$ and $L = \dfrac{2b}{1-b} > 0$.

Now, due to (3.8), by Theorem 3.3 in [11] we obtain that $F_f \neq \emptyset$, while, by (3.9) we obtain the uniqueness of the fixed point. Denote $F_f = \{x^*\}$.

Now by (3.9) with $x := x^*$ and $y := x$, we get the quasi-contraction condition (2.1). The rest of the proof follows by Theorem 2.4.

□

The previous two theorems could now be unified to get a more general saturated principle for the class of nonself strict almost contractions, see also [13], [14], [53], [54], for more related developments.

Theorem 3.4. (Saturated principle for nonself strict almost contractions)
Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ an nonself strict almost contraction, i.e., a mapping for which there exist the constants $l, l_1 \in (0, 1)$ and $L, L_1 \geq 0$ such that

$$d(f(x), f(y)) \leq ld(x, y) + Ld(y, f(x)), \forall x, y \in Y \tag{3.10}$$

and

$$d(f(x), f(y)) \leq l_1 d(x, y) + L_1 d(x, f(x)), \forall x, y \in Y. \tag{3.11}$$

Suppose also that:

(a) f has property (M);

(b) f satisfies Rothe's boundary condition (3.4)

Then:

(i) $F_f = \{x^*\}$. Moreover, if for some $y \in Y$ and $n \in \mathbb{N}^*$, $f^n(y)$ is defined and $f^n(y) = y$, then $y = x^*$.

(ii) $d(x, x^*) \leq \dfrac{1}{1-l_1} d(x, f(x)),\ \forall x \in Y$;

(iii) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that, $y_n \to x^*$ as $n \to \infty$.

(iv) For each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

(v) Let $Z \subset Y$ be a bounded subset with $x^* \in Z$. Let us denote

$$Z_1 := f(Z), Z_2 = f(Z_1 \cap Y), \ldots, Z_{n+1} = f(Z_n \cap Y), \ldots.$$

Then $Z_n \neq \emptyset$, for all $n \in \mathbb{N}^*$.

Proof. We use Theorem 3.3 in [11] to obtain (i) and since, by (3.11) with $x := x^*$ and $y := x$, we get the quasi-contraction condition (2.1), the rest of the proof follows by Theorem 2.4.

□

4. Applications

It is clear that we have many abstract applications to operatorial equations and concrete applications to ODE, PDE, integral equations,...

In general, in any case where one apply contraction principle, by means of saturated principle of contraction we get more information. On the other hand, we can study various problems like data dependence, Ulam stability etc. In what follows, we present only two such kind of applications.

4.1. Data dependence

Let (X, d) be a metric space, $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator like in Theorem 2.2. Let $g : Y \to X$ be another operator (a perturbation of f) for which we suppose that $F_g \neq \emptyset$ and there exists $\eta > 0$ such that

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in Y.$$

By Theorem 2.2 we have

Theorem 4.1. *Under the above conditions, we have that*

$$d(x^*, y^*) \leq \frac{\eta}{1-l},$$

for all $y^* \in F_g$.

Proof. From conclusion (ii) in Theorem 2.2 we have

$$d(x^*, y^*) \leq \frac{1}{1-l} d(y^*, f(y^*)) = \frac{1}{1-l} d(g(y^*), f(y^*)) \leq \frac{\eta}{1-l}.$$

□

For more considerations on data dependence problem, see also [5], [14], [21], [24], [53], [60], [62], [70],...

4.2. Ulam stability of a fixed point equation

Let (X, d) be a metric space, $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator. By definition, see [66], the fixed point equation

$$x = f(x) \tag{4.1}$$

is Ulam-Hyers stable if there exists a constant $c_f > 0$ such that, for each $\varepsilon > 0$ and any solution $y^* \in X$ of the inequation

$$d(y, f(y)) \leq \varepsilon, \tag{4.2}$$

there exists a solution x^* of (4.1) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

We have

Theorem 4.2. *Let f be as in Theorem 2.2. Then the equation* (4.1) *is Ulam-Hyers stable with*

$$c_f = \frac{1}{1-l}.$$

Proof. From conclusion (ii) in Theorem 2.2 we have

$$d(x^*, y^*) \leq \frac{1}{1-l} d(y^*, f(y^*)) = \frac{1}{1-l}\varepsilon.$$

□

For more considerations on Ulam stability problem, see also [14], [56], [62], [66], [70],...

5. Demicontractive mappings in Hilbert spaces

All previous results were given for classes of non self mappings satisfying a certain relevant metric condition which ensured that the unique fixed point x^* could be obtained as the limit of Picard iteration $\{x_n\}$

$$x_{n+1} = f(x_n), n \geq 0 \tag{5.1}$$

with $x_0 \in Y$ the starting value.

By the results in this section, we would like to illustrate how, by weakening a relevant metric condition to a non relevant one in the sense given at the end of Section 2, most of the conclusions of the saturated principle of contraction for non self mappings are lost.

However, this drawback is compensated by the generality of such a weaker contraction condition. Moreover, in such situations, instead of the Picard type iteration (5.1) associated to f, which is generally not convergent or, even if it converges, its limit is not a fixed point of the operator f, we have to consider some more complex iterative methods, like Mann iteration, see [5]. These iterative methods can be constructed in the presence of a linear structure of the ambient space. This is one of the main reasons we shall work in a Hilbert space in this section.

Let $\mathcal{H}$ be a real Hilbert space, with scalar product denoted by $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be an open subset of $\mathcal{H}$, and $f : C \to \mathcal{H}$ a mapping (possible, nonself) with nonempty set of fixed points in C, that is, $F_f \neq \emptyset$.

Recall that, according to [38], f is said to be *demicontractive* on C if

$$\|f(x) - p\|^2 \leq \|x - p\|^2 + k\|x - f(x)\|^2, \ \forall x \in C, \ \forall p \in F_f, \ 0 < k < 1. \tag{5.2}$$

We notice that if f is an l-quasicontraction or quasi-nonexpansive, then f is demicontractive, but the reverse is no more true.

It is also well known, see [46, 48] that condition (5.2) is equivalent to the following one

$$\langle x - f(x), x - p\rangle \geq \lambda\|x - f(x)\|^2, \ \forall x \in C, \ \forall p \in F_f, \ \lambda = (1-k)/2. \tag{5.3}$$

Note that the demicontractive condition in the form (5.3) has been introduced independently by the second author [45] in 1977, the same year in which the paper by Hicks and Kubicek [38] has been published, but under a different name: the mapping f is said to satisfy condition (A).

Remind, see [45], that a self mapping $T : C \to C$ is said to be *demiclosed at* 0 if, for a sequence $\{u_k\}$ in C which converges weakly to $u \in C$ and for which $\{Tu_k\}$ converges strongly to zero, we have $Tu = 0$.

Theorem 5.1. *Suppose that $f : C \to \mathcal{H}$ is demicontractive on C and $I - f$ is demiclosed at zero. Let $p \in F_f$ be a fixed point of f and $B(p, r)$ a closed ball belonging to C, $B(p, r) = \{x : \|x - p\| \leq r\} \subset C$. Then the sequence $\{x_n\}$ generated by Mann iteration*

$$x_{n+1} = (1 - t_n)x_n + t_n f(x_n), \ x_0 \in B(p, r),$$

where the control sequence $\{t_n\}$ *satisfies* $0 < a \leq t_n \leq b < 2\lambda$, *remains in* $B(p, r)$ *and converges weakly to a fixed point of* f. *If, in addition,* f *satisfies the* quasi-expansive *condition:*

$$\|x - p\| \leq \beta\|x - f(x)\|, \ \forall x \in C, \ \beta > 0, \tag{5.4}$$

then $\{x_n\}$ *converges in norm to the unique fixed point* p *of* f *in* $B(p, r)$.

The proof follows verbatim the proof of Theorem 1 in [45]. If, for a particular x_n, we have $f(x_n) \in \mathcal{H} \setminus C$, then $x_{n+1} \in C$ (actually $x_{n+1} \in B(p, r)$) and hence the sequence $\{x_n\}$ is well defined.

Let us note that the retraction-displacement condition (5.4) is involved in Theorem 5.1 as an assumption and not as a conclusion, like in the case of Theorem 2.2.

Note also that, for a given numerical control sequence, the iteration function of Mann iteration, i.e., $T_{t_n} = (1 - t_n)I + t_n f$, satisfies condition $(b)'$ in Remark 2.3 which, as it was noticed, can be used to replace condition (b) in Theorem 2.2.

Indeed, take $y = x - mt_n(x - f(x))$ and find a value for m such that $y \neq x$. Consider now the following *generalized Mann iteration* [47]:

$$x_{n+1} = (I - D_n)x_n + D_n f(x_n), \tag{5.5}$$

where I is the identity mapping and $\{D_n\} \subset \mathcal{L}(\mathcal{H})$ is the "generalized control sequence" and consists of a sequence of linear bounded mappings. Usually this sequence is defined as a function of x, $D_x = \mathcal{D}(x)$, and we will use in the sequel the notation $D_n = D_{x_n}$.

Various known iterative methods are particular cases of (5.5). For example, if $D_x = I$ (the identity mapping), then we obtain the Picard iteration; if F is a Fréchet differentiable mapping and $D_x = F'(x)^{-1}$ and $f(x) = x - F(x)$ then (5.5) reduces to the classical Newton method for the equation $F(x) = 0$; if $D_x = F'(x)^{-1}$ and $f(x) = x - F(x) - F(x - D_x F(x))$ then we obtain a three order method of the form:

$$x_{n+1} = x_n - F'(x_n)(F(x_n) + F(x - F'(x_n)^{-1}F(x_n))). \tag{5.6}$$

In fact this is a modified Newton method in which the derivative is re-evaluated after two steps; often it is called "Potra-Ptak" method. Note that (5.6) is a particular case of a multipoint iterative processes with order three of convergence considered by Ezquerro and Hernandez [29].

The class of *generalized demicontractive* mappings is defined by

$$\langle D_x(x - f(x)), x - p\rangle \geq \lambda\|D_x(x - f(x))\|^2, \ \forall x \in C, \ p \in F_f, \tag{5.7}$$

where λ is a positive number.

Theorem 5.2. *Let* $f : C \to \mathcal{H}$ *be a nonlinear mapping with nonempty set of fixed points,* $F_f \neq \emptyset$. *Suppose the following conditions are satisfied:*

(i) D_n *is invertible and* $\|D_n^{-1}\| \leq M$, $\forall n$;

ii) for some fixed point p_0 *of* f *and* $r > 0$, $B(p_0, r) \subset C$;

(iii) f *is demiclosed at zero and satisfies the generalized demicontractivity condition* (5.6) *with accretive coefficient* $0.5 < \lambda \leq 1$, $\forall x \in B(p_0, r)$ *and* $\forall p \in F_f$.

Then the sequence (5.5) *generated by the generalized Mann iteration with starting point* $x_0 \in B(p_0, r)$ *converges weakly to a fixed point of* T.

If, in addition, T *satisfies the quasi-expansivity condition* (5.4) *on* C, *then the sequence* (5.5) *converges strongly to the unique fixed point of* f *in* $B(p_0, r)$.

Proof. Suppose $x_n \in B(p_0, r)$. We have, for any $p \in F_f$,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (2\lambda - 1)\|D_n(x_n - f(x_n))\|^2 \leq \|x_n - p\|^2. \tag{5.8}$$

In particular (5.8) holds for p_0, therefore, $x_{n+1} \in B(p_0, r)$ and thus $\{x_n\} \subset B(p_0, r)$. From (5.8) it follows also that $\|x_n - p\| \to \varrho_p$ (say). Hence

$$\begin{aligned} M^{-1}\|x_n - f(x_n)\|^2 &\leq \|D_n^{-1}\|^{-1}\|x_n - f(x_n)\|^2 \leq \|D_n(x_n - f(x_n))\|^2 \\ &\leq (2\lambda - 1)^{-1}(\|x_{n+1} - p\|^2 - \|x_n - p\|^2) \to 0, \end{aligned}$$

and therefore $\|x_n - f(x_n)\| \to 0$. As $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_j}\}$ which converge weakly to some q. In particular $\|x_{n_j} - f(x_{n_j})\| \to 0$ and from the demiclosedness at zero of f we deduce that $q \in F_f$.

Suppose there exist two sequences, say $\{u_k\}$ and $\{v_k\}$, which converge weakly to u and v, respectively. As above, $u, v \in F_f$. From $\|x_n - u\| \to \varrho_u$ and $\|x_n - v\| \to \varrho_v$ it results that $\|u_k - u\| \to \varrho_u$, $\|u_k - v\| \to \varrho_u$ and $\|u_k - v\| \to \varrho_v$, $\|v_k - v\| \to \varrho_u$. Let e_k be defined by

$$e_k = \|u_k - u\|^2 - \|v_k - u\|^2 - \|u_k - v\|^2 - \|v_k - v\|^2.$$

It is obvious that $e_k \to 0$, as $n \to \infty$.

On the other hand, $e_k = -2\langle u_k - v_k, u - v\rangle \to -2\|u - v\|^2$ and so $u = v$. Hence, we deduce that the whole sequence $\{x_n\}$ converges weakly to some $q \in Fix(T)$.

Finally, if f is quasi-expansive on $B(p_0, r)$ then, because $\|x_n - f(x_n)\| \to 0$, it results that $\|x_n - p\| \to 0$. □

Remark 5.3. *As we can see from the last part of the proof of Theorem 5.2, conclusion (iii) in Theorem 2.2 holds in this context, too.*

For an interesting recent study that illustrate the relationship between the class of demicontractive mappings involved in Theorem 5.1 and Theorem 5.2, on the one hand, and the class of Kannan contractions, involved in Theorem 2.2, on the other hand, we refer to [48].

For other iterative algorithms in the case of non self operators, see [10].

6. The estimation of local convergence radius

In the case of finite dimensional spaces, the condition (5.4) is superfluous. Therefore, in finite dimensional spaces, based on Theorem 5.2, see also [46], we can design the following algorithm for estimating the local convergence radius for the generalized Mann iteration. The main steps of the algorithm are:

1. Apply a search line algorithm (for example of the type half-step algorithm) on the positive real axis to find the largest value for r;

2. At every step 1, solve the following constraint optimization problem

$$m = \min_{x \in B(p,r)} \frac{\langle D_x(x - T(x)), x - p\rangle}{\|D_x(x - T(x))\|^2},$$

and verify condition $m > 0.5$.

We applied this algorithm for various iterative methods and for a number of mappings in one or several variables. Interesting results were obtained in the case of (5.6) method. For example, for the following three mapping in two variables:

$$(a)\ F(x) = \begin{pmatrix} 0.8x_1 - cos(x_1) + x_2^2 + 1 \\ x_1^3 + 0.8x_2 \end{pmatrix},$$

$$(b)\ F(x) = \begin{pmatrix} 3x_1^2 - x_1x_2^3 + 3x_2 \\ 2x_1 + x_2^3 - 0.2x_2 \end{pmatrix},$$

$$(c)\ F(x) = \begin{pmatrix} x_1x_2^3 - x_1 + 2x_2^2 \\ x_1^2 + sin(x_2) \end{pmatrix},$$

the results are given in Figure 1.

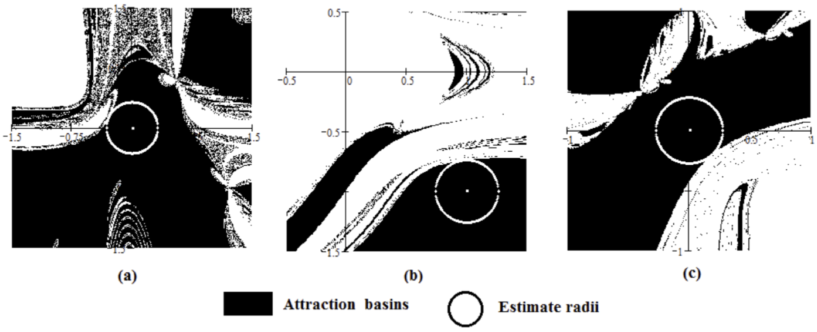

Figure 1: The estimate of local convergence radii by using the proposed algorithm

The black areas represents the whole attraction basins, while the white circles represents the local convergence balls. It can be seen that the proposed algorithm gives convergence radii very close to the maximum ones and sometimes it gives even the maximum convergence radii.

7. Other research directions

We end the paper by indicating some open problems suggested by the above presented results, to indicate some future directions of research.

1. Obtain saturated principles for non self mappings in the case $d(x, y) \in \mathbb{R}^m_+$;
2. Obtain saturated principles for non self mappings in the case $d(x, y) \in s(\mathbb{R}_+)$;
3. Study the generalized retract of a non self operator (see [65]);
4. Study nonself Picard operators (see [60], [24]);
5. Obtain fiber contraction principles for non self operators;
6. Obtain saturated principles for nonself operators on ordered metric spaces.

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